Approximation in Geometry

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Approximation of convex bodies by polytopes

Two notions of distance of convex sets

Convex body: compact convex set with nonempty interior in $\mathbb{R}^d$

Convex polytope: the convex hull of finitely many points in $\mathbb{R}^d$.
May assume: nonempty interior.

Hausdorff distance of two convex sets K and L:

$$\delta_H(K, L) = \inf\{\delta > 0 \ : \ K + \mathbf{B}^d(o, \delta) \subseteq L, L + \mathbf{B}^d(o, \delta) \subseteq K\}.$$

Geometric distance of K and L:

$$d(K, L) = \inf\{\alpha/\beta \ : \ \alpha, \beta > 0, \beta K \subseteq L \subseteq \alpha K\}.$$

This definition is sensitive to the *choice of the origin*.

Simplest approach: The packing bound

Claim (Maximal packing of balls yields Hausdorff approximation — Exercise)

If $\Lambda \subset \mathbb{R}^d$ is such that $\Lambda + \mathbf{B}^d(o, \varepsilon/2)$ is a maximal packing of $\varepsilon/2$ radius balls in $K + \mathbf{B}^d(o, \varepsilon/2)$, then $P = \mathrm{conv}(\Lambda)$ satisfies $\delta_H(P, K) \leq \varepsilon$.

Claim (Volume bound for size of a packing of balls — Ex.)

If $\Lambda + \mathbf{B}^d(o, \varepsilon/2) \subseteq K + \mathbf{B}^d(o, \varepsilon/2)$, then Λ is of cardinality at most $\frac{\mathrm{vol}_d\left(K + \mathbf{B}^d(o, \varepsilon/2)\right)}{\mathrm{vol}_d\left(\mathbf{B}^d(o, \varepsilon/2)\right)}$.

Theorem (Approximation in geometric distance — Ex.)

For any $\varepsilon > 0$ and d, there is a convex polytope P with $\lesssim \left(\frac{3}{\varepsilon}\right)^d$ vertices that is $(1+\varepsilon)$-close to $K + t$ in the geometric distance with an appropriate translation vector $t \in \mathbb{R}^d$.

Approxiation in the geometric distance through hitting caps

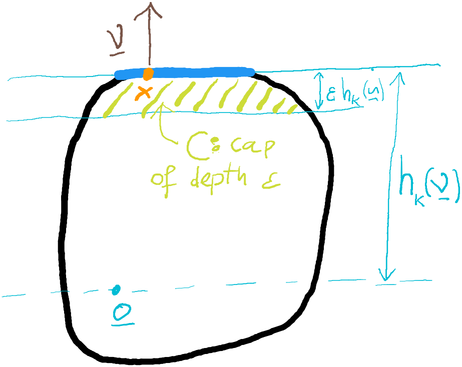
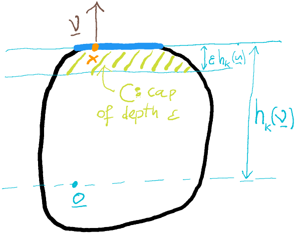


Figure: A cap

Approxiation in the geometric distance through hitting caps



Claim

K smooth convex body, $o \in \mathrm{int}\,(K)$, $\varepsilon \in (0,1)$.
$X \subset K$ finite set.
Then $P = \mathrm{conv}\,(X)$ satisfies $d(K,P) \leq \frac{1}{1-\varepsilon}$ if and only if, X intersects every *cap of depth* ε, that is, every set of the form $\mathrm{cap}\,(x,\varepsilon) = \{y \in K \ : \ \langle y,\nu\rangle \geq (1-\varepsilon)\,\langle x,\nu\rangle\}$, where $x \in \partial K$ and ν is an outer unit normal vector of K at x.

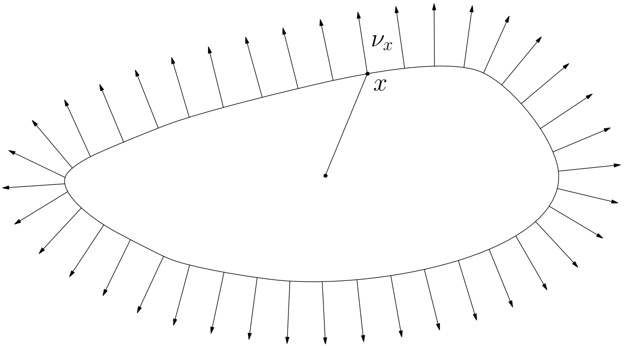
The Bronshteĭn–Ivanov net

K – convex body with smooth boundary, $o \in K, K \subset \mathbf{B}^d(o,R)$.

$S := \{x + \nu_x : x \in \partial K\}$, where ν_x : the outer unit normal to ∂K at x.

$\{x_j + \nu_{x_j} \; : \; j \in [N]\}$ – maximal ρ-separated set in S, i.e., any two elements are at distance $\geq \rho$ (see Figure 1).

$\{x_j : 1 \leq j \leq N\}$ – the *Bronshtein–Ivanov net* of mesh ρ, where $\rho \in (0, 1/2)$.



Nice body $\longrightarrow$ B-I net $\longrightarrow$ approximation

K is a *nice convex body*, if it has smooth boundary, $\mathbf{B}^d(o,1) \subset K \subset \mathbf{B}^d(o,R)$, and for every boundary point $x \in \partial K$, there is a ball of radius Θ containing K whose boundary sphere touches K at x.

Theorem (The B-I net yields approximation in the geometric distance)

If K is a nice convex body with $R = d^2$ and $\Theta = d^5$, then there is a convex polytope P with no more than $d^{100d}\varepsilon^{-\frac{d-1}{2}}$ vertices satisfying $P \subseteq K \subseteq (1+\varepsilon)P$.

We prove this result.

Claim (Exercise)

In the B–I construction, for every $x \in \partial K$, we can find j such that $|x - x_j|^2 + |\nu_x - \nu_{x_j}|^2 \leq \rho^2$.

Lemma (1. Upper bound on the size of a B–I net)

We have $N \leq 2^d (R+3)^d \rho^{-d+1}$.

Lemma (2. Caps of nice bodies are of small diameter)

Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Assume that K is a nice convex body, $x \in \partial K$, and ν is the outer normal to ∂K at x. If $y \in K$ and $\langle y, \nu \rangle \geq (1-\varepsilon)\langle x, \nu \rangle$, then $|y - x| \leq \sqrt{2\Theta R \varepsilon}$.

Lemma (3. Cap contained in cap)

Fix $\varepsilon, \rho \in \left(0, \frac{1}{2}\right)$. Let K be a nice convex body, $x, x', y \in \partial K$; and ν and ν' the outer unit normals to ∂K at x and x' respectively. Assume that $|x - x'|^2 + |\nu - \nu'|^2 \leq \rho^2$ and $\langle y, \nu \rangle \geq (1 - \frac{\varepsilon}{2}) \langle x, \nu \rangle$. Then

$$\langle y, \nu' \rangle \geq \left(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon R + |y - x|)\right) \langle x', \nu' \rangle.$$

Proof of the Theorem

Set $\rho = \frac{\sqrt{\varepsilon}}{10\sqrt{\Theta R}}$.

L1: Cardinality of a B−I net is OK.

L3: If $|x - x'|^2 + |\nu - \nu'|^2 \leq \rho^2$, then

$$\mathrm{Cap}(x, \varepsilon/2) \subseteq \mathrm{Cap}\left(x', \frac{\varepsilon}{2} + 2\rho(\rho + \varepsilon R + |y - x|)\right).$$

By L2,

$$\mathrm{Cap}(x, \varepsilon/2) \subseteq \mathrm{Cap}\left(x', \varepsilon\right).$$

By the Claim, for every $x' \in \partial K$ there is x in the B−I net satisfying the condition of L3.

Thus, points of the B−I net will pierce every ε-cap. □

Proof of Lemma 1

Let $s', s'' \geq 0$.

$$|x' + \nu' + s'\nu' - x'' - \nu'' - s''\nu''|^2 = |x' + \nu' - x'' - \nu''|^2 + |s'\nu' - s''\nu''|^2 + 2s'\langle \nu', x' - x'' \rangle + 2s''\langle \nu'', x'' - x' \rangle + 2(s' + s'')(1 - \langle \nu', \nu'' \rangle) \geq |x' + \nu' - x'' - \nu''|^2.$$

Thus, if the balls of radius $\frac{\rho}{2}$ centered at $x' + \nu'$ and $x'' + \nu''$ are disjoint, so are the balls of radius $\frac{\rho}{2}$ centered at $x' + (1 + s')\nu'$ and $x'' + (1 + s'')\nu''$. From here we conclude that the balls of radius $\frac{\rho}{2}$ centered at the points $x_j + (1 + k\rho)\nu_{x_j}$, $0 \leq k \leq \frac{1}{\rho}$ are all disjoint (see Figure 2) and contained in $\mathbf{B}^d(0, R + 3)$.

Proof of Lemma 1 cont'd

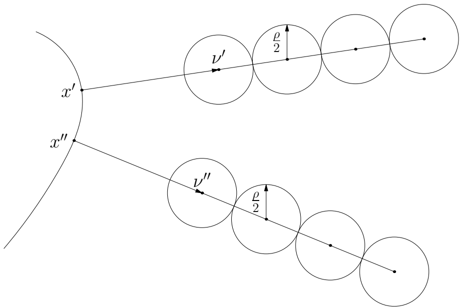


Figure: The disjoint balls

The total number of these balls is at least $\frac{N}{\rho}$ (since for every point x_j in the net, there is a chain of at least $\frac{1}{\rho}$ balls corresponding to different values of k). Hence, $\frac{N}{\rho} \leq \left(\frac{R+3}{\frac{\rho}{2}}\right)^d$. □

Proof of Lemma 2

L2: Cap of a nice body is small diameter.

Let Q be the ball of radius Θ containing K whose boundary sphere touches K at x, that is, $Q = \mathbf{B}^d(x - \Theta\nu, \Theta)$.

Since $o \in K \subset \mathbf{B}^d(0, R)$, we have $0 \leq \langle x, \nu \rangle \leq R$. Thus,

$$\Theta^2 \geq |y - x + \Theta\nu|^2 = |y - x|^2 + 2\Theta\langle y - x, \nu \rangle + \Theta^2,$$

so

$$|y - x|^2 \leq 2\Theta\langle x - y, \nu \rangle \leq 2\Theta\varepsilon\langle x, \nu \rangle \leq 2\Theta R\varepsilon. \quad \square$$

L3: If $|x - x'|^2 + |\nu - \nu'|^2 \le \rho^2$, then $y \in \mathrm{Cap}(x, \varepsilon/2)$ $\Longrightarrow y \in \mathrm{Cap}\left(x', \dots\right)$.

$$
\begin{aligned}
\langle y, \nu' \rangle &= \langle x, \nu' \rangle + \langle y - x, \nu' \rangle = \\
&\langle x', \nu' \rangle + \langle x - x', \nu' \rangle + \langle y - x, \nu \rangle + \langle y - x, \nu' - \nu \rangle \ge \\
&\langle x', \nu' \rangle + \langle x - x', \nu' - \nu \rangle + \langle y - x, \nu \rangle + \langle y - x, \nu' - \nu \rangle \ge \\
&\langle x', \nu' \rangle - \rho^2 - \frac{\varepsilon}{2} \langle x, \nu \rangle - \rho |y - x|.
\end{aligned}
$$

We used $\langle x - x', \nu \rangle \ge 0$. Since $\mathbf{B}^d(o, 1) \subset K \subset \mathbf{B}^d(o, R)$,

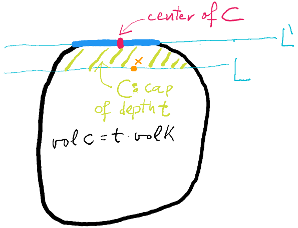
$$
\langle x, \nu \rangle = \langle x, \nu' \rangle + \langle x, \nu - \nu' \rangle \le \langle x', \nu' \rangle + \rho R
$$

and $\langle x', \nu' \rangle \ge 1 > \frac{1}{2}$. So

$$
\begin{aligned}
\langle y, \nu' \rangle &\ge \left(1 - \frac{\varepsilon}{2}\right) \langle x', \nu' \rangle - \rho \left(\rho + \frac{\varepsilon R}{2} + |y - x| \right) \ge \\
&\left(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon R + |y - x|)\right) \langle x', \nu' \rangle. \qquad \square
\end{aligned}
$$

Economic cap covering

Another notion of depth: by volume



Center of cap C: centroid of $C \cap L'$.

Magnified cap, C^λ: the image of C under the magnification about the center of C by factor $\lambda > 0$.

$$\mathrm{depth}_K(x) = \min\{\mathrm{vol}_d(K \cap H) \;:\; H \text{ is a half-space containing } x\}.$$

$\mathrm{cap}(x)$*: the minimal cap of $x \in K$:* $K \cap H$ with minimum volume among all half-spaces H containing x.

Fix $t > 0$.
Floating body: $K_{\geq t} = \{x \in K \;:\; \mathrm{depth}_K(x) \geq t\}$,

Wet part: $K_{\leq t} = \{x \in K \;:\; \mathrm{depth}_K(x) \leq t\}$.

Goal: Cover the wet part by caps

Theorem (Economic cap covering: Bárány and Larman)

$\mathrm{vol}_d(K) = 1$, and $0 < \varepsilon < (2d)^{-2d}$.
Then there are caps $C_1, \ldots, C_m$ and pairwise disjoint convex sets $C'_1, \ldots, C'_m$ such that $C'_i \subseteq C_i$, for each i, and

1. $\bigcup_{i=1}^m C'_i \subseteq K_{\leq\varepsilon} \subseteq \bigcup_{i=1}^m C_i$,
2. $\mathrm{vol}_d(C'_i) > c(d)\varepsilon$ and $\mathrm{vol}_d(C_i) < C(d)\varepsilon$ for each i,
3. for each cap C with $C \cap K_{\geq\varepsilon} = \emptyset$ there is a C_i containing C.

Macbeath regions

Macbeath region of K at $x \in K$ with parameter $\lambda > 0$:

$$M_K(x, \lambda) = x + \lambda[(K - x) \cap (x - K)].$$

Theorem (Bárány)

$\mathrm{vol}_d(K) = 1$ and $t \in (0, t_0)$ (where t_0 depends only on d).
Then there is a polytope P with $K_{\geq t} \subseteq P \subseteq K$ with no more than

$$C(d)\frac{\mathrm{vol}_d(K_{\leq t})}{t}$$

facets, where $C(d) > 0$ depends only on d.

Sketch of the proof

Set $\tau = \lambda t$, where $\lambda = 6^{-d}$.
Choose $x_1, \ldots, x_m \in \partial K_{\geq \tau}$ maximal with respect to the property that the $M(x_i, 1/2)$ are pairwise disjoint.
One can show that

$$c(d)m < \frac{\mathrm{vol}_d\left(K_{\leq \tau}\right)}{\tau} < C(d)\frac{\mathrm{vol}_d\left(K_{\leq t}\right)}{t},$$

for some $c(d), C(d) > 0$.
Remove the magnified (by factor 6) minimal caps from K to obtain

$$P = K \setminus \bigcup_{i=1}^{m} \mathrm{cap}\,(x_i)^6 .$$

It can be shown that (1) no $z \in \partial K$ belongs to P, and (2) $K_{\geq t} \subseteq P$. □

VC-dimension and ε-nets

VC-dimension: a measure of complexity of a set family

Vapnik–Chervonenkis dimension of a set family $\mathcal{F} \subset 2^V$ on a set V is the size of the largest set A such that $\mathcal{F}|_A = \{F \cap A \ : \ F \in \mathcal{F}\}$ is the power set 2^A of A.

Example: Any family of half-spaces in $\mathbb{R}^d$ has low (at most $d + 1$) VC dimension.

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Example: Any family of half-spaces in $\mathbb{R}^d$ has low (at most $d+1$) VC dimension.

Theorem (ε-net Theorem)

$0 < \varepsilon < 1/e$, and let $D \in \mathbb{Z}_+$, $\mathcal{F}$ a family of some measurable subsets of a probability space (U, μ), where $\mu(F) \geq \varepsilon$ for all $F \in \mathcal{F}$. Assume $\dim_{\mathrm{VC}}(\mathcal{F}) \leq D$. Set

$$t := \left\lceil 3\frac{D}{\varepsilon} \ln \frac{1}{\varepsilon} \right\rceil .$$

Choose t elements $X_1, \ldots, X_t$ of V randomly, independently according to μ.
Then $\{X_1, \ldots, X_t\}$ is a transversal of $\mathcal{F}$ with probability at least $1 - (200\varepsilon)^D$.

Approximation by polytope using the ε-net theorem

Theorem

Fix $\vartheta \in (0,1)$, set

$$t = \left\lceil 3\frac{(d+1)e}{(1-\vartheta)^d} \ln \frac{e}{(1-\vartheta)^d} \right\rceil .$$

Then for any centered convex body K in $\mathbb{R}^d$, if t points $X_1, \dots, X_t$ of K are chosen randomly, independently and uniformly, then

$$\vartheta K \subseteq \operatorname{conv}(X_1, \dots, X_t) \subseteq K$$

with probability at least $1 - \left[200\left(\frac{(1-\vartheta)^d}{e}\right)\right]^{d+1}$.

Proof: Hit all caps using the ε-net theorem. We need a measure according to which all caps are big.

Caps are of big volume

Theorem (Grünbaum’s theorem, exercise)

Centroid of K is o.
F — a half-space containing o. Then

$$\mathrm{vol}_d\,(K \cap F) \geq \left(\frac{d}{d+1}\right)^d \mathrm{vol}_d\,(K) \quad > \frac{\mathrm{vol}_d\,(K)}{e}.$$

Lemma (Stability of Grünbaum’s theorem)

Centroid of K is o.
F — a half-space that supports ϑK from outside, with $0 < \vartheta < 1$.
Then

$$\mathrm{vol}_d\,(K)\,\frac{(1-\vartheta)^d}{e} \leq \mathrm{vol}_d\,(K \cap F)\,. \tag{1}$$

A better measure using polarity

K – smooth.
Polar of $K \subset \mathbb{R}^d$: $K^\circ = \{x \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } y \in K\}$.
For $C \subset \partial K$, set $C^* = \{x^* \in \partial K^\circ : x \in C\}$.
Consider the "cones" $\mathrm{Cone}(C) = \{rx : x \in C, 0 \leq r \leq 1\}$ and $\mathrm{Cone}(C^*) = \{ry : y \in C^*, 0 \leq r \leq 1\}$.

$$\mu(C) = \frac{1}{2}\left(\frac{\mathrm{vol}_d(\mathrm{Cone}(C))}{\mathrm{vol}_d(K)} + \frac{\mathrm{vol}_d(\mathrm{Cone}(C^*))}{\mathrm{vol}_d(K^\circ)}\right).$$

Lemma

Assume that K (a smooth, convex body) contains o in $\mathrm{int}\,(K)$, and satisfies the Santaló bound $\mathrm{vol}_d\,(K)\,\mathrm{vol}_d\,(K^\circ) \leq e^{O(d)}d^{-d}$. Then μ is a probability measure on ∂K invariant under linear automorphisms of $\mathbb{R}^d$ and $\mu(\mathrm{cap}\,(x,\varepsilon)) \geq e^{O(d)}\varepsilon^{\frac{d-1}{2}}$ for all $x \in \partial K$ and all $\varepsilon \in (0, \frac{1}{2})$.

Best bound for fine approximation

Theorem (N., Nazarov, Ryabogin)

Let K be a convex body in $\mathbb{R}^d$ with the center of mass at the origin, and let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then there exists a convex polytope P with at most $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.

Open:

- Good approximation in the intermediate range (not so fine, not so rough)?
- Total complexity instead of number of vertices?

Quantitative Helly-type questions

Bárány, Katchalski, Pach '82

Quantitative Volume Theorem [BKP'82]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that any 2d of them have intersection of volume at least 1.
Then $\cap\mathcal{F}$ is of volume at least d^{-2d^2}.

Later: d^{-2d^2} can be replaced by $Cd^{-3d/2}$.

Open: Can we obtain $Cd^{-d/2}$? We know this as an upper bound.

John decomposition of the identity

Definition

We say that a set of vectors $w_1, \dots, w_m \in \mathbb{R}^d$ with weights $c_1, \dots, c_m > 0$ form a *John's decomposition of the identity*, if

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I,$$

where I is the identity operator on $\mathbb{R}^d$.

Lemma (John's theorem)

K convex body in $\mathbb{R}^d$.
If $\mathbf{B}^d$ is the max. volume ellipsoid in K then there are *contact points* $w_1, \dots, w_m \in \partial\mathbf{B}^d \cap \partial K$ (and weights $c_1, \dots, c_m > 0$) that form a John's decomposition of the identity.

John decomposition of the identity

Note: If $w_1, \ldots, w_m \in \partial \mathbf{B}^d$ (with weights $c_1, \ldots, c_m > 0$) form a John's decomposition of the identity, then $\{w_1, \ldots, w_m\}^* \subset d\mathbf{B}$.
By polarity: $\frac{1}{d}\mathbf{B} \subset \operatorname{conv}(\{w_1, \ldots, w_m\})$.

Lemma (Dvoretzky-Rogers lemma)

$w_1, \ldots, w_m \in \partial \mathbf{B}^d$ (with $c_1, \ldots, c_m > 0$) a John's decomposition of the identity. Then there is an orthonormal basis $z_1, \ldots, z_d$ of $\mathbb{R}^d$, and $\{v_1, \ldots, v_d\} \subseteq \{w_1, \ldots, w_m\}$:

$$v_i \in \operatorname{span}\{z_1, \ldots, z_i\}, \quad \text{and} \quad \sqrt{\frac{d-i+1}{d}} \leq \langle v_i, z_i \rangle \leq 1, (i = 1, \ldots, d).$$

Lemma (Pivovarov's estimate, 2010)

Select d vectors $v_1, \ldots, v_d$ randomly from the contact points (each time each point chosen with probability c_i/d). Then the expected volume of the random simplex is

$$\mathbb{E} \operatorname{vol}_d (S_1) = \frac{1}{d!} \cdot \frac{\sqrt{d!}}{d^{d/2}}. \tag{2}$$

Proof of selecting $2d$ with "volume at least $c^d d^{2d}$"

Theorem [N.]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that any 2d of them have intersection of volume at least 1.
Then $\cap\mathcal{F}$ is of volume at least Cd^{-2d}.

Equivalently:

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Then

$$\frac{\mathrm{vol}_d\left(\cap\mathcal{G}\right)}{\mathrm{vol}_d\left(\cap\mathcal{F}\right)} \leq cd^{2d}$$

for some $2d$-member subfamily $\mathcal{G}$ of $\mathcal{F}$.

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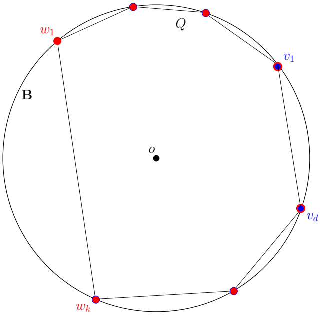
May assume:

- $\mathcal{F}$ consists of closed half-spaces, ie., $P := \cap\mathcal{F}$ is a polytope.
- $\mathbf{B}^d \subset P$ is the ellipsoid of maximal volume in P.

By John's Thm.: There are contact points $w_1, \ldots, w_m \in \partial \mathbf{B} \cap \partial P$ (with $c_1, \ldots, c_m > 0$) that form a John's decomposition of the identity.

$Q := \operatorname{conv}(\{w_1. \ldots, w_m\})$.

By Dvoretzky-Rogers Lemma: There is an ONB $z_1, \ldots, z_d$ of $\mathbb{R}^d$, and $\{v_1, \ldots, v_d\} \subseteq \{w_1, \ldots, w_m\}$ st. $\{v_1, \ldots, v_d\}$ is "nicely aligned" with $z_1, \ldots, z_d$.



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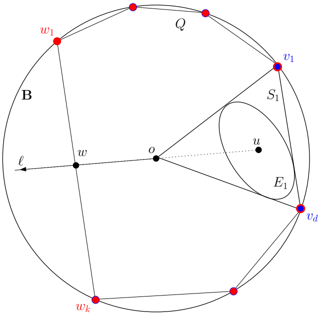
$S_1 := \operatorname{conv}(\{o, v_1, v_2, \ldots, v_d\})$.
E_1: the largest volume ellipsoid in S_1.
u: center of E_1.
ℓ: ray from origin toward $-u$.
w: $\ell \cap \partial Q$.



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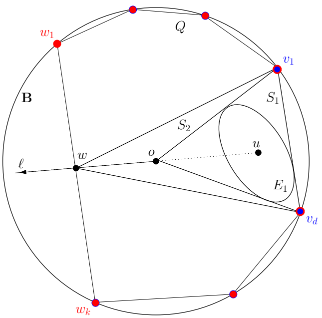


By the "Note": $o \in int(Q)$. In fact, $\frac{1}{d}\mathbf{B} \subset Q$.

Hence, $|w| \geq 1/d$.

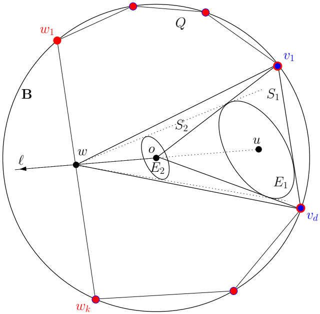
$S_2 := \text{conv}(\{w, v_1, v_2, \ldots, v_d\})$.

By the Claim: $|w| \geq 1/d$.

$S_2 := \text{conv}(\{w, v_1, v_2, \ldots, v_d\})$.
E_2: contraction of E_1 with center w, ratio $\lambda = \frac{|w|}{|w-u|}$.

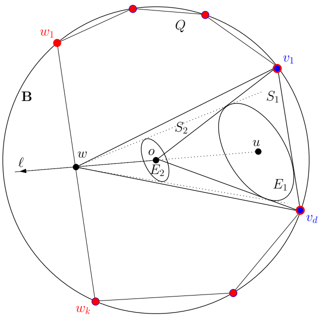
Now,

- E_2 is centered at the origin
- $\lambda \geq \frac{1}{d+1}$
- $E_2 \subseteq S_2$.



By the Claim: $|w| \geq 1/d$.

$S_2 := \text{conv}(\{w, v_1, v_2, \ldots, v_d\})$.
E_2: contraction of E_1 with center w, ratio $\lambda = \frac{|w|}{|w-u|}$.



Now,

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- $\lambda \geq \frac{1}{d+1}$
- $E_2 \subseteq S_2$.

By Caratheodory's theorem, (re-indexing) $w \in \text{conv}(\{w_1, \ldots, w_d\})$.

$$E_2 \subset S_2 \subset \text{conv}(\{w_1, \ldots, w_k, v_1, \ldots, v_d\}).$$

$X := \{w_1, \ldots, w_k, v_1, \ldots, v_d\}$.
$\mathcal{G}$: the family of those half-space which support **B** at the points of X.

Finally, $|\mathcal{G}| \leq 2d$, and $\mathcal{G} \subseteq \mathcal{F}$, and $\cap\mathcal{G} = X^* \subset E_2^*$.

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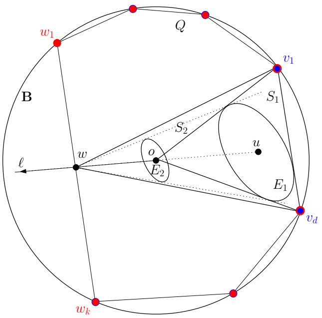

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Finally, $|\mathcal{G}| \leq 2d$, and $\mathcal{G} \subseteq \mathcal{F}$, and $\cap\mathcal{G} = X^* \subset E_2^*$.

After finally, volumes: E_2 not small $\Rightarrow$ E_2^* not big $\Rightarrow$ $\cap\mathcal{G}$ not big.

Remarks

Easy: Cd^{cd} is sharp.

Brazitikos '16+ improved Cd^{-2d} to $Cd^{-1.5d}$.
The modification: Replace E_1 by $S_1 \cap (2g - S_1)$, where $g = \mathrm{centroid}(S_1)$.

Quantitative Colorful Helly Theorem

Theorem (Damásdi, Földvári, N.)

$\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ – finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful choice of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \leq k \leq 2d$ with $1 \leq i_1 < \ldots < i_{2d} \leq 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ contains an ellipsoid of volume at least 1.

Then, there exists an $1 \leq i \leq 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume at least $d^{-O(d^2)}$.

Open:

- $2d$ in place of $3d$ should hold.
- $d^{-O(d)}$ in place of $d^{-O(d^2)}$?

Back to geometric distance — Quantitative Steinitz theorem

The problem: Bound $r(d)$

Steinitz's theorem

For any $Q \subset \mathbb{R}^d$, if $o \in \mathrm{int}\,(\mathrm{conv}\,(Q))$, then there are at most $2d$ points of Q whose convex hull contains the origin in the interior.

Quantitative Steinitz theorem: Bárány, Katchalski, Pach '82

There exists $r(d) > 0$ such that for any $Q \subset \mathbb{R}^d$, if $\mathbf{B}^d \subseteq \mathrm{conv}\,(Q)$, then there is $Q' \subseteq Q$ of size at most $2d$ with $r(d)\mathbf{B}^d \subset \mathrm{conv}\,(Q')$. In fact, $r(d) > d^{-2d}$.

Conjecture [Bárány, Katchalski, Pach '82]

$$r(d) \approx cd^{1/2}.$$

Results

Polynomial lower bound on $r(d)$ [Ivanov, N.]

$$r(d) > 1/(6d^2).$$

Upper bound on $r(d)$ [Ivanov, N.]

$$r(d) < 2/d^{1/2}.$$

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Polynomial lower bound on $r(d)$ [Ivanov, N.]

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Stronger:

The convex hull of a few unit vectors is small

$u_1, \ldots, u_n \in \mathbb{R}^d$, $|u_i| = 1$. $\varepsilon > 0$. Then

$$\operatorname{conv}(\{\pm u_i\}) \not\supset \left(\frac{\sqrt{n}}{d} + \varepsilon\right) \mathbf{B}^d.$$

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Similar flavor:

Conjecture

Let $\{u_1, \ldots, u_{2d}\}$ be unit vectors in $\mathbb{R}^d$. Then there is a point in

$$\bigcap_{i=1}^{2d} \{x \in \mathbb{R}^d \ : \ \langle u_i, x \rangle \leq 1\}$$

with norm $\sqrt{d}$.

Preparations for the proof of $r(d) > 1/(6d^2)$

Goal: Q convex polytope, $Q \supset \mathbf{B}^d$. Find $2d$ vertices whose conv contains $\frac{1}{6d^2}\mathbf{B}^d$.

Almendra–Hernández, Ambrus, Kendall, '22

$\lambda > 0$, and $L \subset \mathbb{R}^d$ convex polytope with $L \subset -\lambda L$.
Then there exist $2d$ vertices L' of L

$$L \subset -(\lambda + 2)d \cdot \mathrm{conv}\,(L').$$

Note: Choose o as centroid, or center of John's ellipsoid, or Santaló point, etc. $\Longrightarrow \lambda \leq d$.

Notation

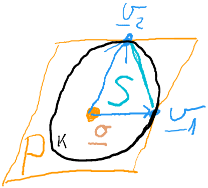
For $v \in \mathbb{R}^d \setminus \{o\}$,

$$H_v = \left\{x \in \mathbb{R}^d \;:\; \langle x, v\rangle \leq 1\right\}.$$

Proof of A-HAK

Among all simplices with d vertices from L and one vertex at the origin, take a simplex $S = \operatorname{conv}(0, v_1, \ldots, v_d)$ with maximal volume.

$$S = \left\{ x \in \mathbb{R}^d \;:\; x = \alpha_1 v_1 + \ldots + \alpha_d v_d \quad \text{for} \;\; \alpha_i \geq 0 \text{ and } \sum_{i=1}^{d} \alpha_i \leq 1 \right\}. \tag{3}$$



Set $P = \sum_{i \in [d]} [-v_i, v_i]$. It is a paralletope:

$$P = \{ x \in \mathbb{R}^d \;:\; x = \beta_1 v_1 + \ldots + \beta_d v_d, \beta_i \in [-1, 1] \}.$$

Claim: $L \subset P$. (4)

Let $S' = -2dS + (v_1 + \ldots + v_d)$. By (3),

$$S' = \left\{ x \in \mathbb{R}^d \ : \ x = \gamma_1 v_1 + \ldots + \gamma_d v_d \quad \text{for } \gamma_i \le 1 \text{ and } \sum_{i=1}^{d} \gamma_i \ge -d \right\},$$

which yields

$$P \subseteq S'. \tag{5}$$

Let y be the intersection of the ray emanating from 0 in the direction $-(v_1 + \cdots + v_d)$ and the boundary of Q. By Carathéodory's theorem, we can choose $k \le d$ vertices $\{v'_1, \ldots, v'_k\}$ of L such that $y \in \operatorname{conv}(v'_1, \ldots, v'_k)$. Set $L' = \operatorname{conv}(v_1, \ldots, v_d, v'_1, \ldots, v'_k)$. Clearly, $\frac{v_1 + \cdots + v_d}{d} \in S \subset L$. Thus, $0 \in L'$, and consequently,

$$S \subseteq Q'. \tag{6}$$

Since $L \subset -\lambda L$, we also have that

$$\frac{v_1 + \cdots + v_d}{d} \in -\lambda [y, 0] \subset -\lambda L'.$$

Proof of A-HAK completed

$$\frac{v_1 + \cdots + v_d}{d} \in -\lambda[y, 0] \subset -\lambda L'.$$

Combining it with (4), (5), (6), we obtain

$$L \subset P \subset S' = -2dS + (v_1 + \cdots + v_d) \subset -2d\,L' - \lambda d\,L' = -(\lambda + 2)d\,L'. \quad \square$$

Proof of $r(d) > 1/(6d^2)$

$K := Q^\circ \subset \mathbf{B}^d$. $K = \bigcap_{v \in \mathrm{vert} Q} H_v$.

By duality, it suffices to find $2d$ half-spaces H_v with $v \in \mathrm{vert} Q$, whose intersection is contained in the ball $6d^2\mathbf{B}^d$.

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The idea: Duality again! Let c be such that $K - c \subset -d(K - c)$.

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The idea: Duality again! Let c be such that $K - c \subset -d(K - c)$.

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Apply A-HAK'22 for L with $\lambda = d$. There are $w_1, \dots, w_{2d} \in \mathrm{vert} L$:

$$L \subset -(d+2)d \cdot \mathrm{conv}\left(\{w_i \ : \ i \in [2d]\}\right).$$

Since $c \in K \subset \mathbf{B}^d$, one has that $K - c \subset 2\mathbf{B}^d$ thus, $L \supset \frac{1}{2}\mathbf{B}^d$.

So,

$$\frac{1}{2}\mathbf{B}^d \subset L \subset -(d+2)d \cdot \mathrm{conv}\left(\{w_i \ : \ i \in [2d]\}\right).$$

Take polar, and obtain ...

$$(\mathrm{conv}(\{w_i \,:\, i \in [2d]\}))^\circ \subset 2(d+2)d\mathbf{B}^d.$$

Note: $L^\circ = K - c$. Thus, for any $w \in \mathrm{vert} L$, $H_w = H_v - c$ for some $v \in \mathrm{vert} Q$. Thus,

$$(\mathrm{conv}(\{w_i \,:\, i \in [2d]\}))^\circ = \bigcap_{v_i \in [2d]} (H_{v_i} - c)$$

for corresponding $v_i \in \mathrm{vert} Q$.

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for corresponding $v_i \in \mathrm{vert} Q$.

Thus,

$$\bigcap_{v_i \in [2d]} H_{v_i} = \bigcap_{v_i \in [2d]} (H_{v_i} - c) + c \subset 2(d+2)d\mathbf{B}^d + c \subset (2(d+2)d+1)\,\mathbf{B}^d.$$

Thus, $Q' := \operatorname{conv}\left(\{v_i \;:\; i \in [2d]\}\right)$ is good. □

Approximation of sums of matrices

Independent copies of an isotropic vector

$A \in \mathbb{R}^{d\times d}$ (often, we simply have $A = I$) as a (positive) linear combination of some other matrices.

Goal: Small subset of the matrices whose linear combination (with new coefficients) yields a matrix close to A.

A random vector v in $\mathbb{R}^d$ is called *isotropic*, if $\mathbb{E} v \otimes v = I$.

Rudelson's theorem

If we take k independent copies $y_1, \dots, y_k$ of an isotropic random vector y in $\mathbb{R}^d$ for which $|y|^2 \le \gamma$ almost surely, with

$$k = \left\lceil \frac{c\gamma \ln d}{\varepsilon^2} \right\rceil, \text{ then } \mathbb{E} \left\| \frac{1}{k} \sum_{i=1}^{k} y_i \otimes y_i - I \right\| \le \varepsilon,$$

where $\|A\| = \max\{\langle Ax, Ax\rangle^{1/2} : x \in \mathbb{R}^d, \langle x, x\rangle = 1\}$ denotes the *operator norm* of the matrix A.

In the language of John

Recall: John decomposition of the identity:

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I.$$

Rudelson's result applies in this setting: Taking $\alpha_i = c_i/d$, we get a probability distribution on $[m]$.

Let $\sigma = \{i_1, \dots, i_k\}$ be a multiset obtained by k independent draws from $[m]$ according to it, and set

$$\frac{1}{k}\sum_{i\in\sigma} \sqrt{d}u_i \otimes \sqrt{d}u_i.$$

Rudelson: In expectation, this average is not farther than ε from I in the operator norm, provided that k is at least $\frac{cd\ln d}{\varepsilon^2}$.

A slightly more general form

Theorem (Rudelson's theorem)

Let $0 < \varepsilon < 1$ and $Q_1, \ldots, Q_k$ be independent random matrices distributed according to (not necessarily identical) probability distributions $\mathcal{P}_1, \ldots, \mathcal{P}_k$ on the set $\mathcal{P}^d$ of $d \times d$ real positive semi-definite matrices such that $\mathbb{E}Q_i = A$ for some $A \in \mathcal{P}^d$ and all $i \in [k]$. Set $\gamma = \mathbb{E}(\max_{i\in[k]} \|Q_i\|)$, and assume that

$$k \geq \frac{c\gamma(1 + \|A\|) \ln d}{\varepsilon^2},$$

where c is an absolute constant. Then

$$\mathbb{E}\left\|\frac{1}{k}\sum_{i\in[k]} Q_i - A\right\| \leq \varepsilon. \tag{7}$$

A breakthrough: $\ln d$ removed by an algorithmic approach

Batson, Marcus, Spielman, Srivastava, Friedland, Youssef: We may remove $\ln d$ in the special case of a John decomposition.

Open: Can we remove $\ln d$ in the special case of rank 2 orthogonal projections?

Claim

Exercise

1. The set $\mathcal{P}^d$ of positive semi-definite $d \times d$ matrices (with real entries) form a convex cone with apex at the origin in the vector space $\mathbb{R}^{d(d+1)/2}$ of symmetric matrices.
2. Matrices of trace 1 form a hyperplane H_1 containing $\frac{1}{d}I$ in $\mathbb{R}^{d(d+1)/2}$.
3. The set $\mathcal{P}^d \cap H_1$ is a convex body in H_1.

Proof of Rudelson's Theorem

The *Schatten p-norm* of a real $d \times d$ matrix A is defined as

$$\|A\|_{C_p^d} := \left(\sum_{i=1}^{d} (s_i(A))^p\right)^{1/p},$$

where $s_1(A), \ldots, s_d(A)$ is the sequence of eigenvalues of the positive semi-definite matrix $\sqrt{A^*A}$.

Recall: $\|A\| \leq \|A\|_{C_p^d}$ for all $p \geq 1$, and

$$\|A\| \leq \|A\|_{C_p^d} \leq e\,\|A\| \text{ for } p = \ln d. \tag{8}$$

$\mathbf{r}$ denotes a sequence of k *Rademacher variables*, that is, $\mathbf{r} = (r_1, \ldots, r_k)$, where the r_i are random variables uniformly distributed on $\{1, -1\}$, independent of each other and all other random variables in the context.

Lust–Piquard inequality

Theorem (Lust–Piquard)

$2 \le p < \infty$. For any d and any $Q_1, \ldots, Q_k$ (not necessarily positive definite) square matrices of size d we have

$$\left[\mathop{\mathbb{E}}_{\mathbf{r}} \left\| \sum_{j=1}^{k} r_j Q_j \right\|_{C_p^d}^p \right]^{1/p} \le c\sqrt{p} \max \left\{ \left\| \left(\sum_{j=1}^{k} Q_j Q_j^* \right)^{1/2} \right\|_{C_p^d}, \left\| \left(\sum_{j=1}^{k} Q_j^* Q_j \right)^{1/2} \right\|_{C_p^d} \right\}$$

for a universal constant $c > 0$.

For any $d \times d$ matrix Q, the product Q^*Q is positive semi-definite. Since, by Weyl's inequality, the Schatten p-norm is monotone on the cone of positive semi-definite matrices, we may deduce:

$$\left[\mathop{\mathbb{E}}_{\mathbf{r}} \left\| \sum_{j=1}^{k} r_j Q_j \right\|_{C_p^d}^p \right]^{1/p} \le c\sqrt{p} \left\| \left(\sum_{j=1}^{k} Q_j Q_j^* + Q_j^* Q_j \right)^{1/2} \right\|_{C_p^d}. \tag{9}$$

Symmetrization by Rademacher variables

Lemma (Symmetrization by Rademacher variables)

Let $q_1, \ldots, q_k$ be independent random vectors distributed according to (not necessarily identical) probability distributions $\mathcal{P}_1, \ldots, \mathcal{P}_k$ on a normed space X with $\mathbb{E}q_i = q$ for all $i \in [k]$. Then

$$\mathop{\mathbb{E}}_{q_1,\ldots,q_k} \left\| \frac{1}{k} \sum_{\ell=1}^{k} q_\ell - q \right\| \leq \frac{2}{k} \mathop{\mathbb{E}}_{q_1,\ldots,q_k} \mathop{\mathbb{E}}_{\mathbf{r}} \left\| \sum_{\ell=1}^{k} r_\ell q_\ell \right\| .$$

There is no slide 48.