Problem set 2 by Vida Dujmović. September 1, 2023.

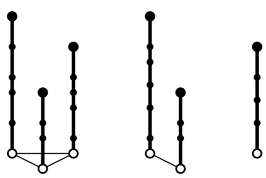
More on Product Structure and treewidth:

A graph is a *planar 3-tree* (also known as *stacked triangulation* in the literature) if it can be defined recursively as follows. A triangle is a planar 3-tree. Embed a planar 3-tree in the plane. Pick any face. Add a vertex to that face and make it adjacent to the three vertices of that face. The resulting graph is a planar 3-tree.

Edge contraction is an operation that removes an edge, vw, from a graph and merges its endpoints v and w into one vertex. A graph H is called a *minor* of the graph G if H can be obtained from G by applying the following operations: deleting edges, deleting vertices and contracting edges.

1. Prove that planar 3-trees have treewidth at most 3. You can use "add a leaf" strategy (that we have seen in the lectures) when building the tree-decomposition.

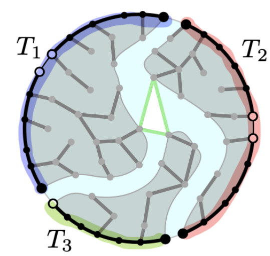
2. A path in a rooted tree is *vertical path* if it is a sub-path of any path from a vertex to the root. Recall that we proved in the lectures the following: given a planar graph G and any rooted BFS spanning tree of G, $V(G)$ can be partitioned into vertical paths such that contracting those vertical paths results in a graph of treewidth at most 8.



tripods with 3,2,1 legs

Modify that proof to obtain the following useful generalizations of the product structure theorem for planar graphs

A *tripod* is a union of up to three vertical paths whose lower endpoints (i.e the endpoints of the vertical paths that are further away from the root) form a clique in G.



a) Prove the following statement: Given a planar graph G and any rooted BFS spanning tree of G, $V(G)$ can be partitioned into *tripods* such that contracting those tripods results in a planar 3-tree (and thus a graph of treewidth at most 3).

b) Show that that (a) implies that every planar graph G is a subgraph of $H \boxtimes P \boxtimes K_3$ where H is a planar 3-tree (and H is a minor of G).

c) A vague question/suggestion: think of what $K_t \boxtimes H$ looks like, where K_t is a complete graph on t vertices.

3. Recall that in the last problem session we proved that planar graphs admit p-centered coloring[1] with at most $O(p^9)$ colors. Debski, Felsner, Micek and Schroder [SODA 2020] proved that planar 3-trees admit p-centered colorings with $O(p^2 \log p)$ colors. Use that result and the $H \boxtimes P \boxtimes K_3$ version of the product structure theorem to improved the $O(p^9)$ to $O(p^3 \log p)$ bound (which is currently the best known bound).

Proper good curves, that is, collinear sets

1. Prove that each n-vertex tree has a proper good curve with at least $n/2$ vertices. (I am not sure what the best bound is).

2. Prove that each n-vertex outerplanar graph has a proper good curve with at least $n/3$ vertices.

   It is possible to improve this bound to $\geq n/2$ by showing that one can always pick a set $S \subseteq V(G)$ of at least $n/2$ vertices from every (edge-maximal) outerplanar graph G, such that no internal edge has both endpoints in S. (An edge is *internal* if it does not lie on the outside face).

3. A set of *at most* n points P in the plain is *subset universal for all n-vertex planar graphs* if every n-vertex planar graph G has a straight line crossing free drawing such that each point P is representing one vertex of G in the drawing. Prove that *every* set of $\lfloor\sqrt{n}\rfloor$ points in the plain is subset universal.

   It is an open problem if $w(\sqrt{n})$ subset universal points for planar graphs exist.

4. Recall the definition of Canonical orderings of planar graphs and the associated Frame graph with a chain or antichain of size at least $\sqrt{n}$. Can you mimic that proof and show that every triconnected cubic n-vertex planar graph has a cycle of length at least $\sqrt{n}$?

5. For those familiar with Schnyder Woods, prove that the Frame graph is the union of two Schnyder trees.

[1] Let p be an integer and $p \geq 1$. A vertex coloring ϕ of a graph G is *p-centered* if for every connected subgraph C of G either ϕ uses more than p colors on C or there is a color that appears exactly once on C.