Problem set

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Applications of the product structure theorem of planar graphs:

1. Let p be an integer and $p \geq 1$. A vertex coloring ϕ of a graph G is *p-centered* if for every connected subgraph C of G either ϕ uses more than p colors on C or there is a color that appears exactly once on C. The *p-centered chromatic number*, $\chi_p(G)$, of G is the minimum integer k such that there is a p-centered coloring of G using k colors.

   Pilipczuk and Siebertz showed that for every $p \geq 1$, and every graph G, $\chi_p(G) \leq \binom{p+t}{t}$ where t is the treewith of G.

   Using that theorem and the product structure theorem of planar graphs (from the lectures) prove that every planar graph G has $\chi_p(G) \leq O(p^9)$.

2. A *queue layout* of a graph G consists of a total order σ of $V(G)$ and a partition of $E(G)$ into sets (called *queue*) such that no two edges in the same stack *nest*; that is, that is, there are no edges vw and xy in a single queue with $v <_\sigma x <_\sigma y <_\sigma w$. The minimum number of queues in a queue layout of G is the *queue-number* of G, $\text{qn}(G)$. Prove that there exists a constant c such that every planar graph has queue number at most c.

   Use the product structure theorem of planar graphs (from the lectures) and the following result by Wiechart: for every graph G, $\text{qn}(G) \leq 2^t - 1$ where t is the treewidth of G.

   Try first to prove that for every graph H and path P, $\text{qn}(H \times P) \leq 3 \cdot \text{qn}(H) + 1$

Treewidth:

3. A graph G is *outerplanar* if G has a drawing with no edge crossings and such that all the vertices of G lie on the outerface. Prove that outerplanar graphs have treewidth at most 2. Hint: Start with edge-maximal outerplanar graphs G (they are combinatorially equivalent to triangulations of polygons) and observe that the dual graph of G (minus the vertex corresponding to the outer face) is a tree. Use that tree to build your tree decomposition.

4. Prove that k by k grid graph has a treewidth at most k.

Separators:

Let S be a subset of vertices of an n-vertex graph G. S is a $1/c$-*separator* of G if removing S from G (along with the edges incident to S) results in a graph in which every connected component has at most n/c vertices. $|S|$ is the *size of the sepearator*.

5. Prove that every n-vertex tree has a 1/2-separator of size 1.

6. a) Prove that every n-vertex outerplanar graph has 1/2-separator of size 3.

   b) Prove that every n-vertex outerplanar graph has 2/3-separator of size 2.

7. Prove that every n-vertex graph G of treewidth k has 1/2-separator of size $k + 1$.

Lipton Tarjan Theorem:

8. With the help of question 7 and the product structure theorem of planar graphs ((from the lectures) try to prove the Lipton–Tarjan's theorem (it states that every n-vertex planar graph has a 1/2-separator of size $O(\sqrt(n))$.