Threshold for the measure of random polytopes

Exercises for the Convex and Discrete Geometry Summer School
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A. Some useful facts about log-concave probability measures

1. Let $f:\mathbb{R}^n\to[0,\infty)$ be a log-concave function with finite, positive integral. Then, there exist constants $A,B>0$ such that $f(x)\leqslant Ae^{-B\|x\|_2}$ for all $x\in\mathbb{R}^n$. In particular, f has finite moments of all orders.

2. (Fradelizi) Let $f:\mathbb{R}^n\to[0,\infty)$ be a centered log-concave density. Then,

$$f(0)\leqslant\|f\|_\infty\leqslant e^n f(0).$$

3. (Grünbaum) Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Then,

$$\frac{1}{e}\leqslant\mu(\{x:\langle x,\theta\rangle\geqslant 0\})\leqslant 1-\frac{1}{e}$$

for every $\theta\in S^{n-1}$.

4. Let $f:\mathbb{R}^n\to[0,\infty)$ be an isotropic log-concave density. Then

$$L_f=\|f\|_\infty^{1/n}\geqslant c,$$

where $c>0$ is an absolute constant.

5. (Borell's lemma) Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then, for any symmetric convex set A in $\mathbb{R}^n$ with $\mu(A)=\alpha\in(0,1)$ and any $t>1$ we have

$$1-\mu(tA)\leqslant\alpha\left(\frac{1-\alpha}{\alpha}\right)^{\frac{t+1}{2}}.$$

6. (Reverse Hölder inequalities for seminorms) Let μ be a log-concave probability measure on $\mathbb{R}^n$. If $g:\mathbb{R}^n\to\mathbb{R}$ is a seminorm then, for any $q>p\geqslant 1$, we have

$$\left(\int_{\mathbb{R}^n}|g|^p\,d\mu\right)^{1/p}\leqslant\left(\int_{\mathbb{R}^n}|g|^q\,d\mu\right)^{1/q}\leqslant c\frac{q}{p}\left(\int_{\mathbb{R}^n}|g|^p\,d\mu\right)^{1/p},$$

where $c>0$ is an absolute constant.

Hints

1. Since $\int f > 0$, we can find $t \in (0,1)$ such that the set $C := \{x : f(x) > t\}$ has positive Lebesgue measure. Note that C is convex and this implies that C has non-empty interior. Working with $f_1(\cdot) = f(\cdot + x_0)$ for some $x_0 \in \mathrm{int}(C)$, we may assume that $rB_2^n \subseteq C$.

Show that $K = \{x : f(x) > t/e\}$ is bounded (note that $0 < |K| < \infty$, K is convex and contains rB_2^n). So, we can find $R > 0$ such that $K \subset \frac{R}{2}B_2^n$. Then, for every x with $\|x\|_2 > R$ we have $R\frac{x}{\|x\|_2} \notin K$, and hence $f(Rx/\|x\|_2) \leqslant t/e$, while $r\frac{x}{\|x\|_2} \in C$, which shows that $f(rx/\|x\|_2) \geqslant t$. Moreover, we may write

$$\frac{Rx}{\|x\|_2} = \frac{\|x\|_2 - R}{\|x\|_2 - r}\frac{rx}{\|x\|_2} + \frac{R - r}{\|x\|_2 - r}x.$$

Use the fact that f is log-concave to show that $f(x) \leqslant te^{-\frac{\|x\|_2 - r}{R - r}} < e^{-\|x\|_2/R}$ for every $x \in \mathbb{R}^n$ with $\|x\|_2 > R$. On the other hand, show that there exists $M > 0$ such that $f(x) \leqslant M$ for every $x \in RB_2^n$. Combining the above, we can find two constants $A, B > 0$, which depend on f, so that $f(x) \leqslant Ae^{-B\|x\|_2}$ for every $x \in \mathbb{R}^n$.

2. We may assume that f is strictly positive and continuously differentiable. From Jensen's inequality and using the assumption that f is centered we have

$$\log f(0) = \log f\left(\int_{\mathbb{R}^n} yf(y)dy\right) \geqslant \int_{\mathbb{R}^n} f(y)\log f(y)dy.$$

Let $x \in \mathbb{R}^n$. Using the fact that f is log-concave we have that

$$-\log f(x) \geqslant -\log f(y) + \langle x - y, \nabla(-\log f)(y)\rangle.$$

Multiplying both terms of the last inequality by $f(y)$, and then integrating with respect to y, we get

$$\begin{aligned}-\log f(x) &\geqslant -\int_{\mathbb{R}^n} f(y)\log f(y)dy + \int_{\mathbb{R}^n} \langle x - y, -\nabla f(y)\rangle\, dy \\ &\geqslant -\int_{\mathbb{R}^n} f(y)\log f(y)dy - n,\end{aligned}$$

where the last inequality follows if we integrate by parts (and since $f(y)$ decays exponentially as $\|y\|_2 \to \infty$). Combining the above, we get

$$\log f(0) \geqslant \int_{\mathbb{R}^n} f(y)\log f(y)dx \geqslant \log f(x) - n,$$

for every $x \in \mathbb{R}^n$. Taking the supremum over all x we get the result.

3. Without loss of generality we may assume that, for some $M > 0$,

$$\mu(\{x : |\langle x, \theta\rangle| > M\}) = 0.$$

The general case then follows by approximating a general log-concave measure by measures which have this property in the direction of θ.

Let $G(t) = \mu(\{x : \langle x, \theta\rangle \leqslant t\})$. Then, G is a log-concave increasing function and we have $G(t) = 0$ for $t \leqslant -M$ and $G(t) = 1$ for $t \geqslant M$. Since μ is centered, we have

$$\int_{-M}^{M} tG'(t)dt = 0,$$

and applying integration by parts we see that

$$\int_{-M}^{M} G(t)dt = M.$$

We want to prove that

$$G(0) \geqslant \frac{1}{e}$$

(that $G(0)\leqslant 1-1/e$ as well will then follow by replacing θ with $-\theta$ and repeating the argument). Observe that $\log G$ is a concave function, therefore

$$G(t)\leqslant G(0)e^{\alpha t}$$

with $\alpha = G'(0)/G(0)$. We may choose M large enough so that $1/\alpha < M$. Then, using that $G(t)\leqslant G(0)e^{\alpha t}$ if $t\leqslant 1/\alpha$ and that, trivially, $G(t)\leqslant 1$ if $t>1/\alpha$, we can write

$$M=\int_{-M}^{M} G(t)dt \leqslant \int_{-\infty}^{1/\alpha} G(0)e^{\alpha t}dt + \int_{1/\alpha}^{M} \mathbf{1}\, dt = \frac{eG(0)}{\alpha} + M - \frac{1}{\alpha}.$$

We conclude that $G(0)\geqslant 1/e$ as claimed.

4. Since f is isotropic, we may write

$$\begin{aligned} n = \int \|x\|_2^2 f(x)\,dx &= \int_{\mathbb{R}^n}\left(\int_0^{\|x\|_2^2} \mathbf{1}\,dt\right) f(x)\,dx \\ &= \int_0^{\infty}\int_{\mathbb{R}^n} \mathbf{1}_{\{x:\|x\|_2^2\geqslant t\}}(x) f(x)\,dx\,dt \\ &= \int_0^{\infty}\int_{\mathbb{R}^n\setminus \sqrt{t}B_2^n} f(x)\,dx\,dt \\ &= \int_0^{\infty}\left(1-\int_{\sqrt{t}B_2^n} f(x)dx\right)dt \\ &\geqslant \int_0^{(\omega_n\|f\|_\infty)^{-2/n}} [1-\omega_n\|f\|_\infty t^{n/2}]\,dt \\ &= (\omega_n\|f\|_\infty)^{-2/n}\frac{n}{n+2}, \end{aligned}$$

where $\omega_n = |B_2^n|$. Since $\omega_n^{-1/n}\simeq \sqrt{n}$, we get $\|f\|_\infty^{1/n}\geqslant c$ for some absolute constant $c>0$.

5. Using the symmetry and convexity of A, check that

$$\frac{2}{t+1}(\mathbb{R}^n\setminus(tA)) + \frac{t-1}{t+1}A \subseteq \mathbb{R}^n\setminus A.$$

for every $t>1$.

6. Write

$$\int_{\mathbb{R}^n} |f|^q\,d\mu = \int_0^{\infty} qs^{q-1}\mu(\{x:|f(x)|\geqslant s\})\,ds.$$

Note that the set $A=\{x\in\mathbb{R}^n : |f(x)|\leqslant 3\|f\|_p\}$ is symmetric and convex. Also, for any $t>0$ we have that $tA=\{x\in\mathbb{R}^n : |f(x)|\leqslant 3t\|f\|_p\}$, while $\mu(A)\geqslant 1-3^{-p}$.