Threshold for the measure of random polytopes

Exercises for the Convex and Discrete Geometry Summer School
Erdős Center - Alfréd Rényi Institute of Mathematics

B. Proof of the varentropy inequality

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave probability density. Write $f = e^{-\psi}$, where $\psi : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. Set

$$V(f) = \mathrm{Var}_{\mu_f}(\ln f) = \int_{\mathbb{R}^n} f(\ln f)^2 - \left(\int_{\mathbb{R}^n} f \ln f\right)^2,$$

where μ_f is the log-concave probability measure with density f.

1. Define $F : (0, \infty) \to \mathbb{R}$ with

$$F(p) = \ln\left(\int_{\mathbb{R}^n} f^p(x)\, dx\right).$$

Show that

$$F''(p) = \frac{1}{p^2} V(f_p),$$

where f_p is the log-concave density

$$f_p = \frac{f^p}{\int_{\mathbb{R}^n} f^p}.$$

2. Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function. Define $w : \mathbb{R}^n \times (0, \infty) \to (-\infty, \infty]$ by

$$w(z, p) = p\psi(z/p).$$

Show that w is convex.

3. Show that the function $G : (0, \infty) \to \mathbb{R}$ defined by

$$G(p) = p^n \int_{\mathbb{R}^n} f^p(x)\, dx$$

is log-concave.

4. Show that $V(f_p) \leqslant n$ for every $p > 0$. In particular, for $p = 1$, we get

$$V(f) \leqslant n.$$

This is the varentropy inequality.

5. Suppose that $f = e^{-\psi}$ as above, and ψ is positively homogeneous of degree 1. Show that $V(f_p) = n$ for every $p > 0$. In particular, for $p = 1$, we get

$$V(f) = n.$$

This shows that the inequality in 4 is sharp.

Hints

1. It should follow by a careful computation of $F''(p)$ and $V(f_p)$.

2. Check that $w(\lambda z_1 + (1-\lambda)z_2, \lambda p_1 + (1-\lambda)p_2) \leqslant \lambda w(z_1, p_1) + (1 - \lambda(w(z_2, p_2)$ for every $z_1, z_2 \in \mathbb{R}^n$, $p_1, p_2 > 0$ and $\lambda \in (0, 1)$.

3. Make the change of variables $x = z/p$ and use the previous exercise. A marginal of a log-concave measure is log-concave, therefore it has a log-concave density.

4. Note that $\ln G(p) = n \ln p + F(p)$. Differentiate twice and use Exerecise 1 and Exercise 3.

5. Show that in this case we have $G(p) = 1$ for all $p > 0$.