Approximation in Geometry

Notes for the lectures given at the
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Lecture 1

Approximation of convex bodies by polytopes

The Minkowski sum of two set A, B in $\mathbb{R}^d$ is denoted by $A + B = \{a + b : \ a \in A, b \in B\}$. The origin is denoted o, the closed Euclidean ball centered at $x \in \mathbb{R}^d$ of radius ρ is denoted by $\mathbf{B}^d(x, \rho)$. The boundary of a convex body K is denoted by ∂K. The *centroid* (or, center of mass) of a convex body K is the point obtained as the following integral

$$\frac{1}{\mathrm{vol}(K)} \int_K x \, \mathrm{d}x,$$

where $\mathrm{d}x$ denotes integration with respect to the Lebesgue measure in the affine hull of K, and $\mathrm{vol}(K)$ is the $\dim(K)$-dimensional volume (Lebesgue measure) of K. In general, we mostly use the notation in Schneider's book, [Sch14].

1.1 Preliminaries

In order to define the problem of approximating a *convex body* (a compact convex set with nonempty interior) by a *convex polytope* (the convex hull of finitely many points in $\mathbb{R}^d$), we need to have some notion of distance between convex sets. We will use two such notions.

The *Hausdorff distance* of two convex sets K and L in $\mathbb{R}^d$ is defined as

$$\delta_H(K, L) = \inf\{\delta > 0 : \ K + \mathbf{B}^d(o, \delta) \supseteq L, L + \mathbf{B}^d(o, \delta) \supseteq K\}.$$

We will define the *geometric distance* of K and L as

$$d(K, L) = \inf\{\alpha/\beta : \ \alpha, \beta > 0, \beta K \subseteq L \subseteq \alpha K\}.$$

Note that this definition is sensitive to the choice of the origin, in other words, it is not translation invariant.

In what follows, K is a given convex body in $\mathbb{R}^d$, and our goal is to find a polytope which is close to K in one of the two distances defined above.

Exercise 1.1.1. *Let K be a smooth convex body containing the origin in its interior and $\varepsilon \in (0, 1)$. Let $X \subset K$ be a finite set. Show that the polytope $P = \mathrm{conv}(X)$ satisfies $d(K, P) \leq \frac{1}{1-\varepsilon}$ if and only if, X intersects every* cap of depth ε, *that is, every set of the form* $\mathrm{cap}(x, \varepsilon) = \{y \in K : \ \langle y, \nu\rangle \geq (1 - \varepsilon)\langle x, \nu\rangle\}$*, where x is an arbitrary boundary point of K and ν is any outer unit normal vector of K at x.*

1.2 The packing bound

Goal: Find a polytope P with few vertices such that $\delta_H(P,K) \le \varepsilon$.

Exercise 1.2.1. *Show that if* $\Lambda \subset \mathbb{R}^d$ *is such that* $\Lambda + \mathbf{B}^d(o, \varepsilon/2)$ *is a maximal packing of* $\varepsilon/2$ *radius balls in* $K + \mathbf{B}^d(o, \varepsilon/2)$*, then* $P = \operatorname{conv}(\Lambda)$ *is a polytope satisfying* $\delta_H(P,K) \le \varepsilon$.

Exercise 1.2.2. *Prove that there is a* Λ *with* $\Lambda + \mathbf{B}^d(o,\varepsilon) \supseteq K$ *of cardinality at most* $\frac{\operatorname{vol}\left(K+\mathbf{B}^d(o,\varepsilon/2)\right)}{\operatorname{vol}\left(\mathbf{B}^d(o,\varepsilon/2)\right)}$.

Exercise 1.2.3. *Prove that for any* $\varepsilon > 0$ *and dimension* d*, there is a polytope* P *with no more than roughly* $\left(\frac{3}{\varepsilon}\right)^d$ *vertices that is* $(1+\varepsilon)$*-close to* $K + t$ *in the geometric distance with an appropriate translation vector* $t \in \mathbb{R}^d$.

1.3 The Bronshteĭn–Ivanov net

Let $\rho \in \left(0, \frac{1}{2}\right)$. Let K be a convex body with smooth boundary containing the origin and contained in $\mathbf{B}^d(o,R)$. Consider the set S of points $\{x + \nu_x : x \in \partial K\}$, where ν_x is the outer unit normal to ∂K at x. Let $\{x_j + \nu_{x_j} : 1 \le j \le N\}$ be a maximal ρ-separated set in S, i.e., a set such that any two of its members are at distance at least ρ (see Figure 1.1). We call the corresponding set $\{x_j : 1 \le j \le N\}$ a *Bronshtein–Ivanov net* of mesh ρ for the body K.

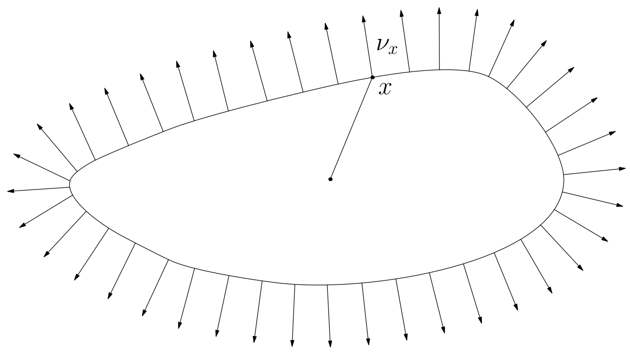

Figure 1.1: The Bronshteĭn–Ivanov net

Exercise 1.3.1. *In the construction above, for every* $x \in \partial K$*, we can find* j *such that* $|x - x_j|^2 + |\nu_x - \nu_{x_j}|^2 \le \rho^2$.

Lemma 1.3.1 (Upper bound on the size of a B–I net)**.** *We have* $N \le 2^d (R+3)^d \rho^{-d+1}$.

Proof. Assume that $s', s'' \ge 0$. Write

$$|x' + \nu' + s'\nu' - x'' - \nu'' - s''\nu''|^2 = |x' + \nu' - x'' - \nu''|^2 + |s'\nu' - s''\nu''|^2 + 2s'\langle \nu', x' - x''\rangle + 2s''\langle \nu'', x'' - x'\rangle + 2(s'+s'')(1 - \langle \nu', \nu''\rangle) \ge |x' + \nu' - x'' - \nu''|^2.$$

Thus, if the balls of radius $\frac{\rho}{2}$ centered at $x' + \nu'$ and $x'' + \nu''$ are disjoint, so are the balls of radius $\frac{\rho}{2}$ centered at $x' + (1+s')\nu'$ and $x'' + (1+s'')\nu''$. From here we conclude that the balls

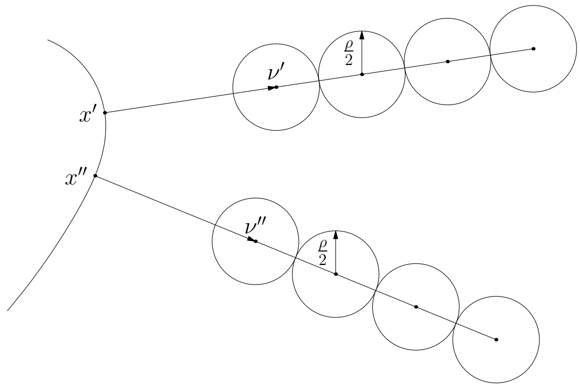

Figure 1.2: The disjoint balls

of radius $\frac{\rho}{2}$ centered at the points $x_j + (1+k\rho)\nu_{x_j}$, $0 \le k \le \frac{1}{\rho}$ are all disjoint (see Figure 1.2) and contained in $\mathbf{B}^d(0, R+3)$.
The total number of these balls is at least $\frac{N}{\rho}$ (for every point x_j in the net, there is a chain of at least $\frac{1}{\rho}$ balls corresponding to different values of k), whence $\frac{N}{\rho} \le \left(\frac{R+3}{\frac{\rho}{2}}\right)^d$ and the desired bound for N follows. $\square$

We will call K a *nice convex body*, if it has smooth boundary, $\mathbf{B}^d(o,1) \subset K \subset \mathbf{B}^d(o,R)$, and for every boundary point $x \in \partial K$, there exists a ball of radius Θ containing K whose boundary sphere touches K at x.

Lemma 1.3.2 (Caps of nice bodies are of small diameter)**.** *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Assume that K is a nice convex body, $x \in \partial K$, and ν is the outer normal to ∂K at x. If $y \in K$ and $\langle y, \nu\rangle \ge (1-\varepsilon)\langle x, \nu\rangle$, then $|y - x| \le \sqrt{2\Theta R\varepsilon}$.*

Proof. Let Q be the ball of radius Θ containing K whose boundary sphere touches K at x. Then $y \in Q$ and ν is the outer unit normal to Q at x, so Q is centered at $x - \Theta\nu$. Note also that, since $o \in K \subset \mathbf{B}^d(0,R)$, we have $0 \le \langle x, \nu\rangle \le R$. Now we have

$$\Theta^2 \ge |y - x + \Theta\nu|^2 = |y-x|^2 + 2\Theta\langle y - x, \nu\rangle + \Theta^2,$$

so

$$|y-x|^2 \le 2\Theta\langle x - y, \nu\rangle \le 2\Theta\varepsilon\langle x, \nu\rangle \le 2\Theta R\varepsilon,$$

as required. $\square$

Lemma 1.3.3 (The cap about x may be replaced by the cap about x')**.** *Fix $\varepsilon, \rho \in \left(0, \frac{1}{2}\right)$. Let K be a nice convex body, $x, x', y \in \partial K$; and ν and ν' the outer unit normals to ∂K at x and x' respectively. Assume that $|x - x'|^2 + |\nu - \nu'|^2 \le \rho^2$ and $\langle y, \nu\rangle \ge \left(1 - \frac{\varepsilon}{2}\right)\langle x, \nu\rangle$. Then*

$$\langle y, \nu'\rangle \ge \left(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon R + |y-x|)\right)\langle x', \nu'\rangle.$$

Proof. We have

$$\begin{aligned}\langle y,\nu'\rangle = \langle x,\nu'\rangle + \langle y-x,\nu'\rangle = \\ \langle x',\nu'\rangle + \langle x-x',\nu'\rangle + \langle y-x,\nu\rangle + \langle y-x,\nu'-\nu\rangle \geq \\ \langle x',\nu'\rangle + \langle x-x',\nu'-\nu\rangle + \langle y-x,\nu\rangle + \langle y-x,\nu'-\nu\rangle \geq \\ \langle x',\nu'\rangle - \rho^2 - \frac{\varepsilon}{2}\langle x,\nu\rangle - \rho|y-x|.\end{aligned}$$

Here, when passing from the second line to the third one, we used the inequality $\langle x-x',\nu\rangle \geq 0$. Note now that, since $\mathbf{B}^d(o,1) \subset K \subset \mathbf{B}^d(o,R)$, we have

$$\langle x,\nu\rangle = \langle x,\nu'\rangle + \langle x,\nu-\nu'\rangle \leq \langle x',\nu'\rangle + \rho R$$

and $\langle x',\nu'\rangle \geq 1 > \frac{1}{2}$. So

$$\langle y,\nu'\rangle \geq \Big(1-\frac{\varepsilon}{2}\Big)\langle x',\nu'\rangle - \rho\Big(\rho + \frac{\varepsilon R}{2} + |y-x|\Big) \geq \Big(1-\frac{\varepsilon}{2} - 2\rho(\rho+\varepsilon R + |y-x|)\Big)\langle x',\nu'\rangle.$$

□

Combining Lemmas 1.3.1, 1.3.2 and 1.3.3, with the choice of $\rho = \frac{\sqrt{\varepsilon}}{10\sqrt{\Theta R}}$, we obtain the following result.

Theorem 1.3.4. *If K is a nice convex body with $R = d^2$ and $\Theta = d^5$, then there is a convex polytope P with no more than $d^{100d}\varepsilon^{-\frac{d-1}{2}}$ vertices satisfying $P \subseteq K \subseteq (1+\varepsilon)P$.*

Fact: Every convex body has an affine image that can be approximated by a nice convex body with $R = d^2$ and $\Theta = d^5$.

References: The packing bound is an old, standard argument, cf. [Nas18]. The Bronshtein–Ivanov net appeared in [BI75].

1.4 Economic cap covering

In this section, we will use caps defined slightly differently. For a convex body K, we define the *depth function* from K to $\mathbb{R}_{\geq 0}$ as

$$\mathrm{depth}_K(x) = \min\{\mathrm{vol}(K\cap H) :\ H \text{ is a half-space containing } x\}.$$

For a point $x \in K$, we will denote its *minimal cap*, that is, the set $K \cap H$ with minimum volume among all half-spaces H containing x with $\mathrm{cap}(x)$.

We will need the technical notion of the *magnified cap* defined as follows. Denote the hyperplane that bounds the minimal half-space in the definition of $\mathrm{cap}(x)$ by L, and let L' denote the supporting hyperplane of $\mathrm{cap}(x)$ parallel to and distinct from L. We will call the centroid of $\mathrm{cap}(x) \cap L'$ the *center of the cap* $\mathrm{cap}(x)$. Finally, for a $\lambda > 0$, we denote by $\mathrm{cap}(x)^\lambda$ the image of $\mathrm{cap}(x)$ under the magnification centered at the center of $\mathrm{cap}(x)$ by factor λ.

The *floating body* (respectively, the *wet part*) of K for $t \geq 0$ is defined as the deep (resp., shallow) points of K, i.e.,

$$K_{\geq t} = \{x \in K :\ \mathrm{depth}_K(x) \geq t\}, \text{ and } K_{\leq t} = \{x \in K :\ \mathrm{depth}_K(x) \leq t\}.$$

Our goal is to cover the wet part by small caps.

Theorem 1.4.1 (Economic cap covering)**.** *There are constants $c(d), C(d) > 0$ depending only on the dimension d such that the following hold. Assume K is a convex body in $\mathbb{R}^d$ with* $\mathrm{vol}\,(K) = 1$*, and* $0 < \varepsilon < (2d)^{-2d}$*. Then there are caps $C_1, \ldots, C_m$ and pairwise disjoint convex sets $C'_1, \ldots, C'_m$ such that $C'_i \subseteq C_i$, for each i, and*

1. $\bigcup_{i=1}^m C'_i \subseteq K_{\leq\varepsilon} \subseteq \bigcup_{i=1}^m C_i$,
2. $\mathrm{vol}\,(C'_i) > c(d)\varepsilon$ *and* $\mathrm{vol}\,(C_i) < C(d)\varepsilon$ *for each* i,
3. *for each cap C with $C \cap K_{\geq\varepsilon} = \emptyset$ there is a C_i containing C.*

The central objects in the proof of the Economic cap covering theorem are Macbeath regions.The *Macbeath region* of K at $x \in K$ with parameter $\lambda > 0$ is the centrally symmetric convex set

$$M_K(x, \lambda) = x + \lambda[(K - x) \cap (x - K)].$$

Theorem 1.4.2 (Bárány)**.** *Let K be a convex body with* $\mathrm{vol}\,(K) = 1$ *and $t \in (0, t_0)$ (where t_0 depends only on d). Then there is a polytope P with $K_{\geq t} \subseteq P \subseteq K$ with no more than*

$$C(d)\frac{\mathrm{vol}\,(K_{\leq t})}{t}$$

facets, where $C(d) > 0$ depends only on d.

Proof. Set $\tau = \lambda t$, where $\lambda = 6^{-d}$, and choose a set of points $\{x_1, \ldots, x_m\}$ from $\partial K_{\geq\tau}$ maximal with respect to the property that the $M(x_i, 1/2)$ are pairwise disjoint. One can show that

$$c(d)m < \frac{\mathrm{vol}\,(K_{\leq\tau})}{\tau} < C(d)\frac{\mathrm{vol}\,(K_{\leq t})}{t},$$

for some $c(d), C(d) > 0$ depending only on d. Now, we remove the magnified (by factor 6) minimal caps from K to obtain

$$P = K \setminus \bigcup_{i=1}^m \mathrm{cap}\,(x_i)^6\,.$$

It can be shown that (1) no $z \in \partial K$ belongs to P, and (2) $K_{\geq t} \subseteq P$. □

References: The Economic Cap Covering theorem is due to Bárány and Larman [BL88, B8́9], using the idea of *Macbeath regions* introduced by Macbeath [Mac52]. For a wonderful exposition, see [Bár07].

1.5 VC-dimension and ε-nets

The *Vapnik–Chervonenkis dimension* (VC-dimension, in short) of a set family $\mathcal{F} \subset 2^V$ on a set V is the size of the largest set A such that $\mathcal{F}|_A = \{F \cap A : F \in \mathcal{F}\}$ is the power set 2^A of A.

Exercise 1.5.1. *Show that if V is a (finite or infinite) subset of $\mathbb{R}^d$ and $\mathcal{G}$ is some (finite or infinite) set of half-spaces in $\mathbb{R}^d$, then the VC-dimension of $\mathcal{F} = \{G \cap V : G \in \mathcal{G}\}$ is at most $d + 1$.*

The main advantage of a set family $\mathcal{F}$ with low VC-dimension is that it is easy to find a small subset U of V that intersects all members of $\mathcal{F}$, provided that there is a measure on V according to which each set in $\mathcal{F}$ is of not too small measure. More precisely, we have the following fundamental result.

Theorem 1.5.1 (ε-net Theorem)**.** *Let $0 < \varepsilon < 1/e$, and let D be a positive integer. Let $\mathcal{F}$ be a family of some measurable subsets of a probability space (U, μ), where the probability of each member F of $\mathcal{F}$ is $\mu(F) \geq \varepsilon$. Assume that the VC-dimension of $\mathcal{F}$ is at most D. Set*

$$t := \left\lceil 3\frac{D}{\varepsilon} \ln \frac{1}{\varepsilon} \right\rceil.$$

Choose t elements $X_1, \ldots, X_t$ of V randomly, independently according to μ. Then $\{X_1,\ldots, X_t\}$ is a transversal of $\mathcal{F}$ with probability at least $1 - \delta$, where

$$\delta := (200\varepsilon)^D.$$

In order to use it for approximation of a convex body by a polytope, we need to find a measure according to which no relevant cap is too small.

First, we recall a classical result of Grünbaum as an exercise.

Exercise 1.5.2 (Grünbaum's theorem)**.** *Let K be a convex body in $\mathbb{R}^d$ with centroid at the origin, and let F be a half-space containing o. Then*

$$\operatorname{vol}(K \cap F) \geq \left(\frac{d}{d+1}\right)^d \operatorname{vol}(K).$$

Note that the right hand side is greater than $\frac{\operatorname{vol}(K)}{e}$.

Lemma 1.5.2 (Stability of Grünbaum's theorem)**.** *Let K be a convex body in $\mathbb{R}^d$ with centroid at the origin. Let $0 < \vartheta < 1$, and F be a half-space that supports ϑK from outside. Then*

$$\operatorname{vol}(K) \frac{(1-\vartheta)^d}{e} \leq \operatorname{vol}(K \cap F). \tag{1.1}$$

Theorem 1.5.3. *Let $\vartheta \in (0, 1)$. Set*

$$t = \left\lceil 3\frac{(d+1)e}{(1-\vartheta)^d} \ln \frac{e}{(1-\vartheta)^d} \right\rceil.$$

Then for any centered convex body K in $\mathbb{R}^d$, if t points $X_1, \ldots, X_t$ of K are chosen randomly, independently and uniformly, then

$$\vartheta K \subseteq \operatorname{conv}(X_1, \ldots, X_t) \subseteq K$$

with probability at least $1 - \delta$, where

$$\delta = \left[200\left(\frac{(1-\vartheta)^d}{e}\right)\right]^{d+1}.$$

References: The Espilon-net theorem was found by Haussler and Welzl [HW87] based on ideas of Vapnik and Cervonenkis [VČ68] and then further developed to its sharpest form by Komlós, Pach and Woeginger [KPW92], cf. [PA95, Mat02, Mus22] for recent developments. Theorem 1.5.3 appeared in [Nas19].

1.6 Other measures

For a Borel set $C \subset \partial K$, let $C^* = \{x^* \in \partial K^\circ : x \in C\}$ denote the corresponding points of the polar of K.

Consider the "cones" $\mathrm{Cone}(C) = \{rx : x \in C, 0 \le r \le 1\}$ and $\mathrm{Cone}(C^*) = \{ry : y \in C^*, 0 \le r \le 1\}$.

$$\mu(C) = \frac{1}{2}\left(\frac{\mathrm{vol}_d(\mathrm{Cone}(C))}{\mathrm{vol}_d(K)} + \frac{\mathrm{vol}_d(\mathrm{Cone}(C^*))}{\mathrm{vol}_d(K^\circ)}\right).$$

Using this measure, we have that each cap C of depth ε is at least of measure

$$\mu(C) \ge C^d \varepsilon^{\frac{d-1}{2}}$$

Recall the notation $\mathrm{cap}\,(x, \varepsilon)$ from Exercise 1.1.1.

Lemma 1.6.1. *Let K be a strictly convex body with smooth boundary. Assume that K contains o in its interior and satisfies the Santaló bound $\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\circ) \le e^{O(d)} d^{-d}$. Then μ is a probability measure on ∂K invariant under linear automorphisms of $\mathbb{R}^d$ and $\mu(\mathrm{cap}\,(x, \varepsilon)) \ge e^{O(d)} \varepsilon^{\frac{d-1}{2}}$ for all $x \in \partial K$ and all $\varepsilon \in (0, \frac{1}{2})$.*

The following result is shown in [NNR20]. For further developments, see [AdFM23, AM23].

Theorem 1.6.2. *Let K be a convex body in $\mathbb{R}^d$ with the center of mass at the origin, and let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then there exists a convex polytope P with at most $e^{O(d)} \varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.*

Lecture 2

Quantitative Helly-type questions

The study of quantitative versions of Helly-type questions was initiated by Bárány, Katchalski and Pach in [BKP82].

2.1 Rough approximation of the volume

The **Quantitative Volume Theorem** from [BKP82] states the following. *Assume that the intersection of any $2d$ members of a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is of volume at least one. Then the volume of the intersection of all members of the family is of volume at least $c(d)$, a constant depending on d only.*

In [BKP82], it is proved that one can take $c(d) = d^{-2d^2}$ and conjectured that it should hold with $c(d) = d^{-cd}$ for an absolute constant $c > 0$. It was confirmed with $c(d) \approx d^{-2d}$ in [Nas16], whose argument was refined by Brazitikos [Bra17], who showed that one may take $c(d) \approx d^{-3d/2}$. For more on quantitative Helly-type results, see the surveys [HW18, DLGMM19]. For recent quantitative variants of the Fractional Helly Theorem and the (p, q)-Theorem cf. [RS17, DLLHRS17, SXS19].

2.1.1 Preliminaries for the proof of QVT

Definition 2.1.1. We say that a set of vectors $w_1, \ldots, w_m \in \mathbb{R}^d$ with weights $c_1, \ldots, c_m > 0$ form a *John's decomposition of the identity*, if

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I, \tag{2.1}$$

where I is the identity operator on $\mathbb{R}^d$.

We recall John's theorem [Joh48] (see also [Bal97]).

Lemma 2.1.2 (John's theorem)**.** *For any convex body K in $\mathbb{R}^d$, there is a unique ellipsoid of maximal volume in K. Furthermore, this ellipsoid is $\mathbf{B}^d(o, 1)$ if, and only if, there are points $w_1, \ldots, w_m \in \partial\mathbf{B}^d(o, 1) \cap \partial K$ (called* contact points*) and corresponding weights $c_1, \ldots, c_m > 0$ that form a John's decomposition of the identity.*

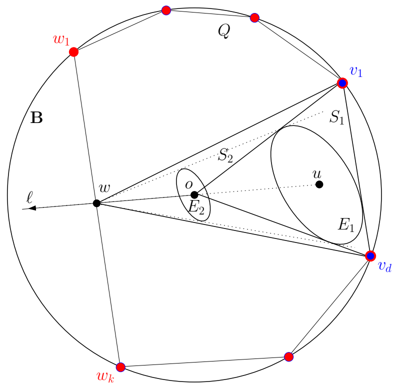

Figure 2.1:

Exercise 2.1.1. *Let Δ denote a regular simplex in $\mathbb{R}^d$ such that the ball $\mathbf{B}^d(o,1)$ is the largest volume ellipsoid in Δ. Prove that*

$$\operatorname{vol}(\Delta) = \frac{d^{d/2}(d+1)^{(d+1)/2}}{d!}. \tag{2.2}$$

A key tool in the proof is the Dvoretzky-Rogers lemma [DR50].

Lemma 2.1.3 (Dvoretzky-Rogers lemma)**.** *Assume that $w_1,\dots,w_m \in \partial\mathbf{B}^d(o,1)$ and $c_1,\dots,c_m > 0$ form a John's decomposition of the identity. Then there is an orthonormal basis $z_1,\dots,z_d$ of $\mathbb{R}^d$, and a subset $\{v_1,\dots,v_d\}$ of $\{w_1,\dots,w_m\}$ such that*

$$v_i \in \operatorname{span}\{z_1,\dots,z_i\}, \quad \textit{and} \quad \sqrt{\frac{d-i+1}{d}} \le \langle v_i, z_i\rangle \le 1, \quad \textit{for all } i=1,\dots,d. \tag{2.3}$$

Exercise 2.1.2. *Prove Lemma 2.1.3.*

2.1.2 Proof of QVT

Without loss of generality, we may assume that $\mathcal{F}$ consists of closed half-spaces, and $0 < \operatorname{vol}(\cap\mathcal{F}) < \infty$ (that is, $\cap\mathcal{F}$ is a convex body in $\mathbb{R}^d$), and $\mathcal{F}$ is a finite family. In short, we will assume that $P = \cap\mathcal{F}$ is a d-dimensional convex polyhedron.

The problem is clearly affine invariant, so we may further assume that $\mathbf{B}^d(o,1) \subset P$ is the ellipsoid of maximal volume in P.

By John's theorem, there are contact points $w_1,\dots,w_m \in \partial\mathbf{B}^d(o,1) \cap \partial\mathbf{B}^d(o,1)$ (and weights $c_1,\dots,c_m > 0$) that form a John's decomposition of the identity. We denote their convex hull by $Q = \operatorname{conv}(\{) w_1.\dots,w_m\}$. Lemma 2.1.3 yields that there is an orthonormal basis $z_1,\dots,z_d$ of $\mathbb{R}^d$, and a subset $\{v_1,\dots,v_d\}$ of the contact points $\{w_1,\dots,w_m\}$ such that (2.3) holds.

Let $S_1 = \operatorname{conv}(\{) o, v_1, v_2,\dots,v_d\}$ be the simplex spanned by these contact points, and let E_1 be the largest volume ellipsoid contained in S_1. We denote the center of E_1 by u. Let ℓ be the ray emanating from the origin in the direction of the vector $-u$. Clearly, the origin is in the interior of Q. In fact, by the remark following Lemma 2.1.2, $\frac{1}{d}\mathbf{B}^d(o,1) \subset Q$. Let w be the point of intersection of the ray ℓ with ∂Q. Then $|w| \ge 1/d$. Let S_2 denote the simplex $S_2 = \operatorname{conv}(\{) w, v_1, v_2,\dots,v_d\}$. See Figure 2.1.

We apply a contraction with center w and ratio $\lambda = \frac{|w|}{|w-u|}$ on E_1 to obtain the ellipsoid E_2. Clearly, E_2 is centered at the origin and is contained in S_2. Furthermore,

$$\lambda = \frac{|w|}{|u|+|w|} \geq \frac{|w|}{1+|w|} \geq \frac{1}{d+1}. \tag{2.4}$$

Since w is on ∂Q, by Caratheodory's theorem, w is in the convex hull of some set of at most d vertices of Q. By re-indexing the vertices, we may assume that $w \in \operatorname{conv}(\{) w_1, \ldots, w_k\}$ with $k \leq d$. Now,

$$E_2 \subset S_2 \subset \operatorname{conv}(\{) w_1, \ldots, w_k, v_1, \ldots, v_d\}. \tag{2.5}$$

Let $X = \{w_1, \ldots, w_k, v_1, \ldots, v_d\}$ be the set of these unit vectors, and let $\mathcal{G}$ denote the family of those half-space which support $\mathbf{B}^d(o,1)$ at the points of X. Clearly, $|\mathcal{G}| \leq 2d$. Since the points of X are contact points of P and $\mathbf{B}^d(o,1)$, we have that $\mathcal{G} \subseteq \mathcal{F}$. By (2.5),

$$\cap \mathcal{G} = X^* \subset E_2^*. \tag{2.6}$$

By (2.3),

$$\operatorname{vol}(S_1) \geq \frac{1}{d!} \cdot \frac{\sqrt{d!}}{d^{d/2}} = \frac{1}{\sqrt{d!} d^{d/2}}. \tag{2.7}$$

Since $\mathbf{B}^d(o,1) \subset \cap \mathcal{F}$, by (2.6) and (2.4), (2.2), (2.7) we have

$$\frac{\operatorname{vol}(\cap \mathcal{G})}{\operatorname{vol}(\cap \mathcal{F})} \leq \frac{\operatorname{vol}(E_2^*)}{\operatorname{vol}(\mathbf{B}^d(o,1))} = \frac{\operatorname{vol}\left(\mathbf{B}^d(o,1)\right)}{\operatorname{vol}(E_2)} \leq (d+1)^d \frac{\operatorname{vol}\left(\mathbf{B}^d(o,1)\right)}{\operatorname{vol}(E_1)} = (d+1)^d \frac{\operatorname{vol}(\Delta)}{\operatorname{vol}(S_1)} \tag{2.8}$$

$$= \frac{d^{d/2}(d+1)^{(3d+1)/2}}{d! \operatorname{vol}(S_1)} = \frac{d^d d^{3d/2} e^{3/2}(d+1)^{1/2}}{(d!)^{1/2}} \leq e^{d+1} d^{2d+\frac{1}{2}},$$

where Δ is as defined above (2.2). This completes the proof.

2.2 Colorful Quantitative Volume Theorem

We state the following colorful variant, which appeared in [DFN21]. See [SXS21] for further results.

Theorem 2.2.1 (Colorful Quantitative Volume Theorem)**.** *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection, $C_i \in \mathcal{C}_i$ for each $i \in [3d]$, the intersection $\bigcap_{k=1}^{3d} C_i$ contains an ellipsoid of volume at least 1. Then, there is an $i \in [3d]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume at least $c^{d^2} d^{-5d^2/2}$ with an absolute constant $c \geq 0$.*

Exercise 2.2.1. *Deduce the following result. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of* **2d** ***sets***, *$C_{i_k} \in \mathcal{C}_{i_k}$ for each $k \in [2d]$ with $1 \leq i_1 < \cdots < i_{2d} \leq 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$* ***is of volume at least 1****. Then, there is an $i \in [3d]$ such that $\bigcap_{C \in \mathcal{C}_i} C$* ***is of volume*** *at least $c^{d^2} d^{-5d^2/2-d}$ with an absolute constant $c \geq 0$.*

2.3 Back to geometric distance

Next, we turn to the problem of (a rough) estimation of a polytope as the convex hull of a well chosen small subset of its vertices. The followig theorem is due to Almendra-Hernández, Ambrus and Kendall [AHAK23], improving a result in [IN22].

Theorem 2.3.1. *Let $\lambda > 0$, and $L \subset \mathbb{R}^d$ be a convex polytope such that $L \subset -\lambda L$. Then there is a set of at most $2d$ vertices of L, whose convex hull L' satisfies*

$$L \subset -(\lambda + 2)d \cdot L'.$$

According to the next exercise, with the right choice of the origin, the assumption of the Theorem always holds with $\lambda = d$.

Exercise 2.3.1. *Show that for any convex set K in $\mathbb{R}^d$, there is a translation $t \in \mathbb{R}^d$ such that $K - t \subseteq d(t - K)$.*

Proof of Proposition 2.3.1. The condition $L \subseteq -\lambda L$ ensures that the origin belongs to the interior of L. Among all simplices with d vertices from the set of vertices of L and one vertex at the origin, consider a simplex $S = \operatorname{conv}(0, v_1, \ldots, v_d)$ with maximal volume. The simplex S can be represented as

$$S = \left\{ x \in \mathbb{R}^d : \; x = \alpha_1 v_1 + \ldots + \alpha_d v_d \quad \text{for} \;\; \alpha_i \geq 0 \text{ and } \sum_{i=1}^{d} \alpha_i \leq 1 \right\}. \tag{2.9}$$

Define $P = \sum_{i \in [d]} [-v_i, v_i]$. It is easy to see that P is a paralletope that can be represented as

$$P = \{x \in \mathbb{R}^d : \; x = \beta_1 v_1 + \ldots + \beta_d v_d \quad \text{for } \beta_i \in [-1, 1]\}. \tag{2.10}$$

Since S is chosen maximally, equation (2.10) shows that for any vertex v of L, $v \in P$. By convexity,

$$L \subset P. \tag{2.11}$$

Let $S' = -2dS + (v_1 + \ldots + v_d)$. By (2.9),

$$S' = \left\{ x \in \mathbb{R}^d : \; x = \gamma_1 v_1 + \ldots + \gamma_d v_d \quad \text{for } \gamma_i \leq 1 \text{ and } \sum_{i=1}^{d} \gamma_i \geq -d \right\},$$

which, together with (2.10), yields

$$P \subseteq S'. \tag{2.12}$$

Let y be the intersection of the ray emanating from 0 in the direction $-(v_1 + \cdots + v_d)$ and the boundary of Q. By Carathéodory's theorem, we can choose $k \leq d$ vertices $\{v'_1, \ldots, v'_k\}$ of L such that $y \in \operatorname{conv}(v'_1, \ldots, v'_k)$. Set $L' = \operatorname{conv}(v_1, \ldots, v_d, v'_1, \ldots, v'_k)$. Clearly, $\frac{v_1 + \cdots + v_d}{d} \in S \subset L$. Thus, $0 \in L'$, and consequently,

$$S \subseteq Q'. \tag{2.13}$$

Since $L \subset -\lambda L$, we also have that

$$\frac{v_1 + \cdots + v_d}{d} \in -\lambda[y, 0] \subset -\lambda L'.$$

Combining it with (2.11), (2.12), (2.13), we obtain

$$L \subset P \subset S' = -2dS + (v_1 + \cdots + v_d) \subset -2d\, L' - \lambda d\, L' = -(\lambda + 2)d\, L', \tag{2.14}$$

Completing the proof of Proposition 2.3.1. □

We say that a convex body K is in *John's position*, if the largest volume ellipsoid contained in K is $\mathbf{B}^d\,(o,1)$. It is well know that in this case, $K \subset \mathbf{B}^d\,(o,d)$.

Corollary 2.3.2. *Let Q be a convex polytope in $\mathbb{R}^d$ in John's position. Then there is a subset of at most $2d$ vertices of Q whose convex hull Q' satisfies*

$$Q \subseteq -2d^2 Q'.$$

Lecture 3

Approximation of sums of matrices

In the previous lecture, we saw how taking a John decomposition of the identity and selecting a small subset of the unit vectors yields geometric results. In this lecture, we consider the following general question. Given a matrix A (often, we simply have $A = I$) as a (positive) linear combination of some other matrices. Can we select a small subset of the matrices whose linear combination (with possibly new coefficients) yields a matrix close to A in some norm.

3.1 Rudelson's theorem: a probabilistic approach

A random vector v in $\mathbb{R}^d$ is called *isotropic*, if $\mathbb{E}v \otimes v = I$, where $\mathbb{E}$ denotes the expectation of a random variable, and I is the identity operator on $\mathbb{R}^d$.

According to Rudelson's theorem [Rud99], if we take k independent copies $y_1, \dots, y_k$ of an isotropic random vector y in $\mathbb{R}^d$ for which $|y|^2 \le \gamma$ almost surely, with

$$k = \left\lceil \frac{c\gamma \ln d}{\varepsilon^2} \right\rceil, \text{ then } \mathbb{E}\left\| \frac{1}{k}\sum_{i=1}^{k} y_i \otimes y_i - I \right\| \le \varepsilon,$$

where $\|A\| = \max\{\langle Ax, Ax\rangle^{1/2} : x \in \mathbb{R}^d, \langle x, x\rangle = 1\}$ denotes the *operator norm* of the matrix A.

Rudelson's result applies in the setting of a John decomposition of the identity, see (2.1): if we take $\alpha_i = c_i/d$, we may interpret α_i as the probability of a random unit vector taking the value u_i. Let $\sigma = \{i_1, \dots, i_k\}$ be a multiset obtained by k independent draws from $[m]$ according to the distribution where Prob(i is drawn) $= \alpha_i$, and consider the following average of matrices $\frac{1}{k}\sum_{i\in\sigma} \sqrt{d}u_i \otimes \sqrt{d}u_i$. It follows that, in expectation, this average is not farther than ε from I in the operator norm, provided that k is at least $\frac{cd\ln d}{\varepsilon^2}$, where c is some constant.

We state it in a slightly more general form as appeared in [INP20].

Theorem 3.1.1 (Rudelson's theorem)**.** *Let $0 < \varepsilon < 1$ and $Q_1, \dots, Q_k$ be independent random matrices distributed according to (not necessarily identical) probability distributions $\mathcal{P}_1, \dots, \mathcal{P}_k$ on the set $\mathcal{P}^d$ of $d \times d$ real positive semi-definite matrices such that $\mathbb{E}Q_i = A$ for some $A \in \mathcal{P}^d$ and all $i \in [k]$. Set $\gamma = \mathbb{E}(\max_{i\in[k]} \|Q_i\|)$, and assume that*

$$k \ge \frac{c\gamma(1 + \|A\|)\ln d}{\varepsilon^2},$$

where c is an absolute constant. Then

$$\mathbb{E}\left\|\frac{1}{k}\sum_{i\in[k]} Q_i - A\right\| \le \varepsilon. \tag{3.1}$$

Exercise 3.1.1. *Show the following.*

1. *The set $\mathcal{P}^d$ of positive semi-definite $d\times d$ matrices (with real entries) form a convex cone with apex at the origin in the vector space $\mathbb{R}^{d(d+1)/2}$ of symmetric matrices.*
2. *Matrices of trace 1 form a hyperplane H_1 containing $\frac{1}{d}I$ in $\mathbb{R}^{d(d+1)/2}$.*
3. *The set $\mathcal{P}^d \cap H_1$ is a convex body in H_1.*

3.1.1 Proof of Rudelson's Theorem

Let $\mathcal{P}^d$ denote the cone of positive semi-definite symmetric matrices in $\mathbb{R}^{d\times d}$. The *Schatten p-norm* of a real $d\times d$ matrix A is defined as

$$\|A\|_{C_p^d} := \left(\sum_{i=1}^{d} (s_i(A))^p\right)^{1/p},$$

where $s_1(A),\ldots,s_d(A)$ is the sequence of eigenvalues of the positive semi-definite matrix $\sqrt{A^*A}$. We recall that $\|A\| \le \|A\|_{C_p^d}$ for all $p\ge 1$, and we also have

$$\|A\| \le \|A\|_{C_p^d} \le e\,\|A\| \ \text{ for } p=\ln d, \tag{3.2}$$

where ln denotes the natural logarithm and e denotes its base.

From this point on, $\mathbf{r}$ denotes a sequence of k *Rademacher variables*, that is, $\mathbf{r}=(r_1,\ldots,r_k)$, where the r_i are random variables uniformly distributed on $\{1,-1\}$, independent of each other and all other random variables in the context.

We state the following inequality due to Lust–Piquard and Pisier [LP86, LPP91], essentially in the form as it appears in the book [Pis98, Theorem 8.4.1].

Theorem 3.1.2 (Lust–Piquard)**.** $2\le p<\infty$. *For any d and any $Q_1,\ldots,Q_k$ (not necessarily positive definite) square matrices of size d we have*

$$\left[\mathbb{E}_{\mathbf{r}}\left\|\sum_{j=1}^{k} r_jQ_j\right\|_{C_p^d}^p\right]^{1/p} \le c\sqrt{p}\max\left\{\left\|\left(\sum_{j=1}^{k} Q_jQ_j^*\right)^{1/2}\right\|_{C_p^d}, \left\|\left(\sum_{j=1}^{k} Q_j^*Q_j\right)^{1/2}\right\|_{C_p^d}\right\}$$

for a universal constant $c>0$.

Note that for any $d\times d$ matrix Q, the product Q^*Q is positive semi-definite. Since, by Weyl's inequality, the Schatten p-norm is monotone on the cone of positive semi-definite matrices, we may deduce from the theorem of Lust–Piquard the following inequality

$$\left[\mathbb{E}_{\mathbf{r}}\left\|\sum_{j=1}^{k} r_jQ_j\right\|_{C_p^d}^p\right]^{1/p} \le c\sqrt{p}\left\|\left(\sum_{j=1}^{k} Q_jQ_j^* + Q_j^*Q_j\right)^{1/2}\right\|_{C_p^d}. \tag{3.3}$$

Lemma 3.1.3 (Symmetrization by Rademacher variables)**.** *Let $q_1, \dots, q_k$ be independent random vectors distributed according to (not necessarily identical) probability distributions $\mathcal{P}_1, \dots, \mathcal{P}_k$ on a normed space X with $\mathbb{E} q_i = q$ for all $i \in [k]$.*
Then

$$\mathop{\mathbb{E}}_{q_1,\dots,q_k} \left\| \frac{1}{k} \sum_{\ell=1}^{k} q_\ell - q \right\| \le \frac{2}{k} \mathop{\mathbb{E}}_{q_1,\dots,q_k} \mathop{\mathbb{E}}_{\mathbf{r}} \left\| \sum_{\ell=1}^{k} r_\ell q_\ell \right\|.$$

Exercise 3.1.2. *Prove Lemma 3.1.3.*

Proof of Theorem 3.1.1. The argument follows very closely Rudelson's.
Denote by $D = \frac{1}{k} \sum_{i \in [k]} Q_i - A$, and $p = \ln d$. Then

$$\mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \|D\| \le \mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \|D\|_{C_p^d} \overset{(S)}{\le} \frac{2}{k} \mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \mathop{\mathbb{E}}_{\mathbf{r}} \left\| \sum_{\ell=1}^{k} r_\ell Q_\ell \right\|_{C_p^d}$$

$$\overset{(H)}{\le} \frac{2}{k} \mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \left[\mathop{\mathbb{E}}_{\mathbf{r}} \left\| \sum_{\ell=1}^{k} r_\ell Q_\ell \right\|_{C_p^d}^{p} \right]^{1/p} \overset{(L\text{-}P)}{\le} \frac{c_0 \sqrt{p}}{k} \mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \left\| \left(\sum_{\ell=1}^{k} Q_\ell^2 \right)^{1/2} \right\|_{C_p^d}$$

$$\overset{(PSD)}{\le} \frac{c_0 \sqrt{p}}{k} \mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \left[\max_{\ell \in [k]} \|Q_\ell\|^{1/2} \cdot \left\| \left(\sum_{\ell=1}^{k} Q_\ell \right)^{1/2} \right\|_{C_p^d} \right]$$

$$\le \frac{c_1 \sqrt{p}}{k} \mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \left[\max_{\ell \in [k]} \|Q_\ell\|^{1/2} \cdot \left\| \left(\sum_{\ell=1}^{k} Q_\ell \right) \right\|^{1/2} \right]$$

$$\overset{(H)}{\le} \frac{c_1 \sqrt{\gamma p}}{k} \left[\mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \left\| \left(\sum_{\ell=1}^{k} Q_\ell \right) \right\| \right]^{1/2} \le \frac{c_1 \sqrt{\gamma p}}{\sqrt{k}} \left[\mathop{\mathbb{E}}_{Q_1,\dots,Q_k} \|D\| + \|A\| \right]^{1/2},$$

where c_0 and c_1 are positive constants. Here, we use Lemma 3.1.3 in step (S) and the inequality (3.3) in step (L-P). The inequality (PSD) relies on the fact that the matrices Q_i are positive semi-definite, and (H) follows from Hölder's inequality.
Thus, we obtain

$$\mathbb{E} \|D\| \le \frac{c_1 \sqrt{\gamma \ln d}}{\sqrt{k}} \sqrt{\mathbb{E} \|D\| + \|A\|}.$$

Denoting by $\alpha = \left(\frac{c_1 \sqrt{\gamma \ln d}}{\sqrt{k}} \right)^2$, we have

$$(\mathbb{E} \|D\|)^2 - \alpha \mathbb{E} \|D\| - \alpha \|A\| \le 0.$$

Therefore, we get $\mathbb{E} \|D\| \le \alpha + \sqrt{\alpha \|A\|}$, and thus the inequality

$$\mathbb{E} \|D\| \le \frac{c_1^2 \gamma \ln d}{k} + \frac{c_1 \sqrt{\gamma \|A\| \ln d}}{\sqrt{k}} \le \varepsilon$$

holds for $k \ge \frac{c\gamma(1+\|A\|) \ln d}{\varepsilon^2}$ with sufficiently large c. Theorem 3.1.1 is proved. □

3.2 An algorithmic approach

If instead of considering the *expectation* of the average of randomly chosen $u_i \otimes u_i$, we want to show the *existence* of a small subset of the set of $u_i \otimes u_i$ whose average is close to I, then the picture changes, as was shown by a completely different approach introduced in the fundamental paper of Batson, Spielman and Srivastava [BSS14]. It was developed further by Marcus, Spielman and Srivastava [MSS15] (see also [Sri12]), and by Friedland and Youssef [FY17]. In [FY17], the following is shown.

Theorem 3.2.1. *Let $u_1, \dots, u_m$ be unit vectors in $\mathbb{R}^d$ that yield a John decomposition of I. Then there is a (deterministically obtained) multi-subset σ of $[m]$ of size $|\sigma| = \frac{cd}{\varepsilon^2}$ with $\left\| \frac{1}{|\sigma|} \sum_{i \in \sigma} \sqrt{d} u_i \otimes \sqrt{d} u_i - I \right\| < \varepsilon$.*

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