Exercise Session by Ádám Sagmeister and Márton Naszódi

Exercise 1. Let K be a convex body containing the origin in its interior and $\varepsilon \in (0,1)$. Let $X \subset \partial K$ be a finite set. Show that the polytope $P = \operatorname{conv}(X)$ satisfies $d(K,P) \leq \frac{1}{1-\varepsilon}$ if and only if, X intersects every *cap of depth* ε, that is, every set of the form $\operatorname{cap}(x,\varepsilon) = \{y \in K : \langle y, \nu\rangle \geq (1-\varepsilon)\langle x,\nu\rangle\}$, where x is an arbitrary boundary point of K and ν is any outer unit normal vector of K at x.

Exercise 2. Show that if $\Lambda \subset \mathbb{R}^d$ is such that $\Lambda + \mathbf{B}^d(o,\varepsilon/2)$ is a maximal packing of $\varepsilon/2$ radius balls in $K + \mathbf{B}^d(o,\varepsilon/2)$, then $P = \operatorname{conv}(\Lambda)$ is a polytope satisfying $\delta_H(P,K) \leq \varepsilon$.

Exercise 3. Prove that there is a Λ with $\Lambda + \mathbf{B}^d(o,\varepsilon) \supseteq K$ of cardinality at most $\frac{\operatorname{vol}\left(K+\mathbf{B}^d(o,\varepsilon/2)\right)}{\operatorname{vol}\left(\mathbf{B}^d(o,\varepsilon/2)\right)}$.

Exercise 4. Prove that for any $\varepsilon > 0$ and dimension d, there is a polytope P with no more than roughly $\left(\frac{3}{\varepsilon}\right)^d$ vertices that is $(1+\varepsilon)$-close to $K + t$ in the geometric distance with an appropriate translation vector $t \in \mathbb{R}^d$.

Exercise 5. Prove that in the Bronshtein–Ivanov construction, for every $x \in \partial K$, we can find j such that $|x - x_j|^2 + |\nu_x - \nu_{x_j}|^2 \leq \rho^2$.

Exercise 6. Show that if V is a (finite or infinite) subset of $\mathbb{R}^d$ and $\mathcal{G}$ is some (finite or infinite) set of half-spaces in $\mathbb{R}^d$, then the VC-dimension of $\mathcal{F} = \{G \cap V : G \in \mathcal{G}\}$ is at most $d+1$.

Exercise 7. (Grünbaum's theorem) Let K be a convex body in $\mathbb{R}^d$ with centroid at the origin, and let F be a half-space containing o. Then

$$\operatorname{vol}(K \cap F) \geq \left(\frac{d}{d+1}\right)^d \operatorname{vol}(K).$$

Exercise 8. Let Δ denote a regular simplex in $\mathbb{R}^d$ such that the ball $\mathbf{B}^d(o,1)$ is the largest volume ellipsoid in Δ. Prove that

$$\operatorname{vol}(\Delta) = \frac{d^{d/2}(d+1)^{(d+1)/2}}{d!}.$$

Exercise 9. (Dvoretzky–Rogers lemma) Assume that $w_1, \ldots, w_m \in \partial\mathbf{B}^d(o,1)$ and $c_1, \ldots, c_m > 0$ form a John's decomposition of the identity. Then there is an orthonormal basis $z_1, \ldots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \ldots, v_d\}$ of $\{w_1, \ldots, w_m\}$ such that

$$v_i \in \operatorname{span}\{z_1, \ldots, z_i\}, \quad \text{and} \quad \sqrt{\frac{d-i+1}{d}} \leq \langle v_i, z_i\rangle \leq 1, \quad \text{for all } i = 1, \ldots, d.$$

Exercise 10. Deduce the following result. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of **2d sets**, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $k \in [2d]$ with $1 \leq i_1 < \cdots < i_{2d} \leq 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ **is of volume at least 1**. Then, there is an $i \in [3d]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ **is of volume** at least $c^{d^2} d^{-5d^2/2-d}$ with an absolute constant $c \geq 0$.

Exercise 11. Show that for any convex set K in $\mathbb{R}^d$, there is a translation $t \in \mathbb{R}^d$ such that $K - t \subseteq d(t - K)$.

Exercise 12. Show the following.

1. The set $\mathcal{P}^d$ of positive semi-definite $d \times d$ matrices (with real entries) form a convex cone with apex at the origin in the vector space $\mathbb{R}^{d(d+1)/2}$ of symmetric matrices.

2. Matrices of trace 1 form a hyperplane H_1 containing $\frac{1}{d}I$ in $\mathbb{R}^{d(d+1)/2}$.

3. The set $\mathcal{P}^d \cap H_1$ is a convex body in H_1.

Exercise 13. (Symmetrization by Rademacher variables) Let $q_1, \ldots, q_k$ be independent random vectors distributed according to (not necessarily identical) probability distributions $\mathcal{P}_1, \ldots, \mathcal{P}_k$ on a normed space X with $\mathbb{E}q_i = q$ for all $i \in [k]$.

Then

$$\mathbb{E}_{q_1,\ldots,q_k} \left\| \frac{1}{k} \sum_{\ell=1}^{k} q_\ell - q \right\| \leq \frac{2}{k} \mathbb{E}_{q_1,\ldots,q_k} \mathbb{E}_{\mathbf{r}} \left\| \sum_{\ell=1}^{k} r_\ell q_\ell \right\|.$$