Multiplicative Weights Update
Exercise Session 2

1. We are given m vectors $V = \{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$, with the property that there exists a hyperplane through the origin that contains all of them on one side. In fact, we assume something slightly stronger:

   there exists a parameter $\epsilon > 0$ and a vector $u^* \geq 0$ with $\sum_{i=1}^{n} (u^*)_i = 1$, such that

   $$\text{for all } v \in V\colon \quad u^* \cdot v \geq \epsilon.$$

   Then use MWU to find a vector $u \in \mathbb{R}^n$ such that $u \cdot v_i \geq 0$, for all $i \in [m]$.

2. Given a set system $(X, \mathcal{R})$, let OPT be the size of the minimum hitting set for $\mathcal{R}$.

   Then give a MWU algorithm that computes a weight function on the vertices such that each set contains vertices of weight $\frac{1}{\text{OPT}}$-th of the total weight.

3. Recall that statement we proved in the lecture:

   Lemma 1. *Let V be a finite set of n elements, $\mathcal{S}$ a finite collection of subsets of V, and $\alpha \in (0, 1]$ a parameter such that the following is true:*

   for any weight function $w\colon \mathcal{S} \to \mathbb{R}^+$, there exists an element $v \in V$ such that

   $$\sum_{S \in \mathcal{S}\colon v \in S} w(S) \geq \alpha \cdot \left(\sum_{S \in \mathcal{S}} w(S) \right).$$

   Then there exists a weight function $w_V\colon V \to \mathbb{R}^+$ such that for any $S \in \mathcal{S}$,

   $$\sum_{v \in S} w_V(v) \geq \alpha \cdot \left(\sum_{v \in V} w_V(v) \right).$$

   Analyse the following algorithm to give another proof of Lemma 1.

   > Initialize $\omega^1(v) = 1$ for all $v \in V$, and let $\epsilon \in (0, 1)$. Further let $\eta > 0$ be a parameter to be set optimally later. For each iteration $t = 1, \ldots, T$:
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   > (a) let $S^t \in \mathcal{S}$ be a set with weight less than $\big((1-\epsilon)\alpha\big)$-th fraction of the current total weight $\Omega^t = \sum_{v \in V} \omega^t(v)$. If no such set exists, we stop with success.
   >
   > (b) update the weights; that is, for each $v \in S^t$:
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   > $$\omega^{t+1}(v) = \omega^t(v)\big(1 + \eta\big).$$