Multiplicative Weights Update
Exercise Session 1

1. Given a finite set system $(X, \mathcal{R})$ with $n = |X|$ and $m = |\mathcal{R}|$, re-consider the problem of two-coloring the elements of X sequentially to get a coloring of low discrepancy for $\mathcal{R}$. The new weight function, $\omega^t \colon X \to \mathbb{R}^+$, is

$$\omega^t(S) = (1 + \eta_S)^{P_S^t} (1 - \eta_S)^{N_S^t} + (1 - \eta_S)^{P_S^t} (1 + \eta_S)^{N_S^t},$$

   where $\eta_S = \Theta\left(\sqrt{\frac{\ln m}{|S|}}\right)$, P_S^t is the number of elements of S colored with $+1$ at time t, and N_S^t is the number of elements of S colored with -1 at time t.

   Show that there exists a two-coloring of $(X, \mathcal{R})$ where each set $S \in \mathcal{R}$ has discrepancy $O\left(\sqrt{|S| \ln m}\right)$.

2. Given a finite set system $(X, \mathcal{R})$ with $n = |X|$ and $m = |\mathcal{R}|$, and a parameter $\epsilon > 0$, a multi-set $A \subseteq X$ is an ϵ-approximation of $\mathcal{R}$ if for each $S \in \mathcal{R}$, we have

$$|S \cap A| = \frac{|S|\,|A|}{|X|} \pm \epsilon\,|A|.$$

   Use MWU technique to show that there exists an ϵ-approximation of $\mathcal{R}$ of size $O\left(\frac{1}{\epsilon^2} \ln m\right)$.

3. Given a finite set system $(X, \mathcal{R})$ with $n = |X|$ (assume n is even) and $m = |\mathcal{R}|$, consider the following 'discrepancy game' between two players, Alice and Bob.

   The game has $n/2$ iterations, where in each iteration:

   (a) first, Alice choses an uncolored element, say denoted by $a^t \in X$, and colors it with $+1$, and

   (b) then, Bob choses an uncolored element, say denoted by $b^t \in X$, and colors it with -1.

   Show that there is a strategy for Alice such that no matter how Bob plays, the final coloring has discrepancy at most $O\left(\sqrt{|S| \ln m}\right)$ for each $S \in \mathcal{R}$.

   The weight function here for Alice is:

$$\sum_{S \in \mathcal{R}} \frac{(1/2)\exp\left(\eta_S \left(P_S^t - N_S^t\right)\right) + \exp\left(\eta_S \left(N_S^t - P_S^t\right)\right)}{\left((1/2)\exp(\eta_S) + \exp(-\eta_S)\right)^{P_S^t + N_S^t}},$$

   where $\eta_S = \Theta\left(\sqrt{\frac{\ln m}{|S|}}\right)$.