Threshold for the measure of random polytopes

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Abstract

Let μ be a log-concave probability measure on $\mathbb{R}^n$ and for any $N > n$ consider the random polytope $K_N = \text{conv}\{X_1, \ldots, X_N\}$, where $X_1, X_2, \ldots$ are independent random points in $\mathbb{R}^n$ distributed according to μ. We study the question if there exists a threshold for the expected measure of K_N. Our approach is based on the Cramer transform Λ_μ^* of μ. We establish, under some conditions, a sharp threshold for the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N: it is close to 0 if $\ln N \ll \mathbb{E}_\mu(\Lambda_\mu^*)$ and close to 1 if $\ln N \gg \mathbb{E}_\mu(\Lambda_\mu^*)$.

1 Introduction

In these notes we study the question how to obtain a threshold for the expected measure of a random polytope defined as the convex hull of independent random points with a log-concave distribution. The general formulation of the problem is the following. Given a log-concave probability measure μ on $\mathbb{R}^n$, let $X_1, X_2, \ldots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ define the random polytope

$$K_N = \text{conv}\{X_1, \ldots, X_N\}.$$

Then, consider the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N, where $\mu^N = \mu \otimes \cdots \otimes \mu$ (N times). This is an affinely invariant quantity, so we may assume that μ is centered, i.e. the barycenter of μ is at the origin.

Given $\delta \in (0,1)$ we say that μ satisfies a "δ-upper threshold" with constant ϱ_1 if

$$\sup\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant \exp(\varrho_1 n)\} \leqslant \delta \tag{1.1}$$

and that μ satisfies a "δ-lower threshold" with constant ϱ_2 if

$$\inf\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \geqslant \exp(\varrho_2 n)\} \geqslant 1 - \delta. \tag{1.2}$$

Then, we define $\varrho_1(\mu, \delta) = \sup\{\varrho_1 : (1.1) \text{ holds true}\}$ and $\varrho_2(\mu, \delta) = \inf\{\varrho_2 : (1.2) \text{ holds true}\}$. Our main goal is to obtain upper bounds for the difference

$$\varrho(\mu, \delta) := \varrho_2(\mu, \delta) - \varrho_1(\mu, \delta)$$

for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

One may also consider a sequence $\{\mu_n\}_{n=1}^\infty$ of log-concave probability measures μ_n on $\mathbb{R}^n$. Then, we say that $\{\mu_n\}_{n=1}^\infty$ exhibits a "sharp threshold" if there exists a sequence $\{\delta_n\}_{n=1}^\infty$ of positive reals such that $\delta_n \to 0$ and $\varrho(\mu_n, \delta_n) \to 0$ as $n \to \infty$. This terminology may be used to describe a variety of results that have been obtained for specific sequences of measures (in most cases, product measures or rotationally invariant

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measures) starting with the classical work [15] of Dyer, Füredi and McDiarmid, which concerns the uniform measure on the discrete cube and the solid cube.

Our aim is to describe a general approach to the problem, that was proposed in [10], working with an arbitrary log-concave probability measure μ on $\mathbb{R}^n$. We present the main ideas, the progress that has been achieved (especially in the case of the uniform measure on a convex body) and several remaining open questions.

2 The origin of the method: the case of the discrete cube

Our starting point is the work of Dyer, Füredi and McDiarmid [15], who established a threshold for the volume of random 0/1 polytopes.

Let X be a random vector in $\mathbb{R}^n$ whose coordinates are independent and take each of the values 1 and -1 with probability $\frac{1}{2}$. Given $N > n$, we consider N independent copies $X_1, \dots, X_N$ of the random vector X. This procedure defines the random 0/1 polytope

$$K_N = \mathrm{conv}\{X_1, \dots, X_N\}. \tag{2.1}$$

Theorem 2.1 (Dyer-Füredi-McDiarmid). *Let $\kappa = \ln 2 - \frac{1}{2}$ and let K_N the random polytope defined by* (2.1). *For every $\varepsilon \in (0, \kappa)$ we have that*

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}\,|K_N| \colon N \leqslant \exp((\kappa - \varepsilon)n)\right\} = 0 \tag{2.2}$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}\,|K_N| \colon N \geqslant \exp((\kappa + \varepsilon)n)\right\} = 1. \tag{2.3}$$

In this section we present the basic points of the proof of Theorem 2.1. Several of the lemmas that are used will be proved in a more general and stronger form later on. The next function plays a key role in the argument.

Definition 2.2. For every $x \in (-1,1)^n$, we set

$$\varphi(x) := \inf\left\{\mathrm{Prob}\big(X \in H\big) \colon x \in H,\ H \text{ closed half-space}\right\}. \tag{2.4}$$

Also, for every origin symmetric convex body $A \subset (-1,1)^n$ we define

$$\varphi_+(A) = \sup_{x \notin A} \varphi(x) \quad \text{and} \quad \varphi_-(A) = \inf_{x \in A} \varphi(x). \tag{2.5}$$

Note that the infimum in (2.4) is determined by those half-spaces H for which $x \in \partial(H)$.

Lemma 2.3. *Let $N > n$ and let A be an origin symmetric convex body contained in $(-1,1)^n$. Then,*

$$\mathbb{E}(|K_N|) \leqslant |A| + N2^n \varphi_+(A).$$

Proof. We write

$$\mathbb{E}\left(|K_N|\right) = \mathbb{E}\left(|K_N \cap A|\right) + \mathbb{E}\left(|K_N \setminus A|\right) \leqslant |A| + \mathbb{E}\left(|K_N \setminus A|\right). \tag{2.6}$$

Note that if H is a closed half-space containing x, and if $x \in K_N$, then we may find $i \leqslant N$ such that $X_i \in H$ (otherwise, we would have $x \in K_N \subseteq H'$, where H' is the complementary half-space of H). It follows that

$$\mathrm{Prob}\big(x \in K_N\big) \leqslant N \cdot \varphi(x).$$

Using Fubini's theorem we see that

$$\mathbb{E}\left(|K_N \setminus A|\right) = \int_{C \setminus A} \mathrm{Prob}(x \in K_N)\, dx \leqslant \int_{C\setminus A} N\varphi(x)\, dx \leqslant N\, \varphi_+(A)\, |C \setminus A|,$$

where $C = [-1,1]^n$. For the last inequality we use the fact that $\varphi(x) \leqslant \varphi_+(A)$ for every $x \notin A$. Going back to (2.6) we get the lemma. $\square$

Note. For the proof of (2.1) we shall choose suitable A (depending on N and n) such that for $N \leqslant \exp((\kappa-\varepsilon)n)$ we will have simultaneously $|A|/2^n \to 0$ and $N\varphi_+(A) \to 0$ as $n \to \infty$.

The second basic observation is the following.

Lemma 2.4. *Let A be an origin symmetric convex body contained in $(-1,1)^n$. Then,*

$$1 - \mathrm{Prob}\big(K_N \supseteq A\big) \leqslant \binom{N}{n} 2^{-(N-n)} + 2\binom{N}{n}\big(1-\varphi_-(A)\big)^{N-n}.$$

We shall prove a more general version of Lemma 2.4 in Section 5 (see Lemma 5.2). What is important is that Lemma 2.4 allows us to use the function φ in order to prove (2.2). Indeed, if we choose suitable A (depending on N and n) so that for $N \geqslant \exp((\kappa+\varepsilon)n)$ we have simultaneously $|A|/2^n \to 1$ and $1 - \mathrm{Prob}\big(K_N \supseteq A\big) \to 0$ as $n \to \infty$, then we get (2.2).

Given a bounded random variable X, consider the moment generating function of X,

$$M(t) := \mathbb{E}\left(e^{tX}\right) \qquad (t \in \mathbb{R})$$

and the logarithmic moment generating function of X,

$$\Lambda(t) := \ln M(t).$$

Since X is bounded, we see that $M(t) < \infty$ for every $t \in \mathbb{R}$. By the symmetry of X it also follows that M and Λ are even functions. Note that

$$\begin{aligned} e^{\Lambda(\lambda t+(1-\lambda)s)} &= M(\lambda t + (1-\lambda)s) = \mathbb{E}\left(e^{\lambda tX} e^{(1-\lambda)sX}\right) \\ &\leqslant \left(\mathbb{E}\, e^{tX}\right)^{\lambda} \left(\mathbb{E}\, e^{sX}\right)^{1-\lambda} = e^{\lambda\Lambda(t)+(1-\lambda)\Lambda(s)} \end{aligned}$$

for every $t, s \in \mathbb{R}$ and every $0 \leqslant \lambda \leqslant 1$, therefore Λ is convex. It follows that M is also convex. We easily check that M is C^∞ on $\mathbb{R}$. The n-th derivative of M is the function

$$M^{(n)}(t) = \mathbb{E}\left(X^n e^{tX}\right).$$

Returning to our case, where X takes the values ± 1 with probability $\frac{1}{2}$, direct computation shows that

$$M(t) := \mathbb{E}\left[e^{tX}\right] = \cosh(t)$$

and

$$\Lambda(t) := \ln M(t) = \ln\cosh(t).$$

Condider the Legendre transform of Λ: this is the function

$$f(x) := \sup\left\{tx - \Lambda(t) \colon t \in \mathbb{R}\right\}, \qquad x \in (-1,1).$$

Lemma 2.5. *The function f is even and strictly convex on $(-1,1)$. For every $x \in (-1,1)$ we have*

$$f(x) = \tfrac{1}{2}(1+x)\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x).$$

Moreover, $\lim_{x\to\pm1} f(x) = \ln 2$.

Proof. We observe that

$$f(x) = xt - \ln\cosh(t), \quad \text{where } \tanh t = x. \tag{2.7}$$

From $\tanh t = x$ we see that

$$e^{2t} = \frac{1+x}{1-x}, \text{ or equivalently, } t = h(x) = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

We also have

$$\cosh t = \frac{e^t}{1+x}.$$

Going back to (2.7) we see that

$$\begin{aligned} f(x) &= xt - \ln\cosh t = xt - t + \ln(1+x) = \ln(1+x) - (1-x)\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\ &= \ln(1+x) - (1-x)\tfrac{1}{2}\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x), \end{aligned}$$

and the lemma follows. □

As a relatively simple consequence of Markov's inequality we get the next upper bound for $\varphi(x)$ in terms of $\sum_{i=1}^n f(x_i)$, for every $x \in (-1,1)^n$.

Lemma 2.6. *For every $x \in (-1,1)^n$ we have that $\varphi(x) \leqslant \exp\left(-\sum_{i=1}^n f(x_i)\right)$.*

Proof. Let H be a closed half-space such that $x \in \partial(H)$. Then, there exists $t \in \mathbb{R}^n$ such that

$$H = H(t) = \{y : \langle t, y - x\rangle \geqslant 0\}.$$

From Markov's inequality,

$$\begin{aligned} \mathrm{Prob}\big(X \in H(t)\big) &= \mathrm{Prob}\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \leqslant \mathbb{E}\left[\exp\left\{\sum_{i=1}^n t_i(X_i - x_i)\right\}\right] \\ &= \prod_{i=1}^n \mathbb{E}\big[\exp(t_i(X_i - x_i)\big] = \prod_{i=1}^n e^{\Lambda(t_i) - t_i x_i}. \end{aligned}$$

From the definition of $\varphi(x)$ we have

$$\varphi(x) \leqslant \inf_{t \in \mathbb{R}^n} \prod_{i=1}^n e^{\Lambda(t_i) - t_i x_i} = \prod_{i=1}^n e^{-\sup\{t x_i - \Lambda(t) :\, t \in \mathbb{R}\}} = \prod_{i=1}^n e^{-f(x_i)}.$$

This proves the lemma. □

We extend f continuously on $[-1,1]$ setting $f(\pm 1) = \ln 2$ and for every $x = (x_1, \ldots, x_n) \in C$ we set

$$F(x) = \frac{1}{n}\sum_{i=1}^n f(x_i).$$

For every $0 < \alpha < \ln 2$, we define

$$F^\alpha = \{x \in (-1,1)^n \colon F(x) \leqslant \alpha\}.$$

Since f is even and convex on $(-1,1)$, the set F^α is an origin symmetric convex body contained in $(-1,1)^n$. From the definition of F^α we see that $\sum_{i=1}^n f(x_i) = nF(x) = \alpha n$ for all $x \in \partial(F^\alpha)$. Therefore, Lemma 2.6 proves the next fact.

Lemma 2.7. *Let $0 < \alpha < \ln 2$. For every $x \in \partial(F^\alpha)$ we have*

$$\varphi(x) \leqslant \exp(-\alpha n).$$

In other words,

$$\varphi_+(F^\alpha) \leqslant \exp(-\alpha n).$$

Let $U_1, \dots, U_n$ be independent random variables, uniformly distributed in $(-1,1)$. Then, for every $0 < \alpha < \ln 2$,

$$2^{-n}|F^\alpha| = \mathrm{Prob}((U_1,\dots,U_n) \in F^\alpha) = \mathrm{Prob}\left(\frac{1}{n}\sum_{i=1}^n f(U_i) \leqslant \alpha\right).$$

Note that

$$\kappa = \mathbb{E}\left(f(U_i)\right) = \tfrac{1}{2}\int_{-1}^1 f(x)dx = \ln 2 - \tfrac{1}{2}.$$

By the law of large numbers we conclude the following.

Lemma 2.8. *For every $\alpha \in (0,\kappa)$ we have*

$$\lim_{n\to\infty} 2^{-n}|F^\alpha| = 0,$$

and, similarly, for every $\alpha \in (\kappa, \ln 2)$ we have

$$\lim_{n\to\infty} 2^{-n}|F^\alpha| = 1.$$

Now, we can prove the first part of the theorem of Dyer, Füredi and McDiarmid.

Proposition 2.9. *For every $\varepsilon \in (0,\kappa)$,*

$$\lim_{n\to\infty} \sup\left\{2^{-n}E(|K_N|) \colon N \leqslant \exp((\kappa-\varepsilon)n)\right\} = 0.$$

Proof. We choose $\alpha = \kappa - \varepsilon/2$. From Lemma 2.8 we have that

$$\lim_{n\to\infty} 2^{-n}|F^\alpha| = 0.$$

On the other hand, if $N \leqslant \exp((\kappa-\varepsilon)n)$, then Lemma 2.7 gives

$$N\varphi_+(F^\alpha) \leqslant \exp(-\varepsilon n/2).$$

Applying Lemma 2.3 with $A = F^\alpha$ we get

$$2^{-n}\mathbb{E}\left(|K_N|\right) \leqslant 2^{-n}|F^\alpha| + \exp(-\varepsilon n/2),$$

and the right hand side tends to 0 as $n \to \infty$. □

For the proof of (2.2) we need to estimate $\varphi(x)$ from below in order to use Lemma 2.4. The basic technical step is the next proposition, which will be discussed, in a more general context, in Section 7.

Proposition 2.10. *For every $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$, depending only on ε, such that for every $0 < \alpha < \ln 2$ and every $n \geqslant n(\varepsilon)$ we have*

$$\varphi_-(F^\alpha) \geqslant \exp(-\alpha(1+\varepsilon)n).$$

Then, the proof of (2.2) is simple.

Proposition 2.11. *For every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}\left(|K_N|\right) \colon N \geqslant \exp((\kappa+\varepsilon)n)\right\} = 1.$$

Proof. Fix $\varepsilon > 0$ choose $\alpha = \kappa + \varepsilon/3$. Combining Lemma 2.4 with Proposition 2.10 we see that if $n \geqslant n(\varepsilon)$ and $N \geqslant \exp((\kappa+\varepsilon)n) \geqslant \exp((\alpha + 2\varepsilon/3)n)$, then

$$\mathbb{E}\left(|K_N|\right) \geqslant |F^\alpha| \cdot \mathrm{Prob}\big(K_N \supseteq F^\alpha\big) \geqslant |F^\alpha|\,(1 - 2^{-n+1}).$$

Since $\alpha > \kappa$, Lemma 2.8 shows that

$$\lim_{n\to\infty} 2^{-n}|F^\alpha| = 1.$$

The result follows. □

3 Notation and background information

In this section we introduce notation and terminology that we use throughout these notes, and provide background information on isotropic convex bodies and log-concave probability measures. We write $\langle\cdot,\cdot\rangle$ for the standard inner product in $\mathbb{R}^n$ and denote the Euclidean norm by $|\cdot|$. In what follows, B_2^n is the Euclidean unit ball, S^{n-1} is the unit sphere, and σ is the rotationally invariant probability measure on S^{n-1}. Lebesgue measure in $\mathbb{R}^n$ is denoted by $|\cdot|$. The letters c, c', c_j, c_j' etc. denote absolute positive constants whose value may change from line to line.

We refer to Schneider's book [35] for basic facts from the Brunn-Minkowski theory and to the book [2] for basic facts from asymptotic convex geometry. We also refer to [11] for more information on isotropic convex bodies and log-concave probability measures.

2.1. Convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex set $K\subset\mathbb{R}^n$ with non-empty interior. We often consider bounded convex sets K in $\mathbb{R}^n$ with $0\in\text{int}(K)$; since the closure of such a set is a convex body, we shall call these sets convex bodies too. We say that K is centrally symmetric if $-K=K$ and that K is centered if the barycenter $\text{bar}(K)=\frac{1}{|K|}\int_K x\,dx$ of K is at the origin. We shall use the fact that if K is a centered convex body in $\mathbb{R}^n$ then

$$\max_{y\in\mathbb{R}^n}|K\cap(y+\xi^{\perp})|\leqslant e\,|K\cap\xi^{\perp}| \tag{3.1}$$

for all $\xi\in S^{n-1}$, where $\xi^{\perp}=\{x\in\mathbb{R}^n:\langle x,\xi\rangle=0\}$. This is a result of Fradelizi; for a proof see [11, Proposition 6.1.9]. The radial function ϱ_K of K is defined for all $x\neq 0$ by $\varrho_K(x)=\sup\{\lambda>0:\lambda x\in K\}$ and the support function of K is given by $h_K(x)=\sup\{\langle x,y\rangle : y\in K\}$ for all $x\in\mathbb{R}^n$. The polar body K° of a convex body K in $\mathbb{R}^n$ with $0\in\text{int}(K)$ is the convex body

$$K^{\circ}:=\big\{y\in\mathbb{R}^n:\langle x,y\rangle\leqslant 1 \text{ for all } x\in K\big\}.$$

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K>0$, the isotropic constant of K, such that

$$\|\langle\cdot,\xi\rangle\|_{L_2(K)}^2:=\int_K\langle x,\xi\rangle^2dx=L_K^2$$

for all $\xi\in S^{n-1}$.

2.2. Log-concave probability measures. A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A+(1-\lambda)B)\geqslant\mu(A)^{\lambda}\mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda\in(0,1)$. A function $f:\mathbb{R}^n\to[0,\infty)$ is called log-concave if its support $\{f>0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\ln f$ to it is concave. If f has finite positive integral then there exist constants $A,B>0$ such that $f(x)\leqslant Ae^{-B|x|}$ for all $x\in\mathbb{R}^n$ (see [11, Lemma 2.2.1]). In particular, f has finite moments of all orders. It is known (see [6]) that if a probability measure μ is log-concave and $\mu(H)<1$ for every hyperplane H in $\mathbb{R}^n$, then μ has a log-concave density f_{μ}. We say that μ is even if $\mu(-B)=\mu(B)$ for every Borel subset B of $\mathbb{R}^n$ and that μ is centered if

$$\text{bar}(\mu):=\int_{\mathbb{R}^n}\langle x,\xi\rangle d\mu(x)=\int_{\mathbb{R}^n}\langle x,\xi\rangle f_{\mu}(x)dx=0$$

for all $\xi\in S^{n-1}$. We shall use the fact that if μ is a centered log-concave probability measure on $\mathbb{R}^k$ then

$$\|f_{\mu}\|_{\infty}\leqslant e^kf_{\mu}(0). \tag{3.2}$$

This is a result of Fradelizi from [16]. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that the indicator function $\mathbf{1}_K$ of K is the density of a log-concave measure, the Lebesgue measure on K.

Given $\kappa\in[-\infty,1/n]$ we say that a measure μ on $\mathbb{R}^n$ is κ-concave if

$$\mu((1-\lambda)A+\lambda B)\geqslant((1-\lambda)\mu^{\kappa}(A)+\lambda\mu^{\kappa}(B))^{1/\kappa} \tag{3.3}$$

for all compact subsets A, B of $\mathbb{R}^n$ with $\mu(A)\mu(B) > 0$ and all $\lambda \in (0,1)$. The limiting cases are defined appropriately. For $\kappa = 0$ the right hand side in (3.3) becomes $\mu(A)^{1-\lambda}\mu(B)^{\lambda}$ (therefore, 0-concave measures are the log-concave measures). In the case $\kappa = -\infty$ the right hand side in (3.3) becomes $\min\{\mu(A), \mu(B)\}$. Note that if μ is κ-concave and $\kappa_1 \leqslant \kappa$ then μ is κ_1-concave.

Next, let $\gamma \in [-\infty, \infty]$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called γ-concave if

$$f((1-\lambda)x + \lambda y) \geqslant ((1-\lambda)f^{\gamma}(x) + \lambda f^{\gamma}(y))^{1/\gamma}$$

for all $x, y \in \mathbb{R}^n$ with $f(x)f(y) > 0$ and all $\lambda \in (0,1)$. Again, we define the cases $\gamma = 0, +\infty$ appropriately. Borell [7] studied the relation between κ-concave probability measures and γ-concave functions and showed that if μ is a measure on $\mathbb{R}^n$ and the affine subspace F spanned by the support $\mathrm{supp}(\mu)$ of μ has dimension $\dim(F) = n$ then for every $-\infty \leqslant \kappa < 1/n$ we have that μ is κ-concave if and only if it has a non-negative density $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ and ψ is γ-concave, where $\gamma = \frac{\kappa}{1-\kappa n} \in [-1/n, +\infty)$.

Let μ and ν be two log-concave probability measures on $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function which is defined ν-almost everywhere and satisfies

$$\mu(B) = \nu(T^{-1}(B))$$

for every Borel subset B of $\mathbb{R}^n$. We then say that T pushes forward ν to μ and write $T_*\nu = \mu$. It is easy to see that $T_*\nu = \mu$ if and only if for every bounded Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(x) d\mu(x) = \int_{\mathbb{R}^n} g(T(y)) d\nu(y).$$

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx} \right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx}.$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered and $\mathrm{Cov}(\mu) = I_n$, where I_n is the identity $n \times n$ matrix. In this case, $L_\mu = \|f_\mu\|_\infty^{1/n}$. For every μ there exists an affine transformation T such that $T_*\mu$ is isotropic. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 1$. Bourgain [8] established the upper bound $L_n \leqslant c\sqrt[4]{n}\ln n$; later, Klartag, in [23], improved this estimate to $L_n \leqslant c\sqrt[4]{n}$. In a breakthrough work, Chen [13] proved that for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $L_n \leqslant n^{\varepsilon}$ for every $n \geqslant n_0(\varepsilon)$. Subsequently, Klartag and Lehec [26] showed that $L_n \leqslant c(\ln n)^4$, and very recently Klartag [25] achieved the best known bound $L_n \leqslant c\sqrt{\ln n}$.

2.3. Centroid bodies. Let μ be a log-concave probability measure on $\mathbb{R}^n$. For any $t \geqslant 1$ we define the L_t-centroid body $Z_t(\mu)$ of μ as the centrally symmetric convex body whose support function is

$$h_{Z_t(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^t \, d\mu(x) \right)^{1/t}.$$

Note that $Z_t(\mu)$ is always centrally symmetric, and $Z_t(T_*\mu) = T(Z_t(\mu))$ for every $T \in GL(n)$ and $t \geqslant 1$. Note also that a centered log-concave probability measure μ is isotropic if and only if $Z_2(\mu) = B_2^n$. The next result of Paouris (see [11, Theorem 5.1.17]) provides upper bounds for the volume of the L_t-centroid bodies of isotropic log-concave probability measures.

Theorem 3.1. *If μ is a centered log-concave probability measure on $\mathbb{R}^n$, then for every $2\leqslant t\leqslant n$ we have that*

$$|Z_t(\mu)|^{1/n}\leqslant c\sqrt{t/n}[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $c>0$ is an absolute constant. In particular, if μ is isotropic then $|Z_t(\mu)|^{1/n}\leqslant c\sqrt{t/n}$ for all $2\leqslant t\leqslant n$.

A variant of the L_t-centroid bodies of μ is defined as follows. For every $t\geqslant 1$ we consider the convex body $Z_t^+(\mu)$ with support function

$$h_{Z_t^+(\mu)}(y)=\left(2\int_{\mathbb{R}^n}\langle x,y\rangle_+^t f_\mu(x)dx\right)^{1/t},$$

where $a_+=\max\{a,0\}$. When f_μ is even, it is clear that $Z_t^+(\mu)=Z_t(\mu)$. In any case, we easily verify that

$$Z_t^+(\mu)\subseteq 2^{1/t}Z_t(\mu).$$

Moreover, if μ is isotropic then $Z_2^+(\mu)\supseteq cB_2^n$ for an absolute constant $c>0$. One can also check that if $1\leqslant t<s$ then

$$\left(\frac{2}{e}\right)^{\frac{1}{t}-\frac{1}{s}}Z_t^+(\mu)\subseteq Z_s^+(\mu)\subseteq c_1\left(\frac{2e-2}{e}\right)^{\frac{1}{t}-\frac{1}{s}}\frac{s}{t}Z_t^+(\mu).$$

The right-hand side inequality gives

$$\mathbb{E}_\mu(2\langle z,\xi\rangle_+^{2t})=[h_{Z_{2t}^+(\mu)}(\xi)]^{2t}\leqslant C^{2t}[h_{Z_t^+(\mu)}(\xi)]^{2t}=C^{2t}[\mathbb{E}_\mu(2\langle z,\xi\rangle_+^t)]^2, \tag{3.4}$$

for all $\xi\in S^{n-1}$, where $C>1$ is an absolute constant. For a proof of all these claims see [20].

2.4. The bodies $B_t(\mu)$. Let μ be a probability measure on $\mathbb{R}^n$. We define

$$M_\mu(v):=\int_{\mathbb{R}^n}e^{\langle v,x\rangle}d\mu(x)=\exp(\Lambda_\mu(v))$$

where

$$\Lambda_\mu(v)=\ln\left(\int_{\mathbb{R}^n}e^{\langle v,x\rangle}d\mu(x)\right)$$

is the logarithmic Laplace transform of μ. We also define

$$\Lambda_\mu^*(v):=\mathcal{L}(\Lambda_\mu)(v)=\sup_{u\in\mathbb{R}^n}\left\{\langle v,u\rangle-\ln\int_{\mathbb{R}^n}e^{\langle u,x\rangle}d\mu(x)\right\},$$

where, given a convex function $g:\mathbb{R}^n\to(-\infty,\infty]$, the Legendre transform $\mathcal{L}(g)$ of g is defined by

$$\mathcal{L}(g)(x):=\sup_{y\in\mathbb{R}^n}\{\langle x,y\rangle-g(y)\}.$$

The function Λ_μ^* is called the Cramer transform of μ and plays a crucial role in the theory of large deviations. For every $t\geqslant 1$ we define

$$M_t(\mu):=\left\{v\in\mathbb{R}^n:\int_{\mathbb{R}^n}|\langle v,x\rangle|^t d\mu(x)\leqslant 1\right\}.$$

Note that

$$Z_t(\mu):=(M_t(\mu))^\circ=\left\{x\in\mathbb{R}^n:|\langle v,x\rangle|^t\leqslant\int_{\mathbb{R}^n}|\langle v,y\rangle|^t d\mu(y)\ \text{ for all } v\in\mathbb{R}^n\right\}.$$

For every $t>0$ we also set

$$B_t(\mu):=\{v\in\mathbb{R}^n:\Lambda_\mu^*(v)\leqslant t\}.$$

We say that a measure μ on $\mathbb{R}^n$ is α-regular if for any $s \geqslant t \geqslant 2$ and every $v \in \mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} |\langle v, x\rangle|^s d\mu(x)\right)^{1/s} \leqslant \alpha \frac{s}{t} \left(\int_{\mathbb{R}^n} |\langle v, x\rangle|^t d\mu(x)\right)^{1/t}.$$

For all $s \geqslant t$ we have $M_s(\mu) \subseteq M_t(\mu)$ and $Z_t(\mu) \subseteq Z_s(\mu)$. If the measure μ is α-regular, then $M_t(\mu) \subseteq \alpha \frac{s}{t} M_s(\mu)$ and $Z_s(\mu) \subseteq \alpha \frac{s}{t} Z_t(\mu)$ for all $s \geqslant t \geqslant 2$. Moreover, for every centered probability measure μ we have $\Lambda_\mu^*(0) = 0$ by Jensen's inequality, and the convexity of Λ_μ^* implies that $B_t(\mu) \subseteq B_s(\mu) \subseteq \frac{s}{t} B_t(\mu)$ for all $s \geqslant t > 0$.

Recall that, by Borell's lemma, every log-concave probability measure is c-regular (see [11, Theorem 2.4.6] for a proof).

Proposition 3.2. *Every log-concave probability measure is c-regular, where $c \geqslant 1$ is an absolute constant.*

The next proposition compares $B_t(\mu)$ with $Z_t(\mu)$ when μ is α-regular.

Proposition 3.3. *If μ is α-regular for some $\alpha \geqslant 1$, then for any $t \geqslant 2$ we have*

$$B_t(\mu) \subseteq 4e\alpha Z_t(\mu).$$

Proof. We first check that if $u \in M_t(\mu)$ then

$$\Lambda_\mu\left(\frac{tu}{2e\alpha}\right) \leqslant t.$$

We fix $u \in M_t(\mu)$ and set $\tilde{u} := \frac{tu}{2e\alpha}$. Then,

$$\left(\int_{\mathbb{R}^n} |\langle \tilde{u}, x\rangle|^k d\mu(x)\right)^{1/k} = \frac{t}{2e\alpha} \left(\int_{\mathbb{R}^n} |\langle u, x\rangle|^k d\mu(x)\right)^{1/k},$$

which is bounded by $\frac{t}{2e\alpha}$ if $k \leqslant t$ and by $\frac{k}{2e}$ if $k > t$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{\langle \tilde{u}, x\rangle} d\mu(x) &\leqslant \int_{\mathbb{R}^n} e^{|\langle \tilde{u}, x\rangle|} d\mu(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^n} |\langle \tilde{u}, x\rangle|^k d\mu(x) \\ &\leqslant \sum_{k \leqslant t} \frac{1}{k!} \left|\frac{t}{2e\alpha}\right|^k + \sum_{k > t} \frac{1}{k!} \left|\frac{k}{2e}\right|^k \leqslant e^{\frac{t}{2e\alpha}} + 1 \leqslant e^t\end{aligned}$$

and the claim follows.

Now, let $v \notin 4e\alpha Z_t(\mu)$. We can find $u \in M_t(\mu)$ such that $\langle v, u\rangle > 4e\alpha$ and then

$$\Lambda_\mu^*(v) \geqslant \left\langle v, \frac{tu}{2e\alpha}\right\rangle - \Lambda_\mu\left(\frac{tu}{2e\alpha}\right) > \frac{t}{2e\alpha} 4e\alpha - t = t.$$

Therefore, $v \notin B_t(\mu)$. □

By Proposition 3.2, we have that Proposition 3.3 holds true (with an absolute constant in place of $4e\alpha$) for every log-concave probability measure.

2.5. Ball's bodies $K_t(\mu)$. If μ is a log-concave probability measure on $\mathbb{R}^n$ then, for every $t > 0$, we define

$$K_t(\mu) := K_t(f_\mu) = \left\{x \in \mathbb{R}^n : \int_0^\infty r^{t-1} f_\mu(rx)\, dr \geqslant \frac{f_\mu(0)}{t}\right\}.$$

From the definition it follows that the radial function of $K_t(\mu)$ is given by

$$\varrho_{K_t(\mu)}(x) = \left(\frac{1}{f_\mu(0)} \int_0^\infty t r^{t-1} f_\mu(rx)\, dr \right)^{1/t} \tag{3.5}$$

for $x \neq 0$. The bodies $K_t(\mu)$ were introduced by K. Ball who also established their convexity. If μ is also centered then, for every $0 < t \leqslant s$,

$$\frac{\Gamma(t+1)^{\frac{1}{t}}}{\Gamma(s+1)^{\frac{1}{s}}} K_s(\mu) \subseteq K_t(\mu) \subseteq e^{\frac{n}{t} - \frac{n}{s}} K_s(\mu). \tag{3.6}$$

A proof is given in [11, Proposition 2.5.7]. It is easily checked that

$$|K_n(f)|\, f_\mu(0) = \int_{\mathbb{R}^n} f_\mu(x) dx = 1 \tag{3.7}$$

(see e.g. [11, Lemma 2.5.6]) and then we can use the inclusions (3.6) in order to estimate the volume of $K_t(\mu)$. For every $t > 0$ we have

$$e^{-1} \leqslant f_\mu(0)^{\frac{1}{n}+\frac{1}{t}} |K_{n+t}(\mu)|^{\frac{1}{n}+\frac{1}{t}} \leqslant e \frac{n+t}{n}. \tag{3.8}$$

We are mainly interested in the convex body $K_{n+1}(\mu)$. We shall use the fact that $K_{n+1}(\mu)$ is centered (see [11, Proposition 2.5.3 (v)]) and that

$$f_\mu(0) |K_{n+1}(\mu)| \approx 1. \tag{3.9}$$

The last estimate follows immediately from (3.7) and (3.8).

4 Expected value of the half-space depth

Let μ be a probability measure on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x. The function

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}$$

is called Tukey's half-space depth. The first work in statistics where some form of the half-space depth appears is an article of Hodges [21] from 1955. Tukey introduced the half-space depth for data sets in [37] as a tool that enables efficient visualization of random samples in the plane. The term "depth" also comes from Tukey's article. We refer the reader to the survey article of Nagy, Schütt and Werner [28] for an overview of this topic, with an emphasis on its connections with convex geometry, and many references.

Tukey's half-space depth plays a key role in the problem that we study in these notes. In this section we prove the basic results that are relevant to our study, starting with the expectation

$$\mathbb{E}_\mu(\varphi_\mu) := \int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x)$$

of φ_μ with respect to μ. The following question was asked in [27]: Does there exist an absolute constant $c \in (0,1)$ such that $\mathbb{E}_\mu(\varphi_\mu) \leqslant c^n$ for all $n \geqslant 1$ and all log-concave probability measures μ on $\mathbb{R}^n$?

The next theorem from [9] provides an affirmative answer (up to a $\ln n$-term).

Theorem 4.1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$. Then, $\mathbb{E}_\mu(\varphi_\mu) \leqslant \exp\left(-cn/L_\mu^2\right)$ where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants.*

The quantity $\mathbb{E}_\mu(\varphi_\mu)$ is affinely invariant and hence for the proof of Theorem 4.1 we may assume that μ is isotropic. In fact, we can prove a more general result.

Theorem 4.2. *Let μ and ν be two isotropic log-concave probability measures on $\mathbb{R}^n$, $n \geqslant n_0$. Then,*

$$\mathbb{E}_\nu(\varphi_\mu) := \int_{\mathbb{R}^n} \varphi_\mu(x)\, d\nu(x) \leqslant \exp\left(-cn/L_\nu^2\right),$$

where $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants.

Note that if μ and ν are two log-concave probability measures on $\mathbb{R}^n$ with the same barycenter, and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine transformation and $T_*\mu$ is the push-forward of μ defined by $T_*\mu(A) = \mu(T^{-1}(A))$, then $\varphi_{T_*\mu}(x) = \varphi_\mu(T^{-1}(x))$ for all $x \in \mathbb{R}^n$, and hence

$$\int_{\mathbb{R}^n} \varphi_{T_*\mu}(x) dT_*\nu(x) = \int_{\mathbb{R}^n} \varphi_\mu(T^{-1}(x)) dT_*\nu(x) = \int_{\mathbb{R}^n} \varphi_\mu(x) d\nu(x).$$

Therefore, Theorem 4.1 is a consequence of Theorem 4.2. To see this, starting with a log-concave probability measure μ on $\mathbb{R}^n$, we may consider an affine transformation T such that $T_*\mu$ is isotropic and then apply Theorem 4.2 to the measures $T_*\mu$ and $\nu = T_*\mu$.

We will use the next basic (and simple) lemma.

Lemma 4.3. *Let μ be a Borel probability measure on $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we have $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$. In particular,*

Proof. Let $x \in \mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$ the half-space $\{z : \langle z - x, \xi\rangle \geqslant 0\}$ is in $\mathcal{H}(x)$, therefore

$$\varphi_\mu(x) \leqslant \mu(\{z : \langle z, \xi\rangle \geqslant \langle x, \xi\rangle\}) \leqslant e^{-\langle x,\xi\rangle} \mathbb{E}_\mu\big(e^{\langle z,\xi\rangle}\big) = \exp\big(-[\langle x, \xi\rangle - \Lambda_\mu(\xi)]\big),$$

and taking the infimum over all $\xi \in \mathbb{R}^n$ we see that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$, as claimed. □

Proof of Theorem 4.2. Consider two isotropic log-concave probability measures μ, ν on $\mathbb{R}^n$. We will show that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\nu(x) \leqslant e^{-cn/L_\nu^2}$$

for some absolute constant $c > 0$. Using Lemma 4.3 we write

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\nu(x) &\leqslant \int_{\mathbb{R}^n} e^{-\Lambda_\mu^*(x)} f_\nu(x)\, dx = \int_{\mathbb{R}^n} \left(\int_{\Lambda_\mu^*(x)}^\infty e^{-t} dt\right) f_\nu(x) dx \\ &= \int_0^\infty e^{-t} \int_{\mathbb{R}^n} \mathbf{1}_{B_t(\mu)}(x) f_\nu(x) dx\, dt = \int_0^\infty e^{-t} \nu(B_t(\mu))\, dt.\end{aligned}$$

Fix $b \in (2/n, 1/2]$ which will be chosen appropriately. Since $\nu(B_t(\mu)) \leqslant 1$ and also $\nu(B_t(\mu)) \leqslant \|f_\nu\|_\infty |B_t(\mu)|$ for all $t > 0$, we may write

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\nu(x) &\leqslant \int_{bn}^\infty e^{-t} \nu(B_t(\mu)) dt + \|f_\nu\|_\infty \int_0^{bn} e^{-t} |B_t(\mu)|\, dt \\ &\leqslant \int_{bn}^\infty e^{-t}\, dt + L_\nu^n \int_0^2 e^{-t} |B_t(\mu)|\, dt + L_\nu^n \int_2^{bn} e^{-t} |B_t(\mu)|\, dt \\ &\leqslant e^{-bn} + L_\nu^n |B_2(\mu)| + L_\nu^n \int_2^{bn} e^{-t} |B_t(\mu)|\, dt.\end{aligned}$$

Applying Proposition 3.3 and Theorem 3.1 we get

$$|B_t(\mu)|^{1/n} \leqslant c_1 |Z_t(\mu)|^{1/n} \leqslant c_2 \sqrt{t/n}$$

for all $2 \leqslant t \leqslant n$, where $c_1, c_2 > 0$ are absolute constants. It is also known that $L_\nu \geqslant c_3$ where $c_3 > 0$ is an absolute constant (see [11, Proposition 2.3.12] for a proof). So, we may assume that $c_2 L_\nu \geqslant \sqrt{2}$. Choosing $b_0 := 1/(c_2 L_\nu)^2 \leqslant 1/2$ we write

$$L_\nu^n \int_2^{b_0 n} e^{-t} |B_t(\mu)|\, dt \leqslant c_2^n L_\nu^n \int_2^{b_0 n} (t/n)^{n/2} e^{-t} dt = (c_2 L_\nu)^n \int_2^{b_0 n} (t/n)^{n/2} e^{-t} dt,$$

and since $b_0 n \leqslant n/2$ and the function $t \mapsto t^{n/2} e^{-t}$ is increasing on $[0, n/2]$, we get

$$(c_2 L_\nu)^n \int_2^{b_0 n} e^{-t} |B_t(\mu)|\, dt \leqslant (b_0 n - 2) \cdot (c_2 L_\nu)^n b_0^{n/2} e^{-b_0 n} = (b_0 n - 2) e^{-b_0 n}.$$

Moreover, $|B_2(\mu)|^{1/n} \leqslant c_2 \sqrt{2/n}$, therefore

$$L_\nu^n |B_2(\mu)| \leqslant (c_4 L_\nu^2 / n)^{n/2} \leqslant e^{-b_0 n},$$

because $c_4 L_\nu^2 / n \leqslant e^{-2}$ if $n \geqslant n_0$. Combining the above we get

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\nu(x) \leqslant e^{-b_0 n} + e^{-b_0 n} + (b_0 n - 2) e^{-b_0 n},$$

and hence

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\nu(x) \leqslant n \exp\left(-n/(c_2 L_\nu)^2\right)$$

which implies the result. □

Next, we show that the exponential estimate of Theorem 4.1 is sharp. We consider first the case where μ is the uniform measure on a convex body K in $\mathbb{R}^n$ and then the case of an arbitrary log-concave probability measure μ on $\mathbb{R}^n$.

Proposition 4.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\int_K \varphi_{\mu_K}(x) dx \geqslant e^{-cn},$$

where $c > 0$ is an absolute constant and μ_K is the uniform measure on K.

Proof. By translation invariance we may assume that the barycenter of K is at the origin. Let $x \in \frac{1}{2}K$. We will show that $\varphi_{\mu_K}(x) \geqslant \frac{1}{e^2 n} \cdot \frac{1}{2^n}$. It suffices to show that

$$\inf |\{z \in K : \langle z, \xi \rangle \geqslant \langle x, \xi \rangle\}| \geqslant \frac{1}{e^2 n} \cdot \frac{1}{2^n}, \tag{4.1}$$

where the infimum is over all $\xi \in S^{n-1}$, because by the definition of $\varphi_{\mu_K}(x)$ we only have to check the half-spaces $H \in \mathcal{H}(x)$ for which x is a boundary point. Moreover, we may consider only those $\xi \in S^{n-1}$ that satisfy $\langle x, \xi \rangle \geqslant 0$, because if $\langle x, \xi \rangle < 0$ then

$$|\{z \in K : \langle z, \xi \rangle \geqslant \langle x, \xi \rangle\}| \geqslant |\{z \in K : \langle z, \xi \rangle \geqslant 0\}| \geqslant 1/e$$

by Grünbaum's lemma (see [11, Lemma 2.2.6]). Fix $\xi \in S^{n-1}$ with $\langle x, \xi \rangle \geqslant 0$ and set $m = h_K(\xi) = \max\{\langle z, \xi \rangle : z \in K\}$. Since $\langle x, \xi \rangle \leqslant m/2$, it is enough to show that

$$|\{z \in K : \langle z, \xi \rangle \geqslant m/2\}| \geqslant \frac{1}{e^2 n} \cdot \frac{1}{2^n}. \tag{4.2}$$

Consider the function $g(t) = |K(\xi,t)|$, where $K(\xi,t) = \{z \in K : \langle z,\xi\rangle = t\}$, $t \in [0,m]$. The Brunn-Minkowski inequality implies that $g^{\frac{1}{n-1}}$ is concave. Therefore, for every $r \in [0,m]$ we have that

$$g(r) \geqslant \Big(1 - \frac{r}{m}\Big)^{n-1} g(0).$$

We write

$$\begin{aligned}|\{z \in K : \langle z,\xi\rangle \geqslant m/2\}| &= \int_{m/2}^{m} g(r)\,dr \geqslant g(0)\int_{m/2}^{m}\Big(1-\frac{r}{m}\Big)^{n-1} dr \\ &= g(0)m\int_{1/2}^{1}(1-s)^{n-1}ds = \frac{1}{n2^n}g(0)m.\end{aligned}$$

Since K is centered, we know that $\|g\|_\infty \leqslant e\,|K\cap\xi^\perp| = eg(0)$ from (3.1). Then, using also Grünbaum's lemma, we see that

$$\frac{1}{e} \leqslant \int_0^m g(r)\,dr \leqslant \|g\|_\infty m \leqslant eg(0)m,$$

and (4.2) follows. It is now clear that

$$\int_K \varphi_{\mu_K}(x)dx \geqslant \int_{\frac{1}{2}K}\varphi_{\mu_K}(x)dx \geqslant \Big|\frac{1}{2}K\Big|\cdot\frac{1}{e^2n}\cdot\frac{1}{2^n} = \frac{1}{e^2n}\cdot\frac{1}{4^n} \geqslant e^{-cn}$$

for some absolute constant $c > 0$. $\square$

Next, we assume that μ is a log-concave probability measure on $\mathbb{R}^n$. Our aim is to prove the next theorem.

Theorem 4.5. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\int_{\mathbb{R}^n}\varphi_\mu(x)d\mu(x) \geqslant e^{-cn},$$

where $c > 0$ is an absolute constant.

By the affine invariance of $\mathbb{E}_\mu(\varphi_\mu)$ we may assume that μ is centered. The proof is based on a number of observations. The first one is a consequence of the Paley-Zygmund inequality; we just adapt here [11, Lemma 11.3.3] to give a lower bound for $\varphi_\mu(x)$ when $x \in \delta Z_t^+(\mu)$ for some $\delta \in (0,1)$.

Lemma 4.6. *Let $t \geqslant 1$ and $\delta \in (0,1)$. For every $x \in \delta Z_t^+(\mu)$ one has*

$$\varphi_\mu(x) \geqslant \frac{(1-\delta^t)^2}{C_1^t},$$

where $C_1 > 1$ is an absolute constant.

Proof. Let $x \in \delta Z_t^+(\mu)$. As in the proof of Proposition 4.4, it is enough to show that

$$\inf \mu(\{z \in \mathbb{R}^n : \langle z,\xi\rangle \geqslant \langle x,\xi\rangle\}) \geqslant \frac{(1-\delta^t)^2}{C_1^t}, \tag{4.3}$$

where the infimum is over all $\xi \in S^{n-1}$ with $\langle x,\xi\rangle \geqslant 0$.

Since $x \in \delta Z_t^+(\mu)$, we have $\langle x,\xi\rangle \leqslant \delta h_{Z_t^+(\mu)}(\xi)$ for any such $\xi \in S^{n-1}$, so it is enough to show that

$$\mu(\{z \in \mathbb{R}^n : \langle z,\xi\rangle \geqslant \delta h_{Z_t^+(\mu)}(\xi)\}) \geqslant \frac{(1-\delta^t)^2}{C_1^t}. \tag{4.4}$$

We apply the Paley-Zygmund inequality

$$\mu(\{z : g(z) \geqslant \delta^t \mathbb{E}_\mu(g)\}) \geqslant (1-\delta^t)^2 \frac{[\mathbb{E}_\mu(g)]^2}{\mathbb{E}_\mu(g^2)}$$

for the function $g(z) = \langle z, \xi\rangle_+^t$. From (3.4) we see that

$$\mathbb{E}_\mu(g^2) \leqslant C_1^t \, [\mathbb{E}_\mu(g)]^2$$

for some absolute constant $C_1 > 0$, and the lemma follows. $\square$

Definition 4.7. For every $t \geqslant 1$ we consider the convex set

$$R_t(\mu) = \{x \in \mathbb{R}^n : f_\mu(x) \geqslant e^{-t} f_\mu(0)\}.$$

The convexity of $R_t(\mu)$ is an immediate consequence of the log-concavity of f_μ. Note that $R_t(\mu)$ is bounded and $0 \in \mathrm{int}(R_t(\mu))$.

Lemma 4.8. *For every $t \geqslant 5n$ we have $R_t(\mu) \supseteq c_0 K_{n+1}(\mu)$, where $c_0 > 0$ is an absolute constant.*

Proof. Let $t \geqslant 5n$. Given any $\xi \in S^{n-1}$ consider the log-concave function $h : [0,\infty) \to [0,\infty)$ defined by $h(t) = f_\mu(t\xi)$. From [24, Lemma 5.2] we know that

$$\int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} h(r) dr \geqslant (1 - e^{-t/8}) \int_0^\infty r^{n-1} h(r) dr.$$

By the definition of $K_n(\mu)$ we have

$$\int_0^\infty r^{n-1} h(r) dr = \frac{f_\mu(0)}{n} [\varrho_{K_n(\mu)}(\xi)]^n.$$

On the other hand,

$$\int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} h(r) dr \leqslant \|f\|_\infty \int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} dr = \frac{\|f\|_\infty}{n} [\varrho_{R_t(\mu)}(\xi)]^n.$$

Using also the fact that $\|f\|_\infty \leqslant e^n f_\mu(0)$ from (3.2), we get

$$e^n [\varrho_{R_t(\mu)}(\xi)]^n \geqslant (1 - e^{-t/8}) [\varrho_{K_n(\mu)}(\xi)]^n.$$

This shows that $R_t(\mu) \supseteq c_0 K_n(\mu)$, where $c_0 > 0$ is an absolute constant. From (3.6) we know that $K_n(\mu) \approx K_{n+1}(\mu)$, and this completes the proof. $\square$

Our final lemma compares $Z_t^+(\mu)$ with $K_{n+1}(\mu)$ when $t \geqslant 5n$.

Lemma 4.9. *For every $t \geqslant 5n$ we have that $Z_t^+(\mu) \supseteq c_0' K_{n+1}(\mu)$, where $c_0' > 0$ is an absolute constant.*

Proof. From Lemma 4.8 we know that $c_0 K_{n+1}(\mu) \subseteq R_t(\mu)$ for all $t \geqslant 5n$, where $c_0 > 0$ is an absolute constant. Let $\xi \in S^{n-1}$ and set $m := h_{c_0 K_{n+1}(\mu)}(\xi) = c_0 h_{K_{n+1}(\mu)}(\xi)$. Define

$$A_\xi = c_0 K_{n+1}(\mu) \cap \{x : \langle x, \xi\rangle \geqslant m/2\}.$$

Since $K_{n+1}(\mu)$ is centered, the proof of Proposition 4.4 shows that

$$|A_\xi| \geqslant \frac{|c_0 K_{n+1}(\mu)|}{e^2 n \cdot 2^n} \geqslant \frac{|c_0 K_{n+1}(\mu)|}{C^n}$$

for some absolute constant $C > c_0$. Moreover, if $x \in A_\xi$ then $x \in R_t(\mu)$ and hence $f_\mu(x) \geqslant e^{-t} f_\mu(0)$. We write

$$\begin{aligned} 2\int_{\mathbb{R}^n} \langle x,\xi\rangle_+^t d\mu(x) &\geqslant 2\int_{A_\xi} \langle x,\xi\rangle_+^t d\mu(x) \\ &\geqslant 2\left(\frac{m}{2}\right)^t e^{-t} f_\mu(0)|A_\xi| \geqslant 2\left(\frac{m}{2e}\right)^t \left(\frac{c_0}{C}\right)^n f_\mu(0)|K_{n+1}(\mu)|. \end{aligned}$$

Using also the fact that $(c_0/C)^n \geqslant (c_0/C)^t$ because $t \geqslant 5n$, we get

$$2\int_{\mathbb{R}^n} \langle x,\xi\rangle_+^t d\mu(x) \geqslant (c_1 m)^t f_\mu(0)|K_{n+1}(\mu)|,$$

where $c_1 > 0$ is an absolute constant. Finally, $f_\mu(0)|K_{n+1}(\mu)| \approx 1$ by (3.9), which implies that

$$h_{Z_t^+(\mu)}(\xi) \geqslant c_2 m = c_0' h_{K_{n+1}(\mu)}(\xi),$$

where $c_0' = c_2 c_0$, and the lemma is proved. □

Proof of Theorem 4.5. Combining Lemma 4.8 and Lemma 4.9 we see that

$$R_{5n}(\mu) \cap Z_{5n}^+(\mu) \supseteq c_1 K_{n+1}(\mu)$$

for some absolute constant $c_1 > 0$. We apply Lemma 4.6 with $t = 5n$ and $\delta = \frac{1}{2}$. For every $x \in \frac{1}{2} Z_{5n}^+(\mu)$ we have

$$\varphi_\mu(x) \geqslant C_1^{-n}$$

for some absolute constant $C_1 > 1$. It follows that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \geqslant C_1^{-n} \mu\left(\tfrac{1}{2} Z_{5n}^+(\mu)\right).$$

Then, by Lemma 4.9 we have $\frac{1}{2} Z_{5n}^+(\mu) \supseteq \frac{c_1}{2} K_{n+1}(\mu)$. Since $\frac{c_1}{2} K_{n+1}(\mu) \subseteq R_{5n}(\mu)$, we know that $f_\mu(x) \geqslant e^{-5n} f_\mu(0)$ for all $x \in \frac{c_1}{2} K_{n+1}(\mu)$. Using also (3.9), we get

$$\begin{aligned} \mu\left(\tfrac{1}{2} Z_{5n}^+(\mu)\right) \geqslant \mu\left(\frac{c_1}{2} K_{n+1}(\mu)\right) &= \int_{\frac{c_1}{2} K_{n+1}(\mu)} f_\mu(x)\, dx \geqslant e^{-5n} f_\mu(0) \left|\frac{c_1}{2} K_{n+1}(\mu)\right| \\ &= e^{-5n} (c_1/2)^n f_\mu(0)|K_{n+1}(\mu)| \geqslant e^{-5n} c_2^n. \end{aligned}$$

Combining the above we conclude that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \geqslant C_1^{-n} e^{-5n} c_2^n \geqslant e^{-cn},$$

for some absolute constant $c > 0$. □

5 Random polytopes and the half-space depth

Let μ be a log-concave probability measure on $\mathbb{R}^n$. For every convex body A in $\mathbb{R}^n$ with $0 \in \mathrm{int}(A)$ we define

$$\varphi_+(A) = \sup_{x \notin A} \varphi_\mu(x) \quad \text{and} \quad \varphi_-(A) = \inf_{x \in A} \varphi_\mu(x).$$

Recall that $B_t(\mu) = \{v \in \mathbb{R}^n : \Lambda_\mu^*(v) \leqslant t\}$, where Λ_μ^* is the Cramer transform of μ.

Let $X_1, X_2, \ldots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ consider the random polytope $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$. A version of the next lemma appeared in Section 2 (see Lemma 2.3).

Lemma 5.1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. For every convex body A in $\mathbb{R}^n$ and every $N > n$ we have that*

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(A) + N\varphi_+(A).$$

In particular, for every $t > 0$,

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(B_t(\mu)) + N\exp(-t).$$

Proof. We write

$$\begin{aligned}\mathbb{E}_{\mu^N}(\mu(K_N)) &= \mathbb{E}_{\mu^N}(\mu(K_N \cap A)) + \mathbb{E}_{\mu^N}(\mu(K_N \setminus A))\\ &\leqslant \mu(A) + \mathbb{E}_{\mu^N}(\mu(K_N \setminus A)).\end{aligned}$$

Observe that if H is a closed half-space containing x, and if $x \in K_N$, then there exists $i \leqslant N$ such that $X_i \in H$ (otherwise we would have $x \in K_N \subseteq H'$, where H' is the complementary half-space). It follows that

$$\mu^N\big(x \in K_N\big) \leqslant N\varphi_\mu(x).$$

Then, Fubini's theorem shows that

$$\mathbb{E}_{\mu^N}(\mu(K_N \setminus A)) = \int_{\mathbb{R}^n\setminus A} \mu^N(x \in K_N)\, d\mu(x) \leqslant \int_{\mathbb{R}^n\setminus A} N\varphi_\mu(x)\, d\mu(x) \leqslant N\varphi_+(A).$$

The last claim follows if we set $A = B_t(\mu)$ because, by Lemma 4.3, $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x)) \leqslant e^{-t}$ for all $x \notin B_t(\mu)$. $\square$

We also need a basic fact that generalizes Lemma 2.4 and plays a main role in the proof of all the upper thresholds that have been obtained so far.

Lemma 5.2. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. For every convex body A in $\mathbb{R}^n$ and every $N > n$ we have that*

$$1 - \mu^N(K_N \supseteq A) \leqslant 2\binom{N}{n}(1 - \varphi_-(A))^{N-n}.$$

Therefore,

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \geqslant \mu(A)\left(1 - 2\binom{N}{n}(1 - \varphi_-(A))^{N-n}\right).$$

Proof. Note that, with probability equal to 1 the random polytope K_N has non-empty interior. For every subset $J = \{j_1, \ldots, j_n\}$ of $\{1, \ldots, N\}$, of cardinality n, note that $X_{j_1}, \ldots, X_{j_n}$ are affinely independent with probability 1, and define the event L_J as follows: for one of the two closed half-spaces H_1, H_2 they determine, say H_i, we have simultaneously $K_N \subset H_i$ and $\mu\big(\mathbb{R}^n \setminus H_i\big) \geqslant \varphi_-(A)$.

If $A \nsubseteq K_N$, then there exists $x \in \partial(A) \setminus K_N$. Since $x \notin K_N$, there exists a facet F of K_N with the following property: one of the two closed half-spaces H_1 and H_2 determined by F contains K_N but does not contain x. Thus, if H_i is this half-space, we have simultaneously $K_N \subset H_i$ and $\mu\big(\mathbb{R}^n \setminus H_i\big) \geqslant \varphi_\mu(x) \geqslant \varphi_-(A)$. Since the hyperplane bounding H_i is determined by some affinely independent vertices $X_{j_1}, \ldots, X_{j_n}$ of K_N which lie in F, this shows that

$$\{A \nsubseteq K_N\} \subseteq \bigcup_J L_J.$$

It follows that

$$\mathrm{Prob}\big(A \nsubseteq K_N\big) \leqslant \sum_J \mathrm{Prob}(L_J) = \binom{N}{n}\mathrm{Prob}(L'),$$

where $L' := L_{\{1,\ldots,n\}}$. It is not hard to see that

$$\mathrm{Prob}(L') \leqslant 2(1 - \varphi_-(A))^{N-n}.$$

Indeed, $X_1, \ldots, X_n$ determine two closed half-spaces $H_i = H_i(X_1, \ldots, X_n)$, $i = 1, 2$. Let L^i be the event that $\mu\big(\mathbb{R}^n \setminus H_i\big) \geqslant \varphi_-(A)$. Then, with Exp denoting expectation with respect to the measure Prob,

$$\begin{aligned}\mathrm{Prob}(L') &\leqslant \sum_{i=1}^{2} \mathrm{Prob}\big(\big\{X_{n+1}, \ldots, X_N \in H_i\big\} \cap L^i\big) \\ &= \sum_{i=1}^{2} \mathrm{Exp}\big(\mathrm{Prob}\big(\big\{X_{n+1}, \ldots, X_N \in H_i\big\} \mid X_1, \ldots, X_n\big)\mathbf{1}_{L^i}\big) \\ &\leqslant (1 - \varphi_-(A))^{N-n} \sum_{i=1}^{2} \mathrm{Prob}(L^i).\end{aligned}$$

The second claim of the lemma follows from Markov's inequality. □

6 Bounds for the expected measure of random polytopes

Let μ be a log-concave probability measure on $\mathbb{R}^n$. Let $X_1, X_2, \ldots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ consider the random polytope $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$. and the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the μ-measure of K_N. Recall that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine transformation and $T_*\mu$ is the push-forward of μ defined by $T_*\mu(A) = \mu(T^{-1}(A))$ then

$$\mathbb{E}_{(T_*\mu)^N}[(T_*\mu)(K_N)] = \mathbb{E}_{\mu^N}[\mu(K_N)].$$

So, we may assume that μ is isotropic.

Theorem 6.1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$. For any $N \leqslant \exp(c_1 n/L_\mu^2)$ we have that*

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant 2\exp\left(-c_2 n/L_\mu^2\right),$$

where $c_1, c_2 > 0$ and $n_0 \in \mathbb{N}$ are absolute constants.

Proof. Using the estimate $\mu(B_t(\mu)) \leqslant \|f_\mu\|_\infty |B_t(\mu)|$, Proposition 3.3 and Theorem 3.1, from Lemma 5.1 we get

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \left(c_1 \|f_\mu\|_\infty^{1/n} \sqrt{t/n}\right)^n + N\exp(-t)$$

for every $N > n$ and $2 \leqslant t \leqslant n$. Recall that μ is isotropic, therefore $\|f_\mu\|_\infty^{2/n} = L_\mu^2 = O(\sqrt{n})$; in fact, Klartag's theorem gives much more. Then, if $n \geqslant n_0$ where $n_0 \in \mathbb{N}$ is an absolute constant, the choice $t := (c_1 e)^{-2} n/\|f_\mu\|_\infty^{2/n}$ satisfies $2 \leqslant t \leqslant n$ and gives

$$\left(c_1 \|f_\mu\|_\infty^{1/n} \sqrt{t/n}\right)^n \leqslant e^{-n}.$$

Then,

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant e^{-n} + N\exp(-c_2 n/\|f_\mu\|_\infty^{2/n}),$$

where $c_2 = (c_1 e)^{-2}$. It follows that if $N \leqslant \exp(c_3 n/\|f_\mu\|_\infty^{2/n})$ where $c_3 = c_2/2$, then we have

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant e^{-n} + \exp(-c_3 n/\|f_\mu\|_\infty^{2/n})$$

and the result follows from the fact that $\|f_\mu\|_\infty^{2/n} = L_\mu^2 \geqslant c$. □

We pass now to the lower threshold. It was proved in [12] that if μ is an even κ-concave measure on $\mathbb{R}^n$ with $0 < \kappa < 1/n$, supported on a convex body K in $\mathbb{R}^n$, if $X_1, X_2, \dots$ are independent random points in $\mathbb{R}^n$ distributed according to μ and $K_N = \mathrm{conv}\{X_1, \dots, X_N\}$ as before, then for any $M \geqslant C$ and any $N \geqslant \exp\left(\frac{1}{\kappa}(\ln n + 2\ln M)\right)$ we have that

$$\frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|} \geqslant 1 - \frac{1}{M}, \tag{6.1}$$

where $C > 0$ is an absolute constant.

Since the family of log-concave probability measures corresponds to the case $\kappa = 0$, it is natural to ask for analogues of this result for 0-concave, i.e. log-concave, probability measures. It is useful to observe that in the case where $X_1, X_2, \dots$ are uniformly distributed in the Euclidean unit ball the sharp threshold for the problem (see [33] and [3]) is

$$\exp\left((1 \pm \varepsilon)\tfrac{1}{2} n \ln n\right), \qquad \varepsilon > 0.$$

We shall establish a weak lower threshold of this order.

Theorem 6.2. *Let $\delta \in (0,1)$. Then,*

$$\inf_{\mu}\left(\inf\left\{\mathbb{E}\left[\mu((1+\delta)K_N)\right] : N \geqslant \exp\left(C\delta^{-1}\ln(2/\delta)\, n \ln n\right)\right\}\right) \longrightarrow 1$$

as $n \to \infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $C > 0$ is an absolute constant.

This is a weak threshold in the sense that we consider the expected measure of $(1+\delta)K_N$ instead of K_N, where $\delta > 0$ is arbitrarily small. The reason for this is the dependence on δ in the next technical proposition.

Proposition 6.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $\delta \in (0,1)$ and any $t \geqslant C_\delta n \ln n$ we have that*

$$\mu((1+\delta)Z_t^+(\mu)) \geqslant 1 - e^{-c_\delta t}$$

where $C_\delta = C\delta^{-1}\ln(2/\delta)$ and $c_\delta = c\delta$ are positive constants depending only on δ.

Proof. Let $\delta \in (0,1)$ and set $\varepsilon = \delta/5$. Fix $t \geqslant n$ which will be determined. Recall that $b_1 B_2^n \subseteq Z_t^+(\mu) \subseteq b_2 t B_2^n$ for some absolute constants $b_1, b_2 > 0$. This implies that if $v, w \in S^{n-1}$ and $|v - w| \leqslant \frac{b_1 \varepsilon}{b_2 t}$ then

$$h_{Z_t^+(\mu)}(v-w) \leqslant b_2 t|v-w| \quad \text{and} \quad b_1 \leqslant \min\{h_{Z_t^+(\mu)}(v), h_{Z_t^+(\mu)}(w)\},$$

therefore

$$h_{Z_t^+(\mu)}(v-w) \leqslant b_2 t|v-w| \leqslant \varepsilon \min\{h_{Z_t^+(\mu)}(v), h_{Z_t^+(\mu)}(w)\}. \tag{6.2}$$

Set $b := b_2/b_1$ and consider a $\frac{\varepsilon}{bt}$-net N of the Euclidean unit sphere S^{n-1} with cardinality $|N| \leqslant (1+2bt/\varepsilon)^n \leqslant (3bt/\varepsilon)^n$. We define

$$W = \bigcap_{\xi \in N}\left\{x : \langle x, \xi\rangle_+ \leqslant \frac{1}{1+\varepsilon} h_{Z_t^+(\mu)}(\xi)\right\}.$$

Let $x \in W$. Then, $\langle x, \xi\rangle_+ \leqslant \frac{1}{1+\varepsilon} h_{Z_t^+(\mu)}(\xi)$ for all $\xi \in N$. We will show that $(1-\varepsilon)\langle x, w\rangle_+ \leqslant h_{Z_t^+(\mu)}(w)$ for all $w \in S^{n-1}$, which is equivalent to $(1-\varepsilon)x \in Z_t^+(\mu)$. We set

$$\alpha_\mu(x) := \max\left\{\frac{\langle x, w\rangle_+}{h_{Z_t^+(\mu)}(w)} : w \in S^{n-1}\right\}$$

and consider $v \in S^{n-1}$ such that $\langle x, v\rangle_+ = \alpha_\mu(x) \cdot h_{Z_t^+(\mu)}(v)$. There exists $\xi \in N$ such that $|\xi - v| \leqslant \frac{\varepsilon}{bt}$. Using the fact that $\langle x, v-\xi\rangle_+ \leqslant \alpha_\mu(x) h_{Z_t^+(\mu)}(v-\xi)$, we write

$$\langle x, v\rangle_+ \leqslant \langle x, \xi\rangle_+ + \langle x, v-\xi\rangle_+ \leqslant \frac{1}{1+\varepsilon} h_{Z_t^+(\mu)}(\xi) + \alpha_\mu(x) h_{Z_t^+(\mu)}(v-\xi).$$

From (6.2) it follows that

$$\langle x, v\rangle_+ \leqslant \frac{1}{1+\varepsilon} h_{Z_t^+(\mu)}(\xi) + \varepsilon\alpha_\mu(x) h_{Z_t^+(\mu)}(v) = \frac{1}{1+\varepsilon} h_{Z_t^+(\mu)}(\xi) + \varepsilon\langle x, v\rangle_+,$$

which gives

$$\langle x, v\rangle_+ \leqslant \frac{1}{1-\varepsilon^2} h_{Z_t^+(\mu)}(\xi).$$

Moreover,

$$h_{Z_t^+(\mu)}(\xi) \leqslant h_{Z_t^+(\mu)}(v) + h_{Z_t^+(\mu)}(\xi - v) \leqslant h_{Z_t^+(\mu)}(v) + \varepsilon h_{Z_t^+(\mu)}(v) = (1+\varepsilon) h_{Z_t^+(\mu)}(v),$$

which finally gives $\alpha_\mu(x) \leqslant 1/(1-\varepsilon)$. This shows that $(1-\varepsilon)W \subseteq Z_t^+(\mu)$. For every $\xi \in N$ we have

$$\mu(\{x : \langle x, \xi\rangle_+ \geqslant (1+\varepsilon)\|\langle\cdot, \xi\rangle_+\|_t\}) \leqslant (1+\varepsilon)^{-t}.$$

Since $\delta \in (0,1)$ we have $0 < \varepsilon < 1/5$, therefore $\frac{(1+\varepsilon)^2}{1-\varepsilon} \leqslant 1 + 5\varepsilon = 1 + \delta$. Then,

$$\begin{aligned}
\mu((1+\delta)Z_t^+(\mu)) &\geqslant \mu\left(\frac{(1+\varepsilon)^2}{1-\varepsilon} Z_t^+(\mu)\right) \geqslant \mu((1+\varepsilon)^2 W) \\
&= \mu\left(\bigcap_{\xi\in N} \left\{x : \langle x, \xi\rangle_+ \leqslant (1+\varepsilon) h_{Z_t^+(\mu)}(\xi)\right\}\right) \\
&\geqslant 1 - |N| \cdot (1+\varepsilon)^{-t} \geqslant 1 - (C'_\varepsilon t)^n (1+\varepsilon)^{-t},
\end{aligned}$$

where $C'_\varepsilon = 3b/\varepsilon$. It follows that there exists $C_\varepsilon > 1$ such that if $t \geqslant C_\varepsilon n \ln n$ then

$$(C'_\varepsilon t)^n (1+\varepsilon)^{-t} \leqslant (1+\varepsilon)^{-t/2} \leqslant e^{-\varepsilon t/4}. \tag{6.3}$$

To see this, consider the function

$$\ell(t) = \frac{t}{2}\ln(1+\varepsilon) - n\ln(3bt/\varepsilon).$$

It is easily checked that ℓ is increasing on $[2n/\ln(1+\varepsilon), \infty)$. Therefore, if $t \geqslant C_\varepsilon n \ln n$ where $C_\varepsilon = \frac{C}{\varepsilon}\ln\left(\frac{2}{\varepsilon}\right)$ for a large enough absolute constant $C > 0$, one can check that $\ell(t) \geqslant \ell(C_\varepsilon n \ln n) > 0$. This implies (6.3). Since $\varepsilon = \delta/5$, we obtain the assertion of the proposition with the stated dependence of the constants C_δ, c_δ on δ. $\square$

Proof of Theorem 6.2. Let $0 < \delta < 1$ and set $\varepsilon = \delta/3$. From Lemma 4.6 we know that for every $x \in (1-\varepsilon)Z_t^+(\mu)$ we have

$$\varphi_\mu(x) \geqslant \frac{(1-(1-\varepsilon)^t)^2}{C_1^t},$$

where $C_1 > 1$ is an absolute constant. Then, taking into account the fact that $1 - \varepsilon > 2/3$, we get

$$\mu^N\Big(K_N \supseteq (1-\varepsilon)Z_t^+(\mu)\Big) \geqslant 1 - 2\binom{N}{n}\left[1 - \frac{(1-(1-\varepsilon)^t)^2}{C_1^t}\right]^{N-n}.$$

By the mean value theorem we have $1-(1-\varepsilon)^t = t\varepsilon z^{t-1}$ for some $z\in(1-\varepsilon,1)$, and hence $1-(1-\varepsilon)^t \geqslant t\varepsilon(1-\varepsilon)^{t-1}$. Taking also into account the fact that $1-\varepsilon > 2/3$, we get

$$\begin{aligned}\mu^N\Big(K_N \supseteq (1-\varepsilon)Z_t^+(\mu)\Big) &\geqslant 1-2\binom{N}{n}\left[1-\frac{(t\varepsilon(1-\varepsilon)^{t-1})^2}{C_1^t}\right]^{N-n}\\ &\geqslant 1-\left(\frac{2eN}{n}\right)^n \exp\left(-(N-n)\frac{(t\varepsilon)^2}{(3C_1)^t}\right).\end{aligned}$$

This last quantity tends to 1 as $n\to\infty$ if

$$(3C_1)^t n\ln(4eN/n) < (N-n)(t\varepsilon)^2, \tag{6.4}$$

and assuming that $\delta\in(1/n^2,1)$ and $t\geqslant C_\varepsilon n\ln n$ where C_ε is the constant from Proposition 6.3, we check that (6.4) holds true if $N\geqslant \exp(C_2 t)$ for a large enough absolute constant $C_2>0$.

Note that $\varepsilon=\delta/3$ implies that $1+\delta > \frac{1+\varepsilon}{1-\varepsilon}$. Then, if $N\geqslant \exp(C_2C_\varepsilon n\ln n)$ we see that

$$\begin{aligned}\mathbb{E}\left[\mu\left((1+\delta)K_N\right)\right] &\geqslant \mathbb{E}\left[\mu\left(\frac{1+\varepsilon}{1-\varepsilon}K_N\right)\right] \geqslant \mu((1+\varepsilon)Z_t^+(\mu))\times \mu^N\Big(K_N\supseteq(1-\varepsilon)Z_t^+(\mu)\Big)\\ &\geqslant \left(1-e^{-c\varepsilon t}\right)\left[1-\left(\frac{2eN}{n}\right)^n\exp\left(-(N-n)\frac{(t\varepsilon)^2}{(3C_1)^t}\right)\right] \longrightarrow 1\end{aligned}$$

as $n\to\infty$. □

We have already mentioned that Theorem 6.2 provides a weak threshold in the sense that we estimate the expectation $\mathbb{E}_{\mu^N}\big(\mu(1+\delta)K_N\big)$ (for an arbitrarily small but positive value of δ) while the original question is about $\mathbb{E}_{\mu^N}\big(\mu(K_N)\big)$. The next result provides an estimate where "δ is removed", however the dependence on n is worse.

Theorem 6.4. *There exists an absolute constant $C>0$ such that*

$$\inf_\mu\Big(\inf\Big\{\mathbb{E}\big[\mu(K_N)\big] : N\geqslant \exp(C(n\ln n)^2u(n))\Big\}\Big) \longrightarrow 1$$

as $n\to\infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $u(n)$ is any function with $u(n)\to\infty$ as $n\to\infty$.

Proof. Let $\delta_n\in(0,1/2)$ that will be suitably chosen. Our starting observation is the inclusion

$$(1-\delta_n)Z_t^+(\mu)\supseteq(1-2\delta_n)\big((1+\delta_n/4)Z_t^+(\mu)\big)+(2\delta_n)\tfrac14 Z_t^+(\mu),$$

which holds true for any $t\geqslant 1$. Since μ is log-concave, we get

$$\mu((1-\delta_n)Z_t^+(\mu))\geqslant\Big(\mu((1+\delta_n/4)Z_t^+(\mu))\Big)^{1-2\delta_n}\left(\mu\left(\tfrac14 Z_t^+(\mu)\right)\right)^{2\delta_n}. \tag{6.5}$$

Next, we make the additional assumptions that $\delta_n\in(1/n^2,1/2)$ and $t\geqslant C_{\delta_n}n\ln n$, where $C_{\delta_n}=C\frac{1}{\delta_n}\ln\left(\frac{2}{\delta_n}\right)$ is the constant from Proposition 6.3. This implies that

$$\Big(\mu((1+\delta_n/4)Z_t^+(\mu))\Big)^{1-2\delta_n}\geqslant(1-e^{-c\delta_n t})^{1-2\delta_n}\geqslant 1-e^{-c\delta_n t}. \tag{6.6}$$

On the other hand, as in the proof of Theorem 4.5 we see that (by Lemma 4.8 and Lemma 4.9)

$$R_{5n}(\mu)\cap Z_t^+(\mu)\supseteq c_1K_{n+1}(\mu)$$

for some absolute constant $c_1 > 0$. Therefore,

$$\mu\left(\tfrac{1}{4}Z_t^+(\mu)\right) \geqslant \mu\Big(\frac{c_1}{4}K_{n+1}(\mu)\Big) = \int_{\frac{c_1}{4}K_{n+1}(\mu)} f_\mu(x)\,dx \geqslant e^{-5n} f_\mu(0)\Big|\frac{c_1}{4}K_{n+1}(\mu)\Big|$$
$$= e^{-5n}(c_1/4)^n f_\mu(0)|K_{n+1}(\mu)| \geqslant e^{-c_2 n}$$

for an absolute constant $c_2 > 0$. This gives

$$\left(\mu\left(\tfrac{1}{4}Z_t^+(\mu)\right)\right)^{2\delta_n} \geqslant e^{-2c_2\delta_n n} \longrightarrow 1 \tag{6.7}$$

if $\delta_n n = o_n(1)$ as $n \to \infty$. Inserting the estimates (6.6) and (6.7) into (6.5) we get

$$\mu((1-\delta_n)Z_t^+(\mu)) = 1 - o_n(1) \tag{6.8}$$

if $\delta_n n = o_n(1)$ and $t \geqslant C\delta_n^{-1}\ln(2/\delta_n)n\ln n$.

Now, repeating the argument of the proof of Theorem 6.2 we see that

$$\mu^N\Big(K_N \supseteq (1-\delta_n)Z_t^+(\mu)\Big) = 1 - o_n(1)$$

provided that $N \geqslant \exp\left(C_2\delta_n^{-1}\ln(2/\delta_n)n\ln n\right)$, and then

$$\mathbb{E}\left[\mu\left(K_N\right)\right] \geqslant \mu((1-\delta_n)Z_t^+(\mu)) \times \mu^N\Big(K_N \supseteq (1-\delta_n)Z_t^+(\mu)\Big) \longrightarrow 1$$

as $n \to \infty$, provided that we choose $\delta_n = o_n(1/n)$ and $N \geqslant \exp\left(C_2\delta_n^{-1}\ln(2/\delta_n)n\ln n\right)$. This proves the theorem. $\square$

7 Comparing half-space depth with the Cramer transform

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ with density $f := f_\mu$. Recall that the logarithmic Laplace transform of μ on $\mathbb{R}^n$, defined by

$$\Lambda_\mu(\xi) = \ln\Big(\int_{\mathbb{R}^n} e^{\langle\xi,z\rangle} f(z)dz\Big),$$

is a non-negative convex function with $\Lambda_\mu(0) = 0$. Moreover, the set $A(\mu) = \{\Lambda_\mu < \infty\}$ is open and Λ_μ is C^∞ and strictly convex on $A(\mu)$.

Recall also that the Cramer transform of μ is the function

$$\Lambda_\mu^*(x) = \sup_{\xi\in\mathbb{R}^n}\left\{\langle x,\xi\rangle - \Lambda_\mu(\xi)\right\}.$$

For every $t > 0$ we consider the convex set

$$B_t(\mu) := \{x \in \mathbb{R}^n : \Lambda_\mu^*(x) \leqslant t\}.$$

and for any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x and consider Tukey's half-space depth

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}.$$

In Lemma 4.3 we showed that for every $x \in \mathbb{R}^n$ we have

$$\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x)).$$

In particular, for any $t > 0$ and for all $x \notin B_t(\mu)$ we have that $\varphi_\mu(x) \leqslant \exp(-t)$. In other words,

$$\varphi_+(B_t(\mu)) \leqslant e^{-t}.$$

Next, we would like to obtain a lower bound for $\varphi_-(B_t(\mu))$, or equivalently for $\varphi_\mu(x)$ when $x \in B_t(\mu)$. In the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$, our estimate is the following.

Theorem 7.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t>0$ we have that*

$$\inf\{\varphi_{\mu_K}(x) : x\in B_t(\mu_K)\}\geqslant \frac{1}{10}\exp(-t-2\sqrt{n}).$$

The first part of the argument works for any centered log-concave probability measure μ with density f on $\mathbb{R}^n$. For every $\xi\in\mathbb{R}^n$ we define the probability measure μ_ξ with density

$$f_\xi(z)=e^{-\Lambda_\mu(\xi)+\langle\xi,z\rangle}f(z).$$

In the next lemma (see [11, Proposition 7.2.1]) we recall some basic facts for μ_ξ.

Lemma 7.2. *The barycenter of μ_ξ is $x=\nabla\Lambda_\mu(\xi)$ and $\mathrm{Cov}(\mu_\xi)=\mathrm{Hess}\,(\Lambda_\mu)(\xi)$.*

Next, we set

$$\sigma_\xi^2=\int_{\mathbb{R}^n}\langle z-x,\xi\rangle^2 d\mu_\xi(z).$$

Let $t>0$. Since $B_t(\mu)$ is convex, in order to give a lower bound for $\inf\{\varphi_\mu(x):x\in B_t(\mu)\}$ it suffices to give a lower bound for $\mu(H)$, where H is any closed half-space whose bounding hyperplane supports $B_t(\mu)$. In that case,

$$\mu(H)=\mu(\{z:\langle z-x,\xi\rangle\geqslant 0\}) \tag{7.1}$$

for some $x\in\partial(B_t(\mu))$, with $\xi=\nabla\Lambda_\mu^*(x)$, or equivalently $x=\nabla\Lambda_\mu(\xi)$ (see e.g. Theorem 23.5 and Corollary 23.5.1 in [34]). Note that

$$\begin{aligned}\mu(\{z:\langle z-x,\xi\rangle\geqslant 0\}) &= \int_{\mathbb{R}^n}\mathbf{1}_{[0,\infty)}(\langle z-x,\xi\rangle)f(z)\,dz\\ &= e^{\Lambda_\mu(\xi)}\int_{\mathbb{R}^n}\mathbf{1}_{[0,\infty)}(\langle z-x,\xi\rangle)e^{-\langle z,\xi\rangle}\,d\mu_\xi(z)\\ &= e^{\Lambda_\mu(\xi)}e^{-\langle x,\xi\rangle}\int_{\mathbb{R}^n}\mathbf{1}_{[0,\infty)}(\langle z-x,\xi\rangle)e^{-\langle z-x,\xi\rangle}\,d\mu_\xi(z)\\ &\geqslant e^{-\Lambda_\mu^*(x)}\int_0^\infty \sigma_\xi e^{-\sigma_\xi t}\mu_\xi(\{z:0\leqslant\langle z-x,\xi\rangle\leqslant\sigma_\xi t\})\,dt.\end{aligned} \tag{7.2}$$

From Markov's inequality we see that

$$\mu_\xi(\{z:\langle z-x,\xi\rangle\geqslant 2\sigma_\xi\})\leqslant\frac{1}{4}.$$

Moreover, since x is the barycenter of μ_ξ, Grünbaum's lemma (see [11, Lemma 2.2.6]) implies that

$$\mu_\xi(\{z:\langle z-x,\xi\rangle\geqslant 0\})\geqslant\frac{1}{e}.$$

Therefore,

$$\int_0^\infty\sigma_\xi e^{-\sigma_\xi t}\mu_\xi(\{z:0\leqslant\langle z-x,\xi\rangle\leqslant\sigma_\xi t\})\,dt\geqslant\int_2^\infty\sigma_\xi e^{-\sigma_\xi t}\left(\frac{1}{e}-\frac{1}{4}\right)dt\geqslant\frac{4-e}{4e}e^{-2\sigma_\xi}. \tag{7.3}$$

We would like now an upper bound for $\sup_\xi\sigma_\xi$. We can have this when $\mu=\mu_K$ is the uniform measure on a centered convex body K of volume 1 on $\mathbb{R}^n$, using a theorem of Nguyen [30] (proved independently by Wang [39]; see also [17]).

Theorem 7.3. *Let ν be a log-concave probability measure on $\mathbb{R}^n$ with density $g=\exp(-p)$, where p is a convex function. Then,*

$$\mathrm{Var}_\nu(p)\leqslant n.$$

Proof of Theorem 7.1. Set $\mu := \mu_K$. Since $f(z) = \mathbf{1}_K(z)$, the density f_ξ of μ_ξ is proportional to $e^{\langle \xi, z\rangle}\mathbf{1}_K(z)$. Using the fact that

$$\mathbb{E}_{\mu_\xi}(\langle \xi, z\rangle) = \langle \nabla\Lambda_\mu(\xi), \xi\rangle = \langle x, \xi\rangle,$$

from Theorem 7.3 we get that

$$\sigma_\xi^2 = \mathbb{E}_{\mu_\xi}(\langle z - x, \xi\rangle)^2 = \mathrm{Var}_{\mu_\xi}(\langle \xi, z\rangle) \leqslant n.$$

Then, combining (7.1), (7.2) and (7.3), for any bounding hyperplane H of $B_t(\mu)$ we have

$$\begin{aligned}\mu(H) &\geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t}\mu_\xi(0 \leqslant \langle z - x, \xi\rangle \leqslant \sigma_\xi t)\, dt \\ &\geqslant \frac{4-e}{4e} e^{-\Lambda_\mu^*(x) - 2\sigma_\xi} \geqslant \frac{1}{10}\exp(-t - 2\sqrt{n}),\end{aligned}$$

as claimed. □

Theorem 7.1 shows that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$10\varphi_{\mu_K}(x) \geqslant \exp(-\Lambda_{\mu_K}^*(x) - 2\sqrt{n})$$

for all $x \in \mathbb{R}^n$. Setting

$$\omega_{\mu_K}(x) = \ln\left(\frac{1}{\varphi_{\mu_K}(x)}\right) \tag{7.4}$$

and taking into account Lemma 4.3 we have the next two-sided estimate.

Corollary 7.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $x \in \mathrm{int}(K)$ we have that*

$$\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda_{\mu_K}^*(x) \leqslant \omega_{\mu_K}(x). \tag{7.5}$$

Note. A basic question that arises from the results of this section is whether an analogue of (7.5) holds true for any centered log-concave probability measure μ on $\mathbb{R}^n$. This would allow us to apply the next steps of the procedure that our approach suggests to all log-concave probability measures.

8 Moments of the Cramer transform

Our approach to the threshold problem requires to know that, given our centered log-concave probability measure μ on $\mathbb{R}^n$, the Cramer transform Λ_μ^* has finite variance. Our first result provides an affirmative answer in the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$. In fact, the next theorem guarantees that, in a more general case, Λ_μ^* has finite moments of all orders.

Theorem 8.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,*

$$\int_{\mathbb{R}^n} e^{\frac{\kappa\Lambda_\mu^*(x)}{2}}\, d\mu(x) < \infty.$$

In particular, for all $p \geqslant 1$ we have that $\mathbb{E}_\mu\big((\Lambda_\mu^*(x))^p\big) < \infty$.

The proof of Theorem 8.1 is based on the next lemma, which is proved in [12, Lemma 7] in the symmetric case.

Lemma 8.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0,1/n]$ and let μ be a centered κ-concave probability measure with* $\mathrm{supp}(\mu) = K$. *Then,*

$$\varphi_\mu(x) \geqslant e^{-2}\kappa(1-\|x\|_K)^{1/\kappa} \tag{8.1}$$

for every $x \in K$, where $\|x\|_K$ is the Minkowski functional of K.

Sketch of the proof. We modify the argument from [12, Lemma 7] to cover the not necessarily symmetric case. First, consider the case $0 < \kappa < 1/n$. Let X be a random vector distributed according to μ. Given $\theta \in S^{n-1}$ let $b = h_K(\theta)$ and $a = h_K(-\theta)$. If g_θ is the density of $\langle X, \theta\rangle$ then $g_\theta^{\frac{\kappa}{1-\kappa}}$ is concave on $[-a,b]$, therefore

$$g_\theta(t) \geqslant g_\theta(0)\left(1-\frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}}$$

for all $t \in [0,b]$. It follows that, for every $0 < s < b$,

$$\mathbb{P}(\langle X,\theta\rangle \geqslant s) = \int_s^b g_\theta(t)\,dt \geqslant g_\theta(0)\int_s^b \left(1-\frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}}\,dt = \kappa g_\theta(0) b\left(1-\frac{s}{b}\right)^{\frac{1}{\kappa}}.$$

Note that g_θ is a centered log-concave density. Therefore, $g_\theta(0) \geqslant e^{-1}\|g_\theta\|_\infty$ by (3.2) and $\|g_\theta\|_\infty b \geqslant \mathbb{P}(\langle X,\theta\rangle \geqslant 0) \geqslant e^{-1}$ by Grünbaum's lemma [11, Lemma 2.2.6], which implies that $g_\theta(0)b \geqslant e^{-2}$. It follows that

$$\mathbb{P}(\langle X,\theta\rangle \geqslant s) = \int_s^b g_\theta(t)\,dt \geqslant e^{-2}\kappa\left(1-\frac{s}{b}\right)^{\frac{1}{\kappa}}.$$

Now, let $x \in K$. Then $\langle x,\theta\rangle \leqslant \|x\|_K h_K(\theta) = \|x\|_K b$, therefore

$$\mathbb{P}(\langle X,\theta\rangle \geqslant \langle x,\theta\rangle) \geqslant \mathbb{P}(\langle X,\theta\rangle \geqslant \|x\|_K b) \geqslant e^{-2}\kappa\left(1-\|x\|_K\right)^{\frac{1}{\kappa}}.$$

For the case $\kappa = 1/n$ recall that a $1/n$-concave measure is κ-concave for every $\kappa \in (0,1/n)$. This means that (8.1) holds true for all $\kappa \in (0,1/n)$ and letting $\kappa \to 1/n$ we obtain the result. □

Proof of Theorem 8.1. From Lemma 4.3 we know that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$, or equivalently,

$$e^{\frac{\kappa\Lambda_\mu^*(x)}{2}} \leqslant \frac{1}{\varphi_\mu(x)^{\kappa/2}}$$

for all $x \in K$. From Lemma 8.2 we know that

$$\varphi_\mu(x) \geqslant e^{-2}\kappa(1-\|x\|_K)^{1/\kappa}$$

for every $x \in K$. It follows that

$$\int_K e^{\frac{\kappa\Lambda_\mu^*(x)}{2}}\,d\mu(x) \leqslant (e^2/\kappa)^{\kappa/2}\int_K \frac{1}{(1-\|x\|_K)^{1/2}}\,d\mu(x).$$

Recall that the cone probability measure ν_K on the boundary $\partial(K)$ of a convex body K with $0 \in \mathrm{int}(K)$ is defined by

$$\nu_K(B) = \frac{|\{rx : x \in B, 0 \leqslant r \leqslant 1\}|}{|K|}$$

for all Borel subsets B of $\partial(K)$. We shall use the identity

$$\int_{\mathbb{R}^n} g(x)\,dx = n|K|\int_0^\infty r^{n-1}\int_{\partial(K)} g(rx)\,d\nu_K(x)\,dr$$

which holds for every integrable function $g:\mathbb{R}^n\to\mathbb{R}$ (see [29, Proposition 1]). Let f denote the density of μ on K. We write

$$\begin{aligned}\int_K \frac{1}{(1-\|x\|_K)^{1/2}}d\mu(x) &= \int_{\mathbb{R}^n}\frac{f(x)}{(1-\|x\|_K)^{1/2}}\mathbf{1}_K(x)\,dx\\ &= n|K|\int_0^\infty r^{n-1}\int_{\partial(K)}\frac{f(ry)}{(1-\|ry\|_K)^{1/2}}\mathbf{1}_K(ry)\,d\nu_K(y)\,dr\\ &= n|K|\int_0^1\frac{r^{n-1}}{\sqrt{1-r}}\int_{\partial(K)}f(ry)\,d\nu_K(y)\,dr\\ &\leqslant n|K|\|f\|_\infty\int_0^1\frac{r^{n-1}}{\sqrt{1-r}}\,dr = n|K|B(n,1/2)\|f\|_\infty\leqslant c\sqrt{n}\|f\|_\infty<+\infty,\end{aligned}$$

and the proof is complete. $\square$

In the case of the uniform measure $\mu=\mu_K$ on a centered convex body K of volume 1 in $\mathbb{R}^n$ we see that

$$\int_K\left(\Lambda^*_{\mu_K}(x)/2n\right)^p dx\leqslant (c_1p)^p\int_K e^{\frac{\Lambda^*_{\mu_K}(x)}{2n}}dx\leqslant (c_2p)^p\sqrt{n},$$

where $c_1,c_2>0$ are absolute constants. This gives the following estimate for the moments of $\Lambda^*_{\mu_K}$:

$$\|\Lambda^*_{\mu_K}\|_{L^p(\mu_K)}\leqslant cpn^{1+\frac{1}{2p}}$$

for all $p\geqslant 1$. However, essentially repeating the argument that we used for Theorem 8.1 we may obtain sharp estimates in the most interesting case $p=1$ or 2. We need the next lemma.

Lemma 8.3. *Let $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$. Then,*

$$\int_0^1 r^{n-1}\ln(1-r)\,dr=-\frac{1}{n}H_n$$

and

$$\int_0^1 r^{n-1}\ln^2(1-r)\,dr=\frac{1}{n}H_n^2+\frac{1}{n}\sum_{k=1}^n\frac{1}{k^2}.$$

Theorem 8.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n\geqslant 2$. Let $\kappa\in(0,1/n]$ and let μ be a centered κ-concave probability measure with* $\mathrm{supp}(\mu)=K$*. Then,*

$$\mathbb{E}_\mu(\Lambda^*_\mu)\leqslant\left(\mathbb{E}_\mu[(\Lambda^*_\mu)^2]\right)^{1/2}\leqslant\frac{c\ln n}{\kappa}\|f\|_\infty^{1/2},$$

where $c>0$ is an absolute constant and f is the density of μ.

Proof. Following the proof of Theorem 8.1 we write

$$\int_K\left(\Lambda^*_\mu(x)\right)^2 d\mu(x)\leqslant\int_K\ln^2\left(\frac{e^2}{\kappa}\frac{1}{(1-\|x\|_K)^{1/\kappa}}\right)d\mu(x).$$

If f is the density of μ on K and ν_K is the cone measure of K, using the inequality $\ln^2(ab)\leqslant 2(\ln^2 a+\ln^2 b)$

where $a,b>0$, we may write

$$\begin{aligned}\frac{1}{2}\int_K \ln^2\left(\frac{e^2}{\kappa}\frac{1}{(1-\|x\|_K)^{1/\kappa}}\right)d\mu(x) - \ln^2\left(\frac{e^2}{\kappa}\right)\\
&\leqslant \int_{\mathbb{R}^n} f(x)\ln^2\left(\frac{1}{(1-\|x\|_K)^{1/\kappa}}\right)\mathbf{1}_K(x)\,dx\\
&= n|K|\int_0^\infty r^{n-1}\int_{\partial(K)} f(ry)\ln^2\left(\frac{1}{(1-\|ry\|_K)^{1/\kappa}}\right)\mathbf{1}_K(ry)\,d\nu_K(y)\,dr\\
&= \frac{n}{\kappa^2}\int_0^1 r^{n-1}\ln^2(1-r)\int_{\partial(K)} f(ry)\,d\nu_K(y)\,dr\\
&\leqslant \frac{n}{\kappa^2}\|f\|_\infty\int_0^1 r^{n-1}\ln^2(1-r)\,dr.\end{aligned}$$

Since $1\leqslant\int_K f(x)\,dx\leqslant\|f\|_\infty$, using also Lemma 8.3 we get

$$\begin{aligned}\int_K \left(\Lambda_\mu^*(x)\right)^2 d\mu(x) &\leqslant \frac{2n}{\kappa^2}\left(\frac{1}{n}H_n^2+\frac{1}{n}\sum_{k=1}^n\frac{1}{k^2}\right)\|f\|_\infty + 2\ln^2\left(\frac{e^2}{\kappa}\right)\\
&\leqslant \left(\frac{2H_n^2}{\kappa^2}+2\sum_{k=1}^\infty\frac{1}{k^2}+2\ln^2(e^2/\kappa)\right)\|f\|_\infty \leqslant \frac{c_1\ln^2 n}{\kappa^2}\|f\|_\infty,\end{aligned}$$

where $c_1>0$ is an absolute constant. This completes the proof. □

In particular, if we assume that $\mu=\mu_K$ is the uniform measure on a centered convex body then we obtain a sharp two sided estimate.

Theorem 8.5. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n\geqslant 2$. Then,*

$$c_1 n/L_{\mu_K}^2 \leqslant \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)} \leqslant \|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)} \leqslant c_2 n\ln n,$$

where L_{μ_K} is the isotropic constant of the uniform measure μ_K on K and $c_1,c_2>0$ are absolute constants.

The left-hand side inequality of Theorem 8.5 follows easily from the proof Theorem 4.1 (see Lemma 9.1 in the next section). Both the lower and the upper bound are of optimal order with respect to the dimension. This can be seen e.g. from the example of the uniform measure on the cube or the Euclidean ball (see Section 9), respectively.

The next result concerns the one-dimensional case. Let μ be a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure and consider a random variable X, on some probability space $(\Omega,\mathcal{F},P)$, with distribution μ, i.e., $\mu(B):=P(X\in B)$, $B\in\mathcal{B}(\mathbb{R})$. We define

$$\alpha_+=\alpha_+(\mu):=\sup\{x\in\mathbb{R}\colon \mu([x,\infty))>0\}) \quad\text{and}\quad \alpha_-=\alpha_-(\mu):=\sup\{x\in\mathbb{R}\colon \mu((-\infty,-x]))>0\}).$$

Thus, $-\alpha_-,\alpha_+$ are the endpoints of the support of μ. Note that we may have $\alpha_\pm=+\infty$. We define $I_\mu=(-\alpha_-,\alpha_+)$. Recall that

$$\Lambda_\mu^*(x):=\sup\{tx-\Lambda_\mu(t)\colon t\in\mathbb{R}\},\qquad x\in\mathbb{R}.$$

In fact, since $tx-\Lambda_\mu(t)<0$ for $t<0$ when $x\in[0,\alpha_+)$, we have that $\Lambda_\mu^*(x)=\sup\{tx-\Lambda_\mu(t)\colon t\geqslant 0\}$ in this case, and similarly $\Lambda_\mu^*(x):=\sup\{tx-\Lambda_\mu(t)\colon t\leqslant 0\}$ when $x\in(-\alpha_-,0]$. One can also check that $\Lambda_\mu^*(\alpha_\pm)=+\infty$. See [19, Lemma 2.8] for the case $\alpha_\pm<+\infty$. In the case $\alpha_\pm=\pm\infty$, the convexity and monotonicity properties of Λ_μ^* imply again that $\lim\limits_{t\to\pm\infty}\Lambda_\mu^*(t)=+\infty$.

Proposition 8.6. *Let μ be a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure. Then,*

$$\int_{I_\mu} e^{\Lambda_\mu^*(x)/2} d\mu(x) \leqslant 4,$$

where $I_\mu = \mathrm{supp}(\mu)$. In particular, for all $p \geqslant 1$ we have that

$$\int_{I_\mu} (\Lambda_\mu^*(x))^p \, d\mu(x) < +\infty.$$

Proof. Let $F(x) = \mu(-\infty, x]$. For any $x \in [0, \alpha_+)$ and $t \geqslant 0$ we have

$$\min\{F(x), 1 - F(x)\} = \varphi_\mu(x) \leqslant e^{-\Lambda_\mu^*(x)}.$$

It follows that

$$\begin{aligned} \int_{I_\mu} e^{\Lambda_\mu^*(x)/2} d\mu(x) &\leqslant \int_{I_\mu} \frac{1}{\sqrt{\min\{F(x), 1 - F(x)\}}} f(x) \, dx \\ &\leqslant \int_{I_\mu} \frac{1}{\sqrt{F(x)}} f(x) \, dx + \int_{I_\mu} \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx. \end{aligned} \tag{8.2}$$

Write f for the density of μ with respect to Lebesgue measure. Then, $(1 - F)'(x) = -f(x)$, which implies that

$$\int_0^{\alpha_+} \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx \leqslant -\int_0^{\alpha_+} \frac{1}{\sqrt{1 - F(x)}} (1 - F)'(x) dx = -2\sqrt{1 - F(x)}\Big|_0^{\alpha_+} = 2\sqrt{1 - F(0)}$$

since $F(\alpha_+) = 1$. In the same way we check that

$$\int_{-\alpha_-}^0 \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx \leqslant -\int_{-\alpha_-}^0 \frac{1}{\sqrt{1 - F(x)}} (1 - F)'(x) dx = -2\sqrt{1 - F(x)}\Big|_{-\alpha_-}^0 = 2 - 2\sqrt{1 - F(0)}.$$

This shows that

$$\int_{I_\mu} \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx \leqslant 2.$$

In a similar way we obtain the same upper bound for the second summand in (8.2) and the result follows. □

Proposition 8.6 can be extended to products. Let μ_i, $1 \leqslant i \leqslant n$ be centered probability measures on $\mathbb{R}$, all of them absolutely continuous with respect to Lebesgue measure. If $\overline{\mu} = \mu_1 \otimes \cdots \otimes \mu_n$ then $I_{\overline{\mu}} = \prod_{i=1}^n I_{\mu_i}$ and we can easily check that

$$\Lambda_{\overline{\mu}}^*(x) = \sum_{i=1}^n \Lambda_{\mu_i}^*(x_i)$$

for all $x = (x_1, \ldots, x_n) \in I_{\overline{\mu}}$, which implies that

$$\int_{I_{\overline{\mu}}} e^{\Lambda_{\overline{\mu}}^*(x)/2} d\overline{\mu}(x) = \prod_{i=1}^n \left(\int_{I_{\mu_i}} e^{\Lambda_{\mu_i}^*(x_i)/2} d\mu_i(x_i) \right) \leqslant 4^n.$$

In particular, for all $p \geqslant 1$ we have that

$$\int_{I_{\overline{\mu}}} (\Lambda_{\overline{\mu}}^*(x))^p \, d\overline{\mu}(x) < +\infty.$$

We close this section with one more case where we can establish that Λ_μ^* has finite moments of all orders. We consider an arbitrary centered log-concave probability measure on $\mathbb{R}^n$ but we have to impose some conditions on the growth of its one-sided L_t-centroid bodies $Z_t^+(\mu)$. Recall that for every $t \geqslant 1$, the one-sided L_t-centroid body $Z_t^+(\mu)$ of μ is the convex body with support function

$$h_{Z_t^+(\mu)}(y) = \left(2\int_{\mathbb{R}^n} \langle x, y\rangle_+^t f_\mu(x)dx\right)^{1/t},$$

where $a_+ = \max\{a, 0\}$. If μ is isotropic then $Z_2^+(\mu) \supseteq cB_2^n$ for an absolute constant $c > 0$ and if $1 \leqslant t < s$ then

$$\left(\frac{2}{e}\right)^{\frac{1}{t}-\frac{1}{s}} Z_t^+(\mu) \subseteq Z_s^+(\mu) \subseteq c_1 \left(\frac{2e-2}{e}\right)^{\frac{1}{t}-\frac{1}{s}} \frac{s}{t} Z_t^+(\mu).$$

The condition that we need is that the family of the one-sided L_t-centroid bodies grows with some mild rate as $t \to \infty$ (note that the assumption in the next proposition can be satisfied only for log-concave probability measures μ with support $\mathrm{supp}(\mu) = \mathbb{R}^n$).

Proposition 8.7. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Assume that there exists an increasing function $g : [1, \infty) \to [1, \infty)$ with $\lim_{t\to\infty} g(t)/\ln(t + 1) = +\infty$ such that $Z_t^+(\mu) \supseteq g(t)Z_2^+(\mu)$ for all $t \geqslant 2$. Then,*

$$\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^p d\mu(x) < +\infty$$

for every $p \geqslant 1$.

Proof. In Lemma 4.6 we saw that if $t \geqslant 1$ then for every $x \in \frac{1}{2}Z_t^+(\mu)$ we have

$$\varphi_\mu(x) \geqslant e^{-c_1 t},$$

where $c_1 > 1$ is an absolute constant. Since $\Lambda_\mu^*(x) \leqslant \ln \frac{1}{\varphi_\mu(x)}$, this shows that $\Lambda_\mu^*(x) \leqslant c_1 t$ for all $x \in \frac{1}{2}Z_t^+(\mu)$. In other words,

$$\frac{1}{2} Z_{t/c_1}^+(\mu) \subseteq B_t(\mu), \qquad t \geqslant c_1. \tag{8.3}$$

Since $\lim_{t\to\infty} g(t) = +\infty$, there exists $t_0 \geqslant c_1$ such that $\mu\left(\frac{g(t_0/c_1)}{2} Z_2^+(\mu)\right) \geqslant 2/3$. From Borell's lemma [11, Lemma 2.4.5] we know that, for all $t \geqslant t_0$,

$$1 - \mu\left(\frac{g(t/c_1)}{2} Z_2^+(\mu)\right) \leqslant e^{-c_2 g(t/c_1)/g(t_0/c_1)},$$

where $c_2 > 0$ is an absolute constant. We write

$$\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^p d\mu(x) = \int_0^\infty pt^{p-1}\mu(\{x : \Lambda_\mu^*(x) \geqslant t\})\, dt = p\int_0^\infty t^{p-1}(1 - \mu(B_t(\mu)))\, dt.$$

From (8.3) it follows that

$$1 - \mu(B_t(\mu)) \leqslant 1 - \mu\left(\frac{1}{2} Z_{t/c_1}^+(\mu)\right) \leqslant 1 - \mu\left(\frac{g(t/c_1)}{2} Z_2^+(\mu)\right) \leqslant e^{-c_2 g(t/c_1)/g(t_0/c_1)}$$

for all $t \geqslant t_0$. Since $\lim_{t\to\infty} g(t)/\ln(t + 1) = +\infty$, there exists $t_p \geqslant t_0$ such that

$$(p - 1)\ln(t) \leqslant \frac{c_2}{2g(t_0/c_1)} g(t/c_1)$$

for all $t \geqslant t_p$. Assume that $p > 2$. Then, from the previous observations we get

$$\begin{aligned} p\int_{t_p}^\infty t^{p-1}(1 - \mu(B_t(\mu)))\, dt &\leqslant p\int_{t_p}^\infty t^{p-1}\left(1 - \mu\left(\tfrac{g(t/c_1)}{2} Z_2^+(\mu)\right)\right) dt \\ &\leqslant p\int_{t_p}^\infty t^{p-1} t^{-2(p-1)}\, dt = p\int_{t_p}^\infty t^{-(p-1)}\, dt < \infty. \end{aligned}$$

This proves the result for $p > 2$ and then from Hölder's inequality it is clear that the assertion of the proposition is also true for all $p \geqslant 1$. $\square$

Note. It is not hard to construct examples of log-concave probability measures, even on the real line, for which $\mathrm{supp}(\mu) = \mathbb{R}^n$ but the assumption of Proposition 8.7 is not satisfied. Consider for example a measure μ on $\mathbb{R}$ with density $f(x) = c \cdot \exp(-p)$ where p is an even convex function rapidly increasing to infinity, e.g. $p(t) = e^{t^2}$.

However, this does not exclude the possibility that for every centered log-concave probability measure μ on $\mathbb{R}^n$ the function Λ_μ^* has finite second or higher moments.

9 Threshold for the measure: the approach and examples

For any log-concave probability measure μ on $\mathbb{R}^n$ we define the parameter

$$\beta(\mu) = \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2} \tag{9.1}$$

provided that

$$\|\Lambda_\mu^*\|_{L^2(\mu)} = (\mathbb{E}(\Lambda_\mu^*)^2)^{1/2} < \infty.$$

In Theorem 4.1 we saw that if μ is a log-concave probability measure on $\mathbb{R}^n$ then

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right),$$

where $c > 0$ is an absolute constant. In fact, the proof of this estimate starts with Lemma 4.3 and follows from the next stronger result: If $n \geqslant n_0$ then

$$\int_{\mathbb{R}^n} \exp(-\Lambda_\mu^*(x))\, d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right)$$

where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants. Then, Jensen's inequality implies that

$$e^{-\mathbb{E}_\mu(\Lambda_\mu^*)} \leqslant \int_{\mathbb{R}^n} \exp(-\Lambda_\mu^*(x))\, d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right).$$

We will need this lower bound for $\mathbb{E}_\mu(\Lambda_\mu^*)$.

Lemma 9.1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$. Then,*

$$\mathbb{E}(\Lambda_\mu^*) \geqslant cn/L_\mu^2,$$

where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants.

We will also need a number of observations in the case $\mu = \mu_K$ where K is a centered convex body of volume 1 in $\mathbb{R}^n$. The next lemma provides a lower bound for $\mathrm{Var}(\Lambda_{\mu_K}^*)$.

Lemma 9.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\mathrm{Var}(\Lambda_{\mu_K}^*) \geqslant c/L_{\mu_K}^4,$$

where $c > 0$ is an absolute constant.

Proof. Borell has proved in [5, Theorem 1] that if T is a convex body in $\mathbb{R}^n$ and f is a non-negative, bounded and convex function on T, not identically zero and with $\min(f) = 0$, then the function

$$\Phi_f(p) = \ln\left[(n+p)\|f\|_p^p\right]$$

is convex on $[0,\infty)$. Consider a centered convex body K of volume 1 in $\mathbb{R}^n$. Applying Borell's theorem for the function $\Lambda_{\mu_K}^*$ on rK, $r \in (0,1)$ and the triple $p = 0, 1$ and 2, and finally letting $r \to 1^-$, we see that

$$(n+1)^2 \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)}^2 \leqslant n(n+2)\|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)}^2,$$

which implies that

$$\mathrm{Var}(\Lambda_{\mu_K}^*) \geqslant \frac{1}{n(n+2)}\|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)}^2.$$

Then, taking into account Lemma 9.1 we obtain the result. □

Recall the definition of $\omega_{\mu_K} = \ln(1/\varphi_{\mu_K})$ in (7.4) and consider the parameter

$$\tau(\mu_K) = \frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K})}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2}. \tag{9.2}$$

The next lemma shows that we can estimate $\beta(\mu_K)$ if we can compute $\tau(\mu_K)$.

Lemma 9.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\beta(\mu_K) = \left(\tau(\mu_K) + O(L_{\mu_K}^2/\sqrt{n})\right)\left(1 + O(L_{\mu_K}^2/\sqrt{n})\right).$$

Proof. From Corollary 7.4 we know that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then for every $x \in \mathrm{int}(K)$ we have that $\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda_{\mu_K}^*(x) \leqslant \omega_{\mu_K}(x)$. Writing $\Lambda_{\mu_K}^* = \omega_{\mu_K} + h$ where $\|h\|_\infty \leqslant 5\sqrt{n}$ we easily see that

$$\mathrm{Var}_{\mu_K}(\Lambda_{\mu_K}^*) = \mathrm{Var}_{\mu_K}(\omega_{\mu_K}) + O(\sqrt{n}\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*))$$

where $X = O(Y)$ means that $|X| \leqslant cY$ for an absolute constant $c > 0$. Lemma 9.1 and the fact that $\mathbb{E}_{\mu_K}(\omega_{\mu_K}) = \mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*) + O(\sqrt{n})$ imply that

$$\frac{\mathbb{E}_{\mu_K}(\omega_{\mu_K})}{\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*)} = 1 + O(L_{\mu_K}^2/\sqrt{n}).$$

Taking also into account the fact that $L_{\mu_K}^2/\sqrt{n} = O((\ln n)^8/\sqrt{n}) = o(1)$ we get

$$\mathbb{E}_{\mu_K}(\omega_{\mu_K}) \approx \mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*).$$

Combining the above we see that

$$\begin{aligned}
\beta(\mu_K) = \frac{\mathrm{Var}_{\mu_K}(\Lambda_{\mu_K}^*)}{(\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*))^2} &= \frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K}) + O(\sqrt{n}\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*))}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2}\left(\frac{\mathbb{E}_{\mu_K}(\omega_{\mu_K})}{\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*)}\right)^2 \\
&= \left(\frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K})}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2} + O\left(L_{\mu_K}^2/\sqrt{n}\right)\right)\left(1 + O\left(L_{\mu_K}^2/\sqrt{n}\right)\right)^2 \\
&= \left(\tau(\mu_K) + O\left(L_{\mu_K}^2/\sqrt{n}\right)\right)\left(1 + O\left(L_{\mu_K}^2/\sqrt{n}\right)\right),
\end{aligned}$$

as claimed. □

Recall that $B_t(\mu)=\{x\in\mathbb{R}^n:\Lambda_\mu^*(x)\leqslant t\}$. We use Lemma 5.1 in the following way. Let $m:=\mathbb{E}_\mu(\Lambda_\mu^*)$. Then, for all $\varepsilon\in(0,1)$, from Chebyshev's inequality we have that

$$\mu(\{\Lambda_\mu^*\leqslant m-\varepsilon m\})\leqslant\mu(\{|\Lambda_\mu^*-m|\geqslant\varepsilon m\})\leqslant\frac{\mathbb{E}_\mu|\Lambda_\mu^*-m|^2}{\varepsilon^2m^2}=\frac{\beta(\mu)}{\varepsilon^2}.$$

Equivalently,

$$\mu(B_{(1-\varepsilon)m}(\mu))\leqslant\frac{\beta(\mu)}{\varepsilon^2}.$$

Let $\delta\in(\beta(\mu),1)$. We distinguish two cases:

(i) If $\beta(\mu)<1/8$ and $8\beta(\mu)<\delta<1$ then, choosing $\varepsilon=\sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1-\varepsilon)m}(\mu))\leqslant\frac{\delta}{2}.$$

Then, from Lemma 5.1 we see that

$$\begin{aligned}\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)):N\leqslant e^{(1-2\varepsilon)m}\}&\leqslant\mu(B_{(1-\varepsilon)m}(\mu))+e^{(1-2\varepsilon)m}e^{-(1-\varepsilon)m}\\&\leqslant\frac{\delta}{2}+e^{-\varepsilon m}\leqslant\delta,\end{aligned}$$

provided that $\varepsilon m\geqslant\ln(2/\delta)$. Since $m\geqslant c_1n/L_\mu^2$, this condition is satisfied if $n/L_\mu^2\geqslant c_2\ln(2/\delta)\sqrt{\delta/\beta(\mu)}$. By the choice of ε we conclude that

$$\varrho_1(\mu,\delta)\geqslant\left(1-\sqrt{8\beta(\mu)/\delta}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

(ii) If $1/8\leqslant\beta(\mu)<1$ and $\beta(\mu)<\delta<1$ then, choosing $\varepsilon=\sqrt{\frac{2\beta(\mu)}{\beta(\mu)+\delta}}$ we have that

$$\mu(B_{(1-\varepsilon)m}(\mu))\leqslant\frac{\beta(\mu)+\delta}{2}.$$

Then, exactly as in (i), we see that

$$\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)):N\leqslant e^{(1-\sqrt{\varepsilon})m}\}\leqslant\frac{\beta(\mu)+\delta}{2}+e^{(1-\sqrt{\varepsilon})m}e^{-(1-\varepsilon)m}\leqslant\frac{\beta(\mu)+\delta}{2}+e^{-(\sqrt{\varepsilon}-\varepsilon)m}\leqslant\delta,$$

provided that

$$(\sqrt{\varepsilon}-\varepsilon)m\geqslant\ln\left(\frac{2}{\delta-\beta(\mu)}\right). \tag{9.3}$$

Note that $1>\varepsilon\geqslant\sqrt{\beta(\mu)}\geqslant\frac{1}{2\sqrt{2}}$, and hence

$$\sqrt{\varepsilon}-\varepsilon=\frac{\sqrt{\varepsilon}}{(1+\sqrt{\varepsilon})(1+\varepsilon)}(1-\varepsilon^2)\geqslant c_1'(1-\varepsilon^2)=c_1'\frac{\delta-\beta(\mu)}{\delta+\beta(\mu)}\geqslant c_2'(\delta-\beta(\mu)),$$

where $c_i'>0$ are absolute constants. Since $m\geqslant c_1n/L_\mu^2$, we see that if $n/L_\mu^2\geqslant\frac{c_2}{\delta-\beta(\mu)}\ln\left(\frac{2}{\delta-\beta(\mu)}\right)$ then (9.3) is satisfied. Therefore, we conclude that

$$\varrho_1(\mu,\delta)\geqslant\left(1-\sqrt[4]{\frac{2\beta(\mu)}{\beta(\mu)+\delta}}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

We summarize the above in the next theorem.

Theorem 9.4. *Let μ be a log-concave probability measure on $\mathbb{R}^n$.*

(i) *Let $\beta(\mu) < 1/8$ and $8\beta(\mu) < \delta < 1$. If $n/L_\mu^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu)}$ then*

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt{8\beta(\mu)/\delta}\right) \frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

(ii) *Let $1/8 \leqslant \beta(\mu) < 1$ and $\beta(\mu) < \delta < 1$. If $n/L_\mu^2 \geqslant \frac{c_2}{\delta - \beta(\mu)} \ln\left(\frac{2}{\delta - \beta(\mu)}\right)$ then*

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt[4]{\frac{2\beta(\mu)}{\beta(\mu) + \delta}}\right) \frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

For the proof of the lower threshold we use Lemma 5.2. Note that if $m := \mathbb{E}_\mu(\Lambda_\mu^*)$ then as before, for all $\varepsilon \in (0,1)$, from Chebyshev's inequality we have that

$$\mu(\{\Lambda_\mu^* \geqslant m + \varepsilon m\}) \leqslant \mu(\{|\Lambda_\mu^* - m| \geqslant \varepsilon m\}) \leqslant \frac{\beta(\mu)}{\varepsilon^2}.$$

If $\beta(\mu) < 1/2$ and $2\beta(\mu) < \delta < 1$ then, choosing $\varepsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1+\varepsilon)m}(\mu)) \geqslant 1 - \frac{\delta}{2}.$$

Therefore, we will have that

$$\varrho_2(\mu, \delta) \leqslant (1 + 2\varepsilon)m/n$$

if our lower bound for $\varphi_-(B_{(1+\varepsilon)m}(\mu))$ gives

$$2\binom{N}{n}\left(1 - \varphi_-(B_{(1+\varepsilon)m}(\mu))\right)^{N-n} \leqslant \frac{\delta}{2} \tag{9.4}$$

for all $N \geqslant N_0 := \exp((1 + 2\varepsilon)m)$. Recall that in the case of the uniform measure μ_K on a centered convex body K of volume 1, Theorem 7.1 shows that

$$\varphi_-(B_{(1+\varepsilon)m}(\mu_K)) \geqslant \frac{1}{10}\exp(-(1+\varepsilon)m - 2\sqrt{n}).$$

We require that n and m are large enough so that $1/2^n < \delta/2$ and $2\sqrt{n} \leqslant \frac{\varepsilon m}{2}$. Using also the fact that $\binom{N}{n} \leqslant e^{-1}\left(\frac{eN}{n}\right)^n$ we see that (9.4) will be satisfied if we also have

$$\left(\frac{2eN}{n}\right)^n e^{-\frac{N-n}{10}e^{-(1+3\varepsilon/2)m}} < 1.$$

Setting $x := N/n$ we see that this last is equivalent to

$$e^{(1+3\varepsilon/2)m} < \frac{x-1}{10\ln(2ex)}.$$

One can now check that if $N \geqslant \exp((1 + 2\varepsilon)m)$ then all the restrictions are satisfied if we assume that $n/L_{\mu_K}^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu_K)}$. In this way we get the following.

Theorem 9.5. *Let $\beta, \delta > 0$ with $2\beta < \delta < 1$. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ with $\beta(\mu_K) = \beta$ and $n/L_{\mu_K}^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta}$ then*

$$\varrho_2(\mu_K, \delta) \leqslant \left(1 + \sqrt{8\beta/\delta}\right) \frac{\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*)}{n}.$$

An estimate analogous to the one in Theorem 9.4 (ii) is also possible but we shall not go through the details. From the discussion in this section it is clear that our approach is able to provide good bounds for the threshold $\varrho(\mu,\delta)$ if the parameter $\beta(\mu)$ is small, especially if $\beta(\mu)=o_n(1)$ as the dimension increases. We illustrate this with a number of examples.

Example 9.6 (Uniform measure on the cube)**.** Let μ_{C_n} be the uniform measure on the unit cube $C_n=\left[-\frac{1}{2},\frac{1}{2}\right]^n$. Since $\mu_{C_n}=\mu_{C_1}\otimes\cdots\otimes\mu_{C_1}$ we have

$$\mathrm{Var}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}})=n\mathrm{Var}_{\mu_{C_1}}(\Lambda^*_{\mu_{C_1}})\qquad\text{and}\qquad \mathbb{E}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}})=n\mathbb{E}_{\mu_{C_1}}(\Lambda^*_{\mu_{C_1}}).$$

Therefore,

$$\beta(\mu_{C_n})=\frac{\mathrm{Var}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}})}{(\mathbb{E}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}}))^2}=\frac{\beta(\mu_{C_1})}{n}\longrightarrow 0.$$

as $n\to\infty$. Then, Theorem 9.4 and Theorem 9.5 show that for any $\delta\in(0,1)$ there exists $n_0(\delta)$ such that, for any $n\geqslant n_0$,

$$\varrho_1(\mu_{C_n},\delta)\geqslant\left(1-\sqrt{\frac{8\beta_{\mu_{C_n}}}{\delta}}\right)\frac{\mathbb{E}(\Lambda^*_{\mu_{C_n}})}{n}\geqslant\left(1-\frac{c_1}{\sqrt{\delta n}}\right)\mathbb{E}(\Lambda^*_{\mu_{C_1}})$$

and

$$\varrho_2(\mu_{C_n},\delta)\leqslant\left(1+\sqrt{\frac{8\beta_{\mu_{C_n}}}{\delta}}\right)\frac{\mathbb{E}(\Lambda^*_{\mu_{C_n}})}{n}\leqslant\left(1+\frac{c_2}{\sqrt{\delta n}}\right)\mathbb{E}(\Lambda^*_{\mu_{C_1}}),$$

which shows that

$$\varrho(\mu_{C_n},\delta)\leqslant\frac{c}{\sqrt{\delta n}},$$

where $c>0$ is an absolute constant. This estimate provides a sharp threshold for the measure of a random polytope K_N with independent vertices uniformly distributed in C_n. It provides a direct proof of the result of Dyer, Füredi and McDiarmid in [15] with a stronger estimate for the "width of the threshold".

Example 9.7 (Gaussian measure)**.** Let γ_n denote the standard n-dimensional Gaussian measure with density $f_{\gamma_n}(x)=(2\pi)^{-n/2}e^{-|x|^2/2}$, $x\in\mathbb{R}^n$. Note that $\gamma_n=\gamma_1\otimes\cdots\otimes\gamma_1$, and hence we may argue as in the previous example. We may also use direct computation to see that

$$\Lambda_{\gamma_n}(\xi)=\ln\Big(\int_{\mathbb{R}^n}e^{\langle\xi,z\rangle}f_{\gamma_n}(z)dz\Big)=\frac{1}{2}|\xi|^2$$

for all $\xi\in\mathbb{R}^n$ and

$$\Lambda^*_{\gamma_n}(x)=\sup_{\xi\in\mathbb{R}^n}\{\langle x,\xi\rangle-\Lambda_{\gamma_n}(\xi)\}=\frac{1}{2}|x|^2$$

for all $x\in\mathbb{R}^n$. It follows that

$$B_t(\gamma_n)=\{x\in\mathbb{R}^n:\Lambda^*_{\gamma_n}(x)\leqslant t\}=\{x\in\mathbb{R}^n:|x|\leqslant\sqrt{2t}\}=\sqrt{2t}B_2^n.$$

Note that if $x\in\partial(B_t(\gamma_n))$ then

$$\varphi_{\gamma_n}(x)=\frac{1}{\sqrt{2\pi}}\int_{\sqrt{2t}}^{\infty}e^{-u^2/2}du\geqslant\frac{c}{\sqrt{t}}e^{-t}$$

for all $t\geqslant 1$ (see [22, p. 17] for a refined form of the lower bound that we use). By the standard concentration estimate for the Euclidean norm with respect to γ_n (see [38, Theorem 3.1.1]), we have $\|\,|x|-\sqrt{n}\,\|_{\psi_2}\leqslant C$, where $C>0$ is an absolute constant, or equivalently, for any $s>0$,

$$\gamma_n(\{x\in\mathbb{R}^n:|\,|x|-\sqrt{n}\,|\geqslant s\sqrt{n}\})\leqslant 2\exp(-cs^2n),$$

where $c>0$ is an absolute constant. This shows that

$$\max\{\gamma_n((1-s)\sqrt{n}B_2^n), 1-\gamma_n((1+s)\sqrt{n}B_2^n)\}\leqslant 2\exp(-cs^2n)$$

for every $s\in(0,1)$. Let $\varepsilon\in(0,1/2)$. Applying Lemma 5.1 with $t=(1-\varepsilon)\,n/2$ and $N\leqslant\exp((1/2-\varepsilon)n)$ we see that

$$\begin{aligned}\mathbb{E}_{\gamma_n^N}(\gamma_n(K_N)) &\leqslant \gamma_n(\sqrt{(1-\varepsilon)n}B_2^n)+\exp(-\varepsilon n/2)\\ &\leqslant 2\exp(-c\varepsilon^2 n)+\exp(-\varepsilon n/2),\end{aligned}$$

using the fact that $\sqrt{1-\varepsilon}\leqslant 1-\varepsilon/2$. It follows that, for any $\delta\in(0,1)$, if we choose $\varepsilon=c_1\sqrt{\ln(4/\delta)}/\sqrt{n}$ we have

$$\sup\left\{\mathbb{E}_{\gamma_n^N}\big(\gamma_n(K_N)\big) : N\leqslant e^{\left(\frac{1}{2}-\varepsilon\right)n}\right\}\leqslant\delta,$$

and hence

$$\varrho_1(\gamma_n,\delta)\geqslant\frac{1}{2}-\frac{c_1\sqrt{\ln(4/\delta)}}{\sqrt{n}}.$$

Now, let $N\geqslant\exp((1/2+\varepsilon)n)$. Applying Lemma 5.2 with $A=B_t(\gamma_n)$ where $t=(1+\varepsilon)n/2$, we see that $\gamma_n(B_t(\gamma_n))=\gamma_n(\sqrt{(1+\varepsilon)n}B_2^n)\geqslant 1-2\exp(-c\varepsilon^2 n)$, because $\sqrt{1+\varepsilon}\geqslant 1+\varepsilon/3$. We also have

$$2\binom{N}{n}\left(1-\varphi_-(B_t(\gamma_n))\right)^{N-n}\leqslant\left(\frac{2eN}{n}\right)^n\exp\left(-\frac{c(N-n)}{\sqrt{n}}e^{-(1+\varepsilon)n/2}\right).$$

Let $\delta\in(0,1)$. We choose $\varepsilon=c_2\sqrt{\ln(4/\delta)}/\sqrt{n}$ and insert these estimates into Lemma 5.2. Arguing as in the proof of (9.4) we see that if $n\geqslant n_0(\delta)$ then

$$\inf\left\{\mathbb{E}_{\gamma_n^N}\big(\gamma_n(K_N)\big) : N\geqslant e^{\left(\frac{1}{2}+\varepsilon\right)n}\right\}\geqslant 1-\delta,$$

and hence

$$\varrho_2(\gamma_n,\delta)\leqslant\frac{1}{2}+\frac{c_2\sqrt{\ln(4/\delta)}}{\sqrt{n}}.$$

Combining the above we get

$$\varrho(\gamma_n,\delta)\leqslant\frac{C\sqrt{\ln(4/\delta)}}{\sqrt{n}},$$

where $C>0$ is an absolute constant.

Finally, we discuss the example of the uniform measure on the Euclidean ball. It was proved in [3] that if $\varepsilon\in(0,1)$ and $K_N=\mathrm{conv}\{x_1,\dots,x_N\}$ where $x_1,\dots,x_N$ are random points independently and uniformly chosen from B_2^n then

$$\lim_{n\to\infty}\sup\left\{\frac{\mathbb{E}|K_N|}{|B_2^n|} : N\leqslant\exp\left((1-\varepsilon)\left(\frac{n+1}{2}\right)\ln n\right)\right\}=0$$

and

$$\lim_{n\to\infty}\inf\left\{\frac{\mathbb{E}|K_N|}{|B_2^n|} : N\geqslant\exp\left((1+\varepsilon)\left(\frac{n+1}{2}\right)\ln n\right)\right\}=1.$$

We shall obtain a similar conclusion with the approach of this work (the estimate below is in fact stronger since it sharpens the width of the threshold from $O(1)$ to $O(1/\sqrt[4]{n})$).

Theorem 9.8. *Let D_n be the centered Euclidean ball of volume 1 in $\mathbb{R}^n$. Then, the sequence $\mu_n:=\mu_{D_n}$ exhibits a sharp threshold with $\varrho(\mu_n,\delta)\leqslant\frac{c}{\sqrt{\delta}\,\sqrt[4]{n}}$ and e.g. if n is even then we have that*

$$\mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*)=\frac{(n+1)}{2}H_{\frac{n}{2}}+O(\sqrt{n})$$

as $n\to\infty$, where $H_m=\sum_{k=1}^m\frac{1}{k}$.

Proof. Note that if K is a centered convex body in $\mathbb{R}^n$ and $r>0$ then $\Lambda^*_{\mu_{rK}}(x)=\Lambda^*_{\mu_K}(x/r)$ for all $x\in\mathbb{R}^n$, where μ_{rK} is the uniform measure on rK. It follows that

$$\frac{1}{|rK|}\int_{rK}[\Lambda^*_{\mu_{rK}}(x)]^p dx=\frac{1}{|K|}\int_K[\Lambda^*_{\mu_K}(x)]^p dx$$

for every $p>0$ and $r>0$. This shows that in order to compute $\beta(\mu_{D_n})$ it suffices to compute the ratio

$$\beta(\mu_{D_n})+1=\frac{\frac{1}{|B_2^n|}\int_{B_2^n}\Lambda^*(x)^2dx}{\left(\frac{1}{|B_2^n|}\int_{B_2^n}\Lambda^*(x)dx\right)^2}$$

where $\Lambda^*:=\Lambda^*_{\mu_{B_2^n}}$. Having in mind Lemma 9.3 we start by computing $\tau(\mu_{B_2^n})$. Set $\omega:=\omega_{\mu_{B_2^n}}$. Then, $\omega(x)=\ln(1/\varphi(x))$ where $\varphi(x)=F(|x|)$,

$$F(r)=c_n\int_r^1(1-t^2)^{\frac{n-1}{2}}dt,\qquad r\in[0,1]$$

and $c_n=\pi^{-1/2}\Gamma(\frac{n}{2}+1)/\Gamma(\frac{n+1}{2})$. From [3, Lemma 2.2] we know that

$$F(r)=(1-r^2)^{\frac{n+1}{2}}h(r,n),$$

where

$$\frac{1}{\sqrt{2\pi(n+2)}}\leqslant h(r,n)\leqslant\frac{1}{r\sqrt{2\pi n}} \tag{9.5}$$

for all $r\in(0,1]$. We assume that n is even (the case where n is odd can be treated in a similar way). Using polar coordinates we compute

$$\begin{aligned}\frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\,dx&=-n\int_0^1 r^{n-1}\ln(F(r))\,dr\\&=-n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\,dr-n\int_0^1 r^{n-1}\ln\left(h(r,n)\right)\,dr.\end{aligned}$$

The leading term is the first one and one can compute it explicitly. After making the change of variables $r^2=u$, we get

$$\begin{aligned}-n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\,dr&=-\frac{n(n+1)}{2}\int_0^1 r^{n-1}\ln(1-r^2)\,dr\\&=-\frac{n(n+1)}{4}\int_0^1 u^{\frac{n-2}{2}}\ln(1-u)\,du=\frac{n(n+1)}{2n}H_{\frac{n}{2}},\end{aligned} \tag{9.6}$$

using also Lemma 8.3. For the second term we recall from (9.5) that $0\leqslant-\ln(h(r,n))\leqslant\frac{1}{2}\ln(2\pi(n+2))\leqslant c_1\ln n$, and hence

$$-n\int_0^1 r^{n-1}\ln\left(h(r,n)\right)\,dr\leqslant c_1\ln n\int_0^1 nr^{n-1}dr=c_1\ln n.$$

Therefore,

$$\frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\,dx=\frac{n+1}{2}H_{\frac{n}{2}}+O(\ln n). \tag{9.7}$$

Using again polar coordinates we write

$$\frac{1}{|B_2^n|}\int_{B_2^n}(\omega(x))^2dx = n\int_0^1 r^{n-1}\ln^2(F(r))\,dr$$
$$= n\int_0^1 r^{n-1}\ln^2((1-r^2)^{\frac{n+1}{2}})\,dr + n\int_0^1 r^{n-1}\ln^2(h(r,n))\,dr$$
$$+ 2n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\ln(h(r,n))\,dr,$$

As before, the leading term is the first one and we can compute it explicitly. After making the change of variables $r^2=u$, we get

$$n\left(\frac{n+1}{2}\right)^2\int_0^1 r^{n-1}\ln^2(1-r^2)\,dr = n\frac{(n+1)^2}{8}\int_0^1 u^{\frac{n-2}{2}}\ln^2(1-u)\,du$$
$$= n\frac{(n+1)^2}{8}\left(\frac{2}{n}H_{\frac{n}{2}}^2+\frac{2}{n}\sum_{k=1}^{n/2}\frac{1}{k^2}\right).$$

On the other hand, from (9.5) we see that if $h(r,n)\leqslant 1$ then $0\leqslant -\ln(h(r,n))\leqslant \frac{1}{2}\ln(2\pi(n+2))\leqslant c_1\ln n$, while if $h(r,n)>1$ then $0\leqslant \ln(h(r,n))\leqslant \ln(1/r)$. Therefore, $\ln^2(h(r,n))\leqslant c_2(\ln n)^2+\ln^2(r)$ for all $r\in(0,1]$. It follows that

$$n\int_0^1 r^{n-1}\ln^2(h(r,n))\,dr\leqslant c_3(\ln n)^2+n\int_0^1 r^{n-1}\ln^2 r\,dr\leqslant c_4(\ln n)^2.$$

Using again the fact that $\ln(h(r,n)^{-1})\leqslant c_1\ln n$ as well as (9.6), we check that

$$2n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\ln(h(r,n))\,dr\leqslant \frac{n(n+1)}{2n}H_{\frac{n}{2}}\cdot c_1\ln n\leqslant c_5 n(\ln n)^2.$$

From these estimates we have

$$\frac{1}{|B_2^n|}\int_{B_2^n}(\omega(x))^2\,dx = \frac{(n+1)^2}{4}H_{\frac{n}{2}}^2+O(n(\ln n)^2). \tag{9.8}$$

From (9.7) and (9.8) we finally get

$$\tau(\mu_{B_2^n}) = O\left(\frac{n(\ln n)^2}{n^2H_{\frac{n}{2}}^2}\right) = O(1/n).$$

Then, Lemma 9.3 and a simple computation show that

$$\beta(\mu_{D_n}) = \left(\tau(\mu_{B_2^n})+O(L_{\mu_{B_2^n}}^2/\sqrt{n})\right)\left(1+O(L_{\mu_{B_2^n}}^2/\sqrt{n})\right) = O(1/\sqrt{n}),$$

because $L_{\mu_{B_2^n}}\approx 1$. Finally, note that by the estimate (7.5) in Corollary 7.4 we have

$$\mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*) = \frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\,dx+O(\sqrt{n}) = \frac{(n+1)}{2}H_{\frac{n}{2}}+O(\sqrt{n})$$

as $n\to\infty$. □

Note. The above discussion leaves open the following basic question: to estimate

$$\beta_n^* := \sup\{\beta(\mu_K) : K \text{ is a centered convex body of volume 1 in } \mathbb{R}^n\}$$

or, more generally,

$$\beta_n := \sup\{\beta(\mu) : \mu \text{ is a centered log-concave probability measure on } \mathbb{R}^n\}.$$

10 Further reading

Special cases of the threshold problem have been studied in various works. In addition to the case of the discrete cube, Dyer, Füredi and McDiarmid established in [15] a sharp threshold for the expected volume of random polytopes with vertices uniformly distributed in the solid cube $B_\infty^n = [-1,1]^n$. If $\kappa = 2\pi/e^{\gamma+1/2}$, where γ is the Euler-Mascheroni constant then for every $\varepsilon \in (0,\kappa)$ one has

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}|K_N| \colon N \leqslant (\kappa-\varepsilon)^n\right\} = 0$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}|K_N| \colon N \geqslant (\kappa+\varepsilon)^n\right\} = 1.$$

The articles [33] and [3], [4] address the same question for a number of cases where X_i have rotationally invariant densities.

Exponential in the dimension upper and lower thresholds are obtained in [18] for the case where X_i are uniformly distributed in a simplex. Let $\Omega_n = \{\mathbf{x} \geqslant \mathbf{0} : x_1 + \cdots + x_n = 1\}$ be the standard embedding of the $(n-1)$-dimensional simplex in n-dimensional space. If $N > C_0^n$, where $C_0 > 0$ is an absolute constant, then

$$\mathbb{E}\,|\mathrm{conv}\{x_1,\ldots,x_N\}| \geqslant (1 - e^{-c_0\sqrt{n}})|\Omega_n|.$$

The basic idea is the following. Given $\alpha \in (0,1)$ consider the caps $C_i(\alpha) = \{\mathbf{x} \in \Omega_n : x_i \geqslant 1-\alpha\}$. The authors describe a set $\Omega(\varepsilon,\gamma)$, where ε,γ are carefully defined parameters, such that $|\Omega(\varepsilon,\gamma)| \geqslant (1-e^{-c_0\sqrt{n}})|\Omega_n|$ and every $\mathbf{x} \in \Omega(\varepsilon,\gamma)$ is very likely to belong to a simplex whose vertices are in the caps $C_1(\alpha),\ldots,C_n(\alpha)$. An exponential number of random points suffices to ensure that, with probability close to 1, there are "many" of them in each $C_i(\alpha)$ so that the convex hull $\mathrm{conv}\{x_1,\ldots,x_N\} \supseteq \Omega(\varepsilon,\gamma)$. This completes the proof. A second main result of the same paper provides a lower threshold. For every $\varepsilon > 0$, if $N < e^{(\gamma-\varepsilon)n}$, where γ is the Euler-Mascheroni constant, then the convex hull of N random points $x_1,\ldots,x_N$ uniformly distributed in Ω_n satisfies $\mathbb{E}\,|\mathrm{conv}\{x_1,\ldots,x_N\}|/|\Omega_n| \to 0$ as $n\to\infty$. To this end, the authors compute the Legendre transform of the log-moment generating function of a random vector X distributed in the simplex.

General upper and lower thresholds (in the spirit of Section 6) were obtained by Chakraborti, Tkocz and Vritsiou in [12] for some general families of distributions. If μ is an even log-concave probability measure supported on a convex body K in $\mathbb{R}^n$ and if $X_1, X_2, \ldots$ are independent random points distributed according to μ, then for any $n < N \leqslant \exp(c_1 n/L_\mu^2)$ we have that

$$\frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|} \leqslant \exp\left(-c_2 n/L_\mu^2\right),$$

where $c_1, c_2 > 0$ are absolute constants. A lower threshold (that we have already mentioned) is also established in [12] for the case where μ is an even κ-concave measure on $\mathbb{R}^n$ with $0 < \kappa < 1/n$, supported on a convex body K in $\mathbb{R}^n$. If $X_1, X_2, \ldots$ are independent random points in $\mathbb{R}^n$ distributed according to μ and $K_N = \mathrm{conv}\{X_1,\ldots,X_N\}$ as before, then for any $M \geqslant C$ and any $N \geqslant \exp\left(\frac{1}{\kappa}(\ln n + 2\ln M)\right)$ we have that

$$\frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|} \geqslant 1 - \frac{1}{M},$$

where $C > 0$ is an absolute constant.

In [19] a threshold for $\mathbb{E}_{\mu^N}|K_N|/(2\alpha)^n$ was established for the case where X_i have independent identically distributed coordinates supported on a bounded interval, under some mild additional assumptions (see below for a more precise description). This result was generalized by Pafis in [31] as follows. Let μ be an even Borel probability measure on the real line and let $X_1,\ldots,X_n$ be independent and identically distributed random variables, defined on some probability space $(\Omega,\mathcal{F},P)$, each with distribution μ. Consider the random vector $\vec{X} = (X_1,\ldots,X_n)$ and, for a fixed N satisfying $N > n$, consider N independent copies $\vec{X}_1,\ldots,\vec{X}_N$ of $\vec{X}$. The distribution of $\vec{X}$ is $\mu_n := \mu\otimes\cdots\otimes\mu$ (n times) and the distribution of $(\vec{X}_1,\ldots,\vec{X}_N)$ is $\mu_n^N := \mu_n\otimes\cdots\otimes\mu_n$ (N times). The goal is to obtain a sharp threshold for the expected μ_n-measure of the random polytope

$$K_N := \mathrm{conv}\left\{\vec{X}_1,\ldots,\vec{X}_N\right\}.$$

Assume that μ is non-degenerate, i.e. $\mathrm{Var}(X) > 0$. Let

$$x^* = x^*(\mu) := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$$

be the right endpoint of the support of μ and set $I_\mu = (-x^*, x^*)$. Note that since μ is non-degenerate and even, we have that $x^* > 0$. As usual, let

$$M_\mu(t) := \mathbb{E}\big(e^{tX}\big) := \int_{\mathbb{R}} e^{tx}\, d\mu(x), \qquad t \in \mathbb{R}$$

denote the moment generating function of X, and let $\Lambda_\mu(t) := \ln M_\mu(t)$ be its logarithmic moment generating function. Finally, consider the Legendre transform $\Lambda_\mu^* : I_\mu \to \mathbb{R}$ of Λ_μ.

We say that μ is *admissible* if it is non-degenerate, i.e. $\mathrm{Var}_\mu(X) > 0$, and satisfies the following conditions:

(i) There exists $r > 0$ such that $\mathbb{E}\big(e^{tX}\big) < \infty$ for all $t \in (-r, r)$; in particular, X has finite moments of all orders.

(ii) One of the following holds: (1) $x^* < +\infty$ and $P(X = x^*) = 0$, or (2) $x^* = +\infty$ and $\{\Lambda_\mu < \infty\} = \mathbb{R}$, or (3) $x^* = +\infty$, $\{\Lambda_\mu < \infty\}$ is bounded and μ is log-concave.

Finally, we say that μ satisfies *the Λ^*-condition* if

$$\lim_{x \uparrow x^*} \frac{-\ln \mu([x, \infty))}{\Lambda_\mu^*(x)} = 1.$$

Theorem 10.1. *Let μ be an admissible even probability measure on $\mathbb{R}$ that satisfies the Λ^*-condition. Then, for any $\delta \in \big(0, \frac{1}{2}\big)$ and any $\varepsilon \in (0,1)$ there exists $n_0(\mu, \delta, \varepsilon)$ such that*

$$\varrho_1(\mu_n, \delta) \geqslant (1-\varepsilon)\mathbb{E}_\mu(\Lambda_\mu^*) \quad \textit{and} \quad \varrho_2(\mu_n, \delta) \leqslant (1+\varepsilon)\mathbb{E}_\mu(\Lambda_\mu^*)$$

for every $n \geqslant n_0(\mu, \delta, \varepsilon)$. In particular, $\{\mu_n\}_{n=1}^\infty$ exhibits a sharp threshold, i.e. $\lim_{n\to\infty} \varrho(\mu_n, \delta) = 0$, with "threshold constant" $\mathbb{E}_\mu(\Lambda_\mu^*)$.

An application of Theorem 10.1 is also given to the case of the product p-measure $\nu_p^n := \nu_p^{\otimes n}$. For any $p \geqslant 1$ we denote by ν_p the probability distribution on $\mathbb{R}$ with density $(2\gamma_p)^{-1}\exp(-|x|^p)$, where $\gamma_p = \Gamma(1 + 1/p)$. We show that ν_p satisfies the Λ^*-condition.

Theorem 10.2. *For any $p \geqslant 1$ we have that*

$$\lim_{x \to \infty} \frac{-\ln(\nu_p[x, \infty))}{\Lambda_{\nu_p}^*(x)} = 1.$$

Note that the measure ν_p is admissible for all $1 \leqslant p < \infty$; it satisfies condition (ii-3) if $p = 1$ and condition (ii-2) for all $1 < p < \infty$. Therefore, Theorem 10.2 implies that if K_N is the convex hull of $N > n$ independent random vectors $\vec{X}_1, \ldots, \vec{X}_N$ with distribution ν_p^n then the expected measure $\mathbb{E}_{(\nu_p^n)^N}(\nu_p^n(K_N))$ exhibits a sharp threshold at $N = \exp((1 \pm \varepsilon)\mathbb{E}_{\nu_p}(\Lambda_{\nu_p}^*)n)$.

The variant of this question that was studied in [19] dealt with the case where μ is an even, compactly supported, Borel probability measure on the real line, $\mu_n(K_N)$ is replaced by the volume of K_N, and

$$\kappa = \kappa(\mu) := \frac{1}{2x^*} \int_{-x^*}^{x^*} \Lambda_\mu^*(x)dx.$$

If $0 < \kappa(\mu) < \infty$ then one has that, for every $\varepsilon \in (0, \kappa)$,

$$\lim_{n\to\infty} \sup\big\{(2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \leqslant \exp((\kappa - \varepsilon)n)\big\} = 0 \tag{10.1}$$

and if the distribution μ satisfies the Λ^*-condition then one also has

$$\lim_{n\to\infty} \inf\big\{(2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \geqslant \exp((\kappa + \varepsilon)n)\big\} = 1. \tag{10.2}$$

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