Lecture notes: Graph Structure and Geometry

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Contents

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These notes and the associated lectures are inspired by two problems from geometric graph theory. Sometimes, the tools developed to solve a problem turn out to be more important than the problem itself. By the end of the lectures, I hope to convince you that that may be the case here.

1 Two Problems

A graph is a pair $G = (V, E)$ where V is a finite set of *vertices* (also called *nodes*) and E is a set of *edges* where an *edge* is an unordered pairs of vertices, that is, $E = \{\{x, y\} \mid x, y \in V \text{ and } x \neq y\}$. In the edge $\{x, y\}$, x and y are the *endpoints* of the edge and the edge is *incident* to x and y. The *degree* of a vertex is the number of edges incident to that vertex, and the maximum degree over all vertices of a graph G is denoted by $\Delta(G)$. In this document we often denote by n the number of vertices in a graph G, and call such a graph an n-vertex graph.

A *drawing* of a graph is a graph representation where the vertices are represented by a set of distinct points in the plane and where each edge is a simple Jordan arc that intersects no vertex other than its own two endpoints. Two edges *cross* if they intersect at some point other than a common endpoint. A drawing is *planar* (or *crossing-free*) if no pair of edges cross.[1] A graph is *planar* if it has a planar drawing. One can argue that planar graphs are the most studied graphs and they are the key objects in these notes.

Undefined terms and notation can be found in Diestel's text [15], freely available here: Diestel's textbook.

1.1 Problem 1

A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *3D grid drawing*, represents the vertices by distinct points in $\mathbb{Z}^3$ (called *grid-points*), and represents each edge as a line-segment between its endpoints, such that edges only intersect at a common endpoint, and an edge only intersects a vertex that is an endpoint of that edge.

In contrast to the case in the plane, a folklore result states that every graph has a 3D grid drawing. Such a drawing can be constructed using the 'moment curve' algorithm in which vertex v_i, $1 \leqslant i \leqslant n$, is represented by the grid-point (i, i^2, i^3). It can be verified that, even in the complete graph K_n, no two edges cross.

Since every graph has a 3D grid drawing, one is interested in optimizing certain measures of the aesthetic quality of a drawing. If a 3D grid drawing is contained in an axis-aligned box with side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. This definition is formulated so that 2D grid drawings have positive volume (that is area).

Observe that the drawings produced by the moment curve algorithm have $\mathcal{O}(n^6)$ volume, for any n-vertex graph. Cohen *et al.* [11] improved this bound, by proving that if p is a prime with $n < p \leqslant 2n$, and each vertex v_i is represented by the grid-point $(i, i^2 \bmod p, i^3 \bmod p)$, then there is still no crossing. This construction is a generalisation of an analogous two-dimensional technique due to Erdős [33]. Furthermore, Cohen *et al.* [11] proved that the resulting $\mathcal{O}(n^3)$ volume bound is asymptotically optimal in the case of the complete graph K_n. It is therefore of interest to identify fixed graph parameters that allow for 3D grid drawings with small volume. Pach, Thiele, and Tóth [49] proved that every

[1] Planar drawings are equivalent to *planar embeddings*. See here for more on graph embeddings https://en.wikipedia.org/wiki/Graph_embedding.

graph with bounded chromatic number has a 3D grid drawing with volume $O(n^2)$, and this bound is optimal for the complete bipartite graph $K_{n/2,n/2}$.

By the 4-colour theorem, planar graphs have chromatic number 4, thus the aforementioned result implies that planar graphs have a 3D grid drawing with volume $O(n^2)$. More strongly a classical result by de Fraysseix, Pach, and Pollack [12, 13] states that planar graphs have 2D grid drawing in $O(n^2)$ volume (area) and that the bound is best possible. A major open problem in the area was to determine if that volume can be improved in 3D. More specifically in 2001 Felsner, Liotta, and Wismath [36, 37] asked the following problem. The problem remained open for almost 20 years is the first of the two motivational problems behind these lecture notes.

Problem 1. [37] *Does every n-vertex planar graph have a 3D grid drawing with $\mathcal{O}(n)$ volume?*

1.2 Problem 2

A *geometric graph* is a graph whose vertices are distinct points in the plane (not necessarily in general position) and whose edges are straight-line segments between pairs of points. Given a geometric graph $\overline{G}$, the *underlying* graph of $\overline{G}$ is a (combinatorial) graph G isomorphic to $\overline{G}$. When the meaning is clear from the context, we may use G both to denote a geometric graph and its underlying graph. If the underlying graph G of a geometric graph $\overline{G}$ belongs to a class of graphs $\mathcal{K}$, then we say that $\overline{G}$ is a *geometric $\mathcal{K}$ graph*. For example, if $\mathcal{K}$ is the class of planar graphs, then $\overline{G}$ is a geometric planar graph. Two edges in a geometric graph *cross* if they intersect at some point other than a common endpoint. [2] A geometric graph with no pair of crossing edges is called *crossing-free*.

Consider a geometric graph G with vertex set $V(G) = \{p_1, \dots, p_n\}$. A crossing-free geometric graph H with vertex set $V(H) = \{q_1, \dots, q_n\}$ is called an *untangling* of G if for all $i, j \in \{1, 2, \dots, n\}$, q_i is adjacent to q_j in H if and only if p_i is adjacent to p_j in G. Furthermore, if $p_i = q_i$ then we say that p_i is *fixed*, otherwise we say that p_i is *free*. If H is an untangling of G with k vertices fixed, then we say that G can be *untangled* while keeping k vertices fixed. Clearly only geometric planar graphs can be untangled. Moreover, since every planar graph is isomorphic to some crossing-free geometric graph [35, 63], trivially every geometric planar graph can be untangled while keeping at least 0 vertices fixed.

This problem has been implemented as a computer game by John Tantalo (see https://en.wikipedia.org/wiki/Planarity). The goal of the game is to untangle a given geometric planar graph as fast as possible. In this online game however, the player is restricted to keep the vertices within the boundaries of a given fixed rectangle. You can try playing a game, here for example: https://www.jasondavies.com/planarity/. Notice the dramatic rise in difficulty already with geometric graphs on 20 vertices.

At the 5th Czech-Slovak Symposium on Combinatorics in Prague in 1998, Mamoru Watanabe asked if every geometric cycle (that is, all polygons) can be untangled while keeping at least εn vertices fixed, for some $\varepsilon > 0$. Pach and Tardos [48] answered that question in the negative by providing an $\mathcal{O}((n \log n)^{2/3})$ upper bound on the number of fixed vertices. Furthermore, they proved that every geometric cycle can be untangled while keeping at least $\sqrt{n}$ vertices fixed. This lower bound for cycles has been improved to near-tight $\Omega(n^{2/3})$ bound by Cibulka [10]. The second motivational problem behind these lecture notes is the following:

Problem 2. [48] *Can every geometric planar graph be untangled while keeping n^ε vertices fixed, for some $\varepsilon > 0$?*

[2] In the literature on crossing number it is customary to require that intersecting edges cross 'properly' and do not 'touch'. Wed to not make such a requirement in this definition of a geometric graph.

2 Product structure – a tool for Problem 1

2.1 From a geometric to a topological problem

Recall that our goal is to solve Problem 1, that is, to prove that every n-vertex planar graph has a 3D grid drawing in $\mathcal{O}(n)$ volume. Let's consider a possible shape of an axis-aligned box of volume $\mathcal{O}(n)$ containing such a drawing. It could be an $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ box, or an $\mathcal{O}(\sqrt{n}) \times \mathcal{O}(\sqrt{n}) \times \mathcal{O}(1)$ box, or an $\mathcal{O}(n^{1/3}) \times \mathcal{O}(n^{1/3}) \times \mathcal{O}(n^{1/3})$ box, among other possibilities.

3D grid drawings that fit in an $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ box have important properties that make them more suitable for conversion to a topological/combinatorial problem. Specifically, notice that if a graph G has a 3D grid $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing D, then all the vertices of G in D lie on $\mathcal{O}(1)$ parallel lines (parallel to z-axis) – see Figure 1. Moreover each of these lines L_i defines a linear ordering $<_i$ of the vertices on L_i (by increasing z-coordinates). The following is true for these linear orderings. For every pair of edges vw and xy of G if v and x are on L_i; and if y and w on line L_j; and, if $v <_i x$ then $y <_j w$. This observation leads to a definition of following layouts (void of geometry), called *track layouts*. As we will show there is a tight relationship between $\mathcal{O}(n)$ volume 3D grid drawings (specifically $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawings) and track layouts. We first define them.

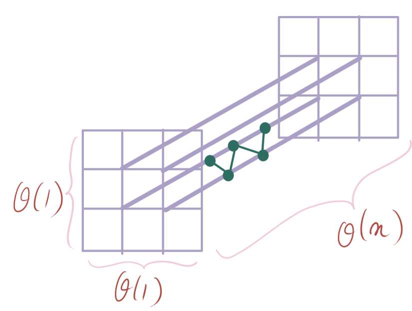

Figure 1: $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing

Let G be a graph. A (proper) *colouring* of G is a partition $\{V_i : i \in I\}$ of $V(G)$, where I is a set of *colours*, such that for every edge vw of G, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Each set V_i is called a *colour class*. A colouring of G with c colours is a *c-colouring*, and we say that G is *c-colourable*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum c such that G is c-colourable.

If $<_i$ is a total order of a colour class V_i, then we call the pair $(V_i, <_i)$ a *track*. If $\{V_i : i \in I\}$ is a colouring of G, and $(V_i, <_i)$ is a track, for each colour $i \in I$, then we say $\{(V_i, <_i) : i \in I\}$ is a *track assignment* of G *indexed by* I. Note that at times it will be convenient to also refer to a colour $i \in I$ and the colour class V_i as a *track*. The precise meaning will always be clear from the context. A *t-track assignment* is a track assignment with t tracks.

As illustrated in Figure 2, an *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct tracks V_i and V_j. A t-track assignment with no X-crossing is called a *t-track layout*. The *track-number* of a graph G, denoted by $\mathsf{tn}(G)$, is the minimum t such that G has a t-track layout.

Let $\{(V_i, <_i) : i \in I\}$ be a t-track layout of a graph G. The *span* of an edge vw of G, with respect to a numbering of the tracks $I = \{1, 2, \ldots, t\}$, is defined to be $|i - j|$ where $v \in V_i$ and $w \in V_j$.

Lemma 1. *If a graph G has an $A \times B \times C$ 3D grid drawing, then G has a $2AB$-track layout.*

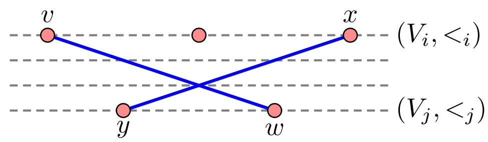

Figure 2: An example of an X-crossing in a track assignment.

Proof. Let $V_{x,y}$ be the set of vertices of G with an x-coordinate of x and a y-coordinate of y, where without loss of generality $1 \leqslant x \leqslant A$ and $1 \leqslant y \leqslant Y$. Order each set $V_{x,y}$ by the z-coordinates of its elements, $\{V_{x,y} : 1 \leqslant x \leqslant A, 1 \leqslant y \leqslant Y\}$. In the resulting ordered set $V_{x,y}$ there may be edges between consecutive vertices in the order (called intralayer edges) – resulting in "improper" AB-track assignment. Without these intralayer edges, there is no X-crossing however in such "improper" track assignment, as otherwise there would be a crossing in the original drawing. To make this track assignment proper (that is to get rid of intralayer edges) we move every second vertex from $V_{x,y}$ to $V'_{x,y}$ such that $V'_{x,y}$ inherits its total order from the original $V'_{x,y}$. The resulting layout is $2AB$-track layout. □

We now prove the converse of Lemma 1. The proof is inspired by the generalisations of the moment curve algorithm by Cohen *et al.* [11] and Pach *et al.* [49], described in Section 1.1. Loosely speaking, Cohen *et al.* [11] allow three 'free' dimensions, whereas Pach *et al.* [49] use the assignment of vertices to colour classes to 'fix' one dimension with two dimensions free. We use an assignment of vertices to tracks to fix two dimensions with one dimension free.

Lemma 2. *If a graph G has a k-track layout, then G has a $k \times 2k \times 2k \cdot n'$ three-dimensional drawing, where n' is the maximum number of vertices in a track.*

Proof. Suppose $\{(V_i, <_i) : 1 \leqslant i \leqslant k\}$ is the given k-track layout. Let p be the smallest prime such that $p > k$. Then $p \leqslant 2k$ by Bertrand's postulate. For each i, $1 \leqslant i \leqslant k$, represent the vertices in V_i by the grid-points

$$\{(i, i^2 \bmod p, t) : 1 \leqslant t \leqslant p \cdot |V_i|, t \equiv i^3 \pmod p)\} ,$$

such that the Z-coordinates respect the given total order $<_i$. Draw each edge as a line-segment between its end-vertices. Suppose two edges e and e' cross such that their end-vertices are at distinct points $(i_\alpha, i_\alpha^2 \bmod p, t_\alpha)$, $1 \leqslant \alpha \leqslant 4$. Then these points are coplanar, and if M is the matrix

$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \bmod p & t_1 \\ 1 & i_2 & i_2^2 \bmod p & t_2 \\ 1 & i_3 & i_3^2 \bmod p & t_3 \\ 1 & i_4 & i_4^2 \bmod p & t_4 \end{pmatrix}$$

then the determinant $\det(M) = 0$. We proceed by considering the number of distinct tracks $N = |\{i_1, i_2, i_3, i_4\}|$.

- $N = 1$: By the definition of an track layout, e and e' do not cross.
- $N = 2$: If either edge is intra-track then e and e' do not cross. Otherwise neither edge is intra-track, and since there is no X-crossing, e and e' do not cross.

• $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $\det(M) = (t_2 - t_1) \cdot \det(M')$, where

$$M' = \begin{pmatrix} 1 & i_2 & i_2^2 \bmod p \\ 1 & i_3 & i_3^2 \bmod p \\ 1 & i_4 & i_4^2 \bmod p \end{pmatrix} .$$

Since $t_1 \neq t_2$, $\det(M') = 0$. However, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is non-zero since i_2, i_3 and i_4 are distinct and p is a prime, a contradiction.

• $N = 4$: Let M' be the matrix obtained from M by taking each entry modulo p. Then $\det(M') = 0$. Since $t_\alpha \equiv i_\alpha^3 \pmod{p}$, $1 \leqslant \alpha \leqslant 4$,

$$M' \equiv \begin{pmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{pmatrix} \pmod{p} .$$

Since each $i_\alpha < p$, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is non-zero since $i_\alpha \neq i_\beta$ and p is a prime. This contradiction proves there are no edge crossings. The produced drawing is at most $k \times 2k \times 2k \cdot n'$. □

Lemma 1 and Lemma 2 imply the following theorem.

We say that a family of graphs has an $\mathcal{O}(1)$ track-number, that is *bounded* track-number, if there exists a constant c such that every graph in the family has track-number at most c. Similarly we say that a family of graphs admits $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ 3D grid drawings if there exists a constant c' such that every n-vertex graph in the family has a $c' \times c' \times c' \cdot n$ 3D grid drawing.

Theorem 3. *A family of graphs has an $\mathcal{O}(1)$ track-number if and only if it admits an $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ 3D grid drawing.*

2.2 Track layouts and queue layouts: the relationship

Theorem 3 tells us that proving that planar graphs have an $\mathcal{O}(1)$ track-number would answer Problem 1.

Before attempting to answer if planar graphs have an $\mathcal{O}(1)$ track-number, I would like give that question some historical perspective. In the early 90's Heath and Rosenberg [46] introduced and studied the so called queue layouts of graphs. A *queue layout* of a graph G consists of a vertex-ordering σ of G, and a partition of $E(G)$ into *queues*, such that no two edges in the same queue are *nested* with respect to σ. That is, there are no edges vw and xy in a single queue with $v <_\sigma x <_\sigma y <_\sigma w$. See Figure 3 (a). A queue layout with q queues is called a q-queue layout and a graph that admits a q-queue layout is called a q-queue graph. The minimum number of queues[3] in a queue layout of G is called the *queue-number* of G, and is denoted by $\text{qn}(G)$. See Figure 3 (b) and (c).

[3]Queue layouts have been extensively studied [34, 41, 42, 46, 51, 56, 59, 60] with applications in parallel process scheduling, fault-tolerant processing, matrix computations, and sorting networks (see [51] for a survey). Queue layouts of directed acyclic graphs [4, 44, 45, 51] and posets [43, 51] have also been investigated.

Heath and Rosenberg [46] characterized 1-queue graphs as the 'arched levelled planar' graphs, and proved that it is NP-complete to recognize such graphs. In a follow-up work Heath *et al.* [42] conjectured that planar graphs have an $\mathcal{O}(1)$ (i.e., bounded) queue layout. That conjecture remained open for 27 years before being resolved with the tools that I will present in the next section, the same tools used to answer Problem 1. The full relationship between linear volume 3D grid drawings, $\mathcal{O}(1)$ track layouts and $\mathcal{O}(1)$ queue layouts will become clear shortly. In the 2000s, it was shown that Heath, Leighton and Rosenberg's [42] conjecture is equivalent to conjecturing that planar graphs have bounded track-number, as shown in the following two lemmas.

Lemma 4. *[28] For every graph G,* $\mathrm{qn}(G) \leqslant \mathsf{tn}(G) - 1$.

The proof of Lemma 4 simply puts the tracks one after another to produce a queue layout. For a simple example consider the transformation of a2-track layout to a1-queue layout in Figure 4

A (partial) converse to Lemma 4 is also true but more complex to prove.

Lemma 5 ([30])**.** *There is a function f such that $\mathsf{tn}(G) \leqslant f(\mathrm{qn}(G))$ for every graph G. In particular, every graph with queue-number at most k has track-number at most $4k \cdot 4^{k(2k-1)(4k-1)}$.*

Lemma 4 and Lemma 5 together say that queue-number and track-number are tied.

2.3 Track layouts and queue layouts of various graph classes

Recall that our goal is to answer Problem 1. Theorem 3, Lemma 4 and Lemma 5 together tell us that proving that planar graphs have $\mathcal{O}(1)$ track-number, or equivalently $\mathcal{O}(1)$ queue-number, implies an affirmative answer to Problem 1.

Let's try to get some intuition about how one can go about constructing track layouts and queue layouts of graphs. We start by considering the simplest of planar graphs, namely, trees.

Queue layouts and track layouts are inherently related to breadth-first (BFS) search. The BFS ordering of the vertices of a tree has no two nested edges, and thus defines a 1-queue layout, as illustrated in Figure 5.

The following result is implicit in the work of Felsner *et al.* [37].

Lemma 6 ([37])**.** *Every tree T has a 3-track layout.*

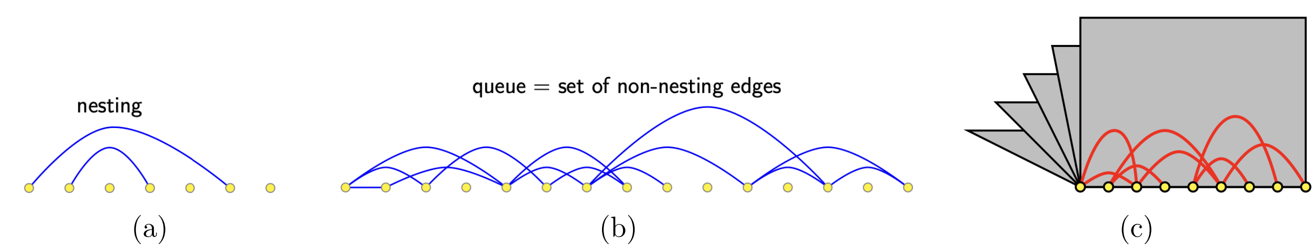

Figure 3: (a) Two nested edges. (b) 1-queue layout. (c) A way to visualize a 5-queue layout as having 1 queue in each page of a book.

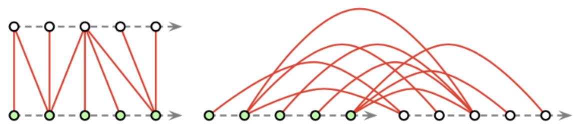

Figure 4: Converting a 2-track layout to 1-queue layout.

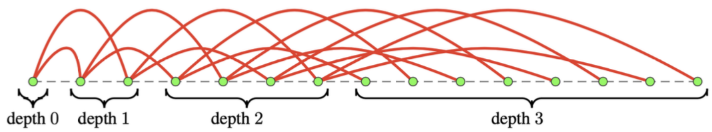

Figure 5: A 1-queue layout of a tree.

Proof by picture. Root T at an arbitrary node r. Start with a planar drawing of T where vertices at distance d from the root are on the line $y = -d$. To get a 1-queue layout of T order the vertices from top to bottom, left to right. To get a 3-track layout "wrap" the drawing onto 3 tracks as in Figure 6 □

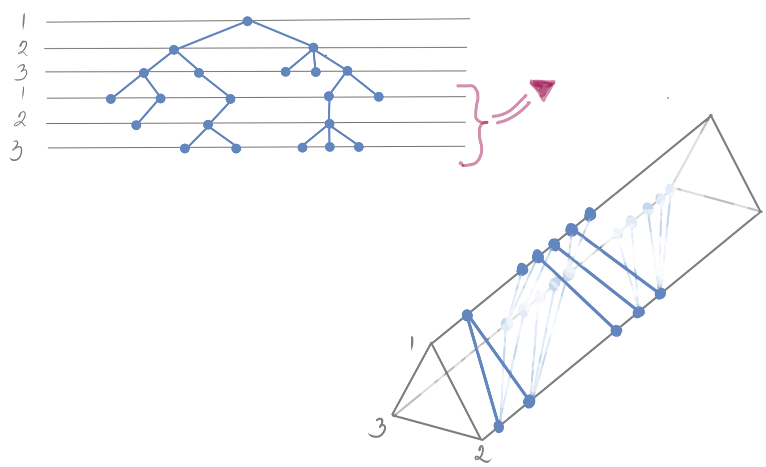

Figure 6: A 3-track layout of a tree.

The next simplest class of planar graphs to consider are outerplanar graphs. A graph is *outerplanar* if it has a planar drawing with all vertices on one face (e.g., the outer face).

Lemma 7 ([37])**.** *Every outerplanar graph has a* 6*-track layout and a* 2*-queue layout.*[4]

What trees and outerplanar graphs have in common is that they are, vaguely speaking, tree-like. More specifically, they have bounded *treewidth*. Trees have treewidth 1 and outerplanar graphs have treewidth 2. Treewidth, first defined by Halin [39], although largely unnoticed until independently rediscovered by Robertson and Seymour [57] and Arnborg and Proskurowski [3], is a measure of the similarity of a graph to a tree (see Section 2.5 for the definition). Treewidth plays a critical role in

[4]These results are implicit in [37] as their definition of track layouts are slightly different.

Robertson and Seymour's graph minor theory and in graph algorithms. Many problems that are NP complete on general graphs have polynomial-time solutions on graphs of bounded treewidth.

It turns out that graphs of bounded treewidth will play an important role in the resolution of Problem 1. We will define treewidth in Section 2.5 but here is why they may be useful. It has been known for a while that graphs of bounded treewidth have bounded queue number and track-number.

Lemma 8 ([28])**.** *Graphs of bounded treewidth have bounded track-number and bounded queue-number.*

However Lemma 8 does not resolve Problem 1 since there are planar graphs that have unbounded treewidth. For example it is known that the $\sqrt{n} \times \sqrt{n}$ planar grid graph has treewidth $\sqrt{n}$. However grids too have queue number 1 as illustrated in Figure 7 below.

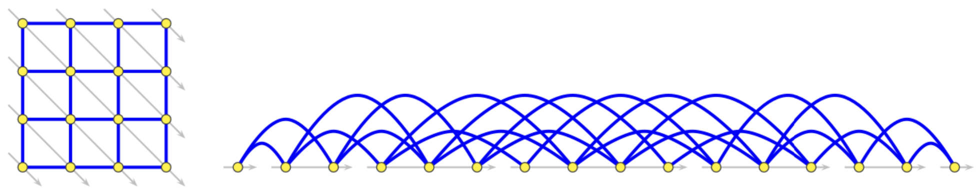

Figure 7: A 1-queue layout of a grid.

2.4 Track layouts and queue layouts via graph products

The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if:

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.

See Figure 8 for an illustration.

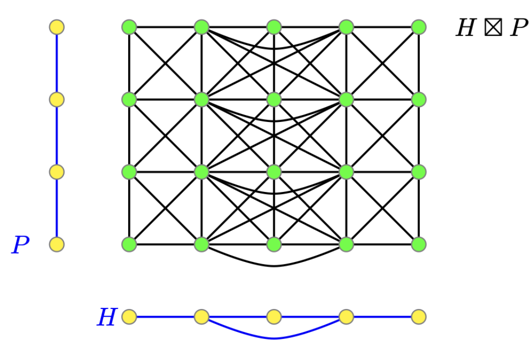

Figure 8: A strong product of a graph H and a path P.

Wood [64] proved the following lemma.

Lemma 9. *[64] For every graph H and every path P,* $\mathrm{qn}(H \boxtimes P) \leqslant 3\,\mathrm{qn}(H) + 1$ *(and thus there is a function f such that* $\mathsf{tn}(H \boxtimes P) \leqslant f(\mathsf{tn}(H))$*).*

Proof by picture. See Figure 9. □

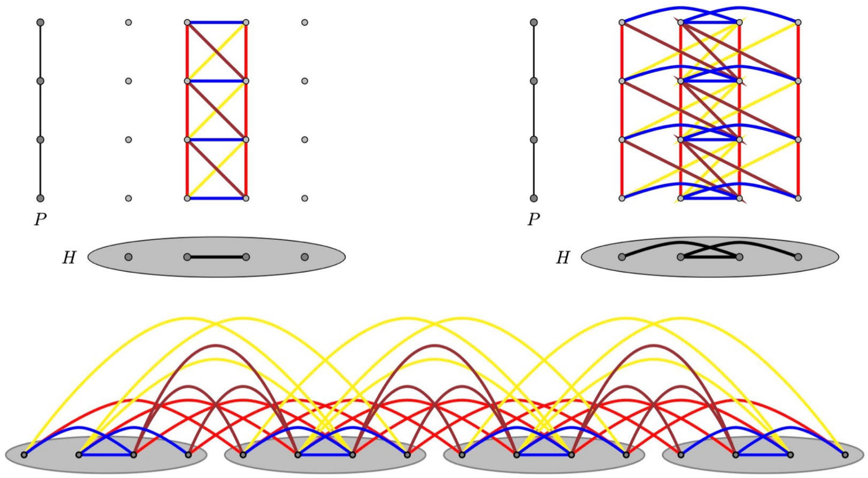

Figure 9: An illustration for the proof of Lemma 9

During a talk in Banff in 2008, Wood made, what he considered a wild conjecture. He conjectured something similar to the following statement: every planar graph is a subgraph of the strong product of a path and a bounded treewidth graph.[5] Eleven years later it turned out that this "wild" conjecture is true. Together with everything presented in the previous sections it not only answered in affirmative Problem 1 but also the 27-year old conjecture by Heath, Leighton and Rosenberg's [42] mentioned already and many other open problems that I did not get to mention.

The next section will be dedicated to proving that "wild conjecture". I should add that at the time we proved this structural result for planar graphs, we expressed it in terms of graph partitions (to be introduced) and it took us a while to realize that what in fact we also proved is an affirmative answer to the "wild" conjecture.

It is the main tool that allowed for resolution of Problem 1 and many other open proplems.

2.5 Product structure of planar graphs

This section is dedicated to proving the main tool used to resolve Problem 1. In particular this section proves the following theorem.

Theorem 10. *[24, 25] Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth at most 8 and some path P.*

Subsequently, the treewidth 8 bound was improved to 6 [62] and a similar theorem was proved for other classes of graphs [25, 29]. That, however, is beyond the scope of these lecture notes.

[5] To be precise, Wood [65] conjectured that for every planar graph G there are graphs X and Y, such that both X and Y have bounded treewidth, Y has bounded maximum degree, and G is a minor of $X \boxtimes Y$, such that the preimage of each vertex of G has bounded radius in $X \boxtimes Y$. Theorem 10 is stronger than this conjecture since it has a subgraph rather than a shallow minor, and Y is a path.

To prove this theorem we need to introduce some useful notation.

Throughout this Section 2, we use the notation $\overrightarrow{X}$ to refer to a particular linear ordering of a set X.

Layerings.

A *layering* of a graph G is an ordered partition $(V_0, V_1, \dots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. If $i = j$ then vw is an *intra-level* edge. If $|i - j| = 1$ then vw is an *inter-level* edge.

If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \text{dist}_G(r, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots)$ is called a *BFS layering* of G. Associated with a BFS layering is a *BFS spanning tree* T obtained by choosing, for each non-root vertex $v \in V_i$ with $i \geqslant 1$, a neighbour w in V_{i-1}, and adding the edge vw to T. Thus $\text{dist}_T(r, v) = \text{dist}_G(r, v)$ for each vertex v of G.

These notions extend to disconnected graphs. If $G_1, \dots, G_c$ are the components of G, and r_j is a vertex in G_j for $j \in \{1, \dots, c\}$, and $V_i := \bigcup_{j=1}^{c} \{v \in V(G_j) : \text{dist}_{G_j}(r_j, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots)$ is called a *BFS layering* of G.

Treewidth.

First we introduce the notion of H-decomposition and tree-decomposition. For graphs H and G, an *H-decomposition* of G consists of a collection $(B_x \subseteq V(G) : x \in V(H))$ of subsets of $V(G)$, called *bags*, indexed by the vertices of H, and with the following properties:

- for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H, and
- for every edge vw of G, there is a vertex $x \in V(H)$ for which $v, w \in B_x$.

The *width* of such an H-decomposition is $\max\{|B_x| : x \in V(H)\} - 1$. The elements of $V(H)$ are called *nodes*, while the elements of $V(G)$ are called *vertices*.

A *tree-decomposition* is a T-decomposition for some tree T. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G. See Figure 10 for illustration. Treewidth is particularly important in structural and algorithmic graph theory; see [5, 40, 55] for surveys.

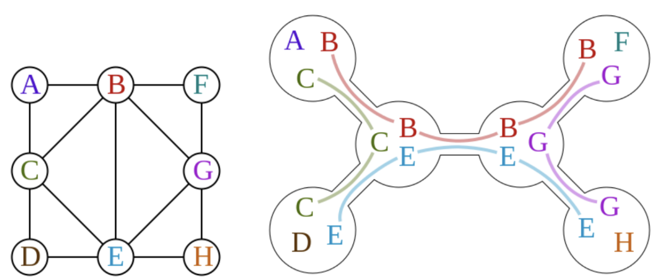

Figure 10: A graph and its tree-decomposition of width 2.

Partitions and Layered Partitions.

The following definitions are central notions in this chapter. A *vertex-partition*, or simply *partition*, of a graph G is a set $\mathcal{P}$ of non-empty sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* (sometimes called the *touching pattern*) of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B.

A partition of G is *connected* if the subgraph induced by each part is connected. In this case, the quotient is the minor of G obtained by contracting each part into a single vertex. Our results for queue layouts do not depend on the connectivity of partitions. But we consider it to be of independent interest that partitions constructed in this paper are connected. Then the quotient is a minor[6] of the original graph.

A partition $\mathcal{P}$ of a graph G is called an *H-partition* if H is a graph that contains a spanning subgraph isomorphic to the quotient $G/\mathcal{P}$. Alternatively, an *H-partition* of a graph G is a partition $(A_x : x \in V(H))$ of $V(G)$ indexed by the vertices of H, such that for every edge $vw \in E(G)$, if $v \in A_x$ and $w \in A_y$ then $x = y$ (and vw is called an *intra-bag* edge) or $xy \in E(H)$ (and vw is called an *inter-bag* edge). The *width* of such an H-partition is $\max\{|A_x| : x \in V(H)\}$. Note that a layering is equivalent to a path-partition.

A *tree-partition* is a T-partition for some tree T. Tree-partitions are well studied with several applications [6, 16, 17, 58, 66]. For example, every graph with treewidth k and maximum degree Δ has a tree-partition of width $O(k\Delta)$; see [16, 66].

A key innovation of this chapter is to consider a layered variant of partitions The *layered width* of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that for some layering $(V_0, V_1, \dots)$ of G, each part in $\mathcal{P}$ has at most ℓ vertices in each layer V_i.

Throughout this paper we consider partitions with bounded layered width such that the quotient has bounded treewidth. We therefore introduce the following definition. A class $\mathcal{G}$ of graphs is said to *admit bounded layered partitions* if there exist $k, \ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most ℓ such that $G/\mathcal{P}$ has treewidth at most k.

The next observation follows immediately from the definitions H-partition of layered width.

Observation 11. *For every graph H, a graph G has an H-partition of layered width at most ℓ if and only if G is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P.*

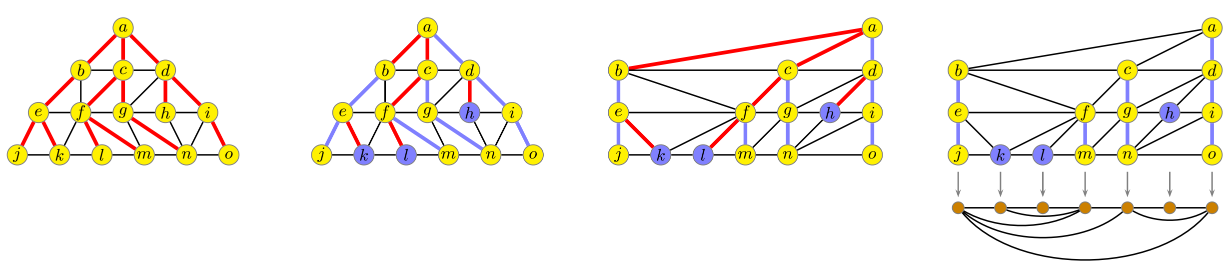

Figure 11: Illustration for Observation 11 of conversion of H-partition of layered width 1 to $H \boxtimes P$.

Thus to prove Theorem 10 it suffices to prove that every planar graph has an H-partition of bounded layered width where H has bounded treewidth.

Thus our goal is to show that planar graphs admit bounded layered partitions, which is achieved in the following key contribution.

Theorem 12. *Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $G/\mathcal{P}$ has treewidth at most 8. Moreover, there is such a partition for every BFS layering of G.*

[6]https://en.wikipedia.org/wiki/Graph_minor

We now set out to prove Theorem 12. The proof is inspired by the following elegant result of Pilipczuk and Siebertz [52]: Every planar graph G has a partition $\mathcal{P}$ into geodesics such that $G/\mathcal{P}$ has treewidth at most 8. Here, a *geodesic* is a path of minimum length between its endpoints. We consider the following particular type of geodesic. If T is a tree rooted at a vertex r, then a non-empty path $(x_1, \dots, x_p)$ in T is *vertical* if for some $d \geqslant 0$ for all $i \in \{1, \dots, p\}$ we have $\text{dist}_T(x_i, r) = d + i$. The vertex x_1 is called the *upper endpoint* of the path and x_p is its *lower endpoint*. Note that every vertical path in a BFS spanning tree is a geodesic. Thus the next theorem strengthens the result of Pilipczuk and Siebertz [52].

Theorem 13. *Let T_0 be a rooted spanning tree in a connected planar graph G_0. Then G_0 has a partition $\mathcal{P}$ into vertical paths in T_0 such that $G_0/\mathcal{P}$ has treewidth at most* 8.

Proof of Theorem 12 assuming Theorem 13. We may assume that G is connected (since if each component of G has the desired partition, then so does G). Let T be a BFS spanning tree of G. By Theorem 13, G has a partition $\mathcal{P}$ into vertical paths in T such that $G/\mathcal{P}$ has treewidth at most 8. Each path in $\mathcal{P}$ is connected and has at most one vertex in each BFS layer corresponding to T. Hence $\mathcal{P}$ is connected and has layered width 1. □

The proof of Theorem 13 is an inductive proof of a stronger statement given in Lemma 15 below. A *plane graph* is a graph embedded in the plane with no crossings. A *near-triangulation* is a plane graph, where the outer-face is a simple cycle, and every internal face is a triangle. For a cycle C, we write $C = [P_1, \dots, P_k]$ if $P_1, \dots, P_k$ are pairwise disjoint non-empty paths in C, and the endpoints of each path P_i can be labelled x_i and y_i so that $y_i x_{i+1} \in E(C)$ for $i \in \{1, \dots, k\}$, where x_{k+1} means x_1. This implies that $V(C) = \bigcup_{i=1}^{k} V(P_i)$.

Proof of Theorem 13 assuming Lemma 15. Let v be the root of T_0. Let G be a plane triangulation containing G_0 as a spanning subgraph with v on the outer-face of G. Let G^+ be the plane triangulation obtained from G by adding one new vertex r into the outer-face of G and adjacent to every vertex on the boundary of the outer-face of G. Let T be the spanning tree of G^+ obtained from T_0 by adding r and the edge rv. Consider T to be rooted at r. The three vertices on the outer-face of G are vertical (singleton) paths in T. Thus G satisfies the assumptions of Lemma 15, which implies that G has a partition $\mathcal{P}$ into vertical paths in T such that $G/\mathcal{P}$ has treewidth at most 8. Note that $G_0/\mathcal{P}$ is a subgraph of $G/\mathcal{P}$ (since $G_0 \subseteq G$ and $T[V(G_0)] = T_0$). Hence $G_0/\mathcal{P}$ has treewidth at most 8. □

Our proof of Lemma 15 employs the following well-known variation of Sperner's Lemma (see [2]):

Lemma 14 (Sperner's Lemma)**.** *Let G be a near-triangulation whose vertices are coloured* $1, 2, 3$*, with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is coloured i. Then G contains an internal face whose vertices are coloured* $1, 2, 3$.

Lemma 15. *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the outer-face of G^+, and let $P_1, \dots, P_k$ for some $k \in \{1, 2, \dots, 6\}$, be pairwise disjoint vertical paths in T such that $F = [P_1, \dots, P_k]$ is a cycle in G^+. Let G be the near-triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F.*

Then G has a partition $\mathcal{P}$ into vertical paths in T where $P_1, \dots, P_k \in \mathcal{P}$, such that the quotient $H := G/\mathcal{P}$ is planar and has a tree-decomposition in which every bag has size at most 9 and some bag contains all the vertices of H corresponding to $P_1, \dots, P_k$.

Proof. The proof is by induction on $n = |V(G)|$. If $n = 3$, then G is a 3-cycle and $k \leqslant 3$. The partition into vertical paths is $\mathcal{P} = \{P_1, \dots, P_k\}$. The tree-decomposition of H consists of a single bag that contains the $k \leqslant 3$ vertices corresponding to $P_1, \dots, P_k$.

For $n > 3$ we wish to make use of Sperner's Lemma on some (not necessarily proper) 3-colouring of the vertices of G. We begin by colouring the vertices of F, as illustrated in Figure 12. There are three cases to consider:

1. If $k = 1$ then, since F is a cycle, P_1 has at least three vertices, so $P_1 = [v, P'_1, w]$ for two distinct vertices v and w. We set $R_1 := v$, $R_2 := P'_1$ and $R_3 := w$.
2. If $k = 2$ then we may assume without loss of generality that P_1 has at least two vertices so $P_1 = [v, P'_1]$. We set $R_1 := v$, $R_2 := P'_1$ and $R_3 := P_2$.
3. If $k \in \{3, 4, 5, 6\}$ then we group consecutive paths by taking $R_1 := [P_1, \dots, P_{\lfloor k/3 \rfloor}]$, $R_2 := [P_{\lfloor k/3 \rfloor + 1}, \dots, P_{\lfloor 2k/3 \rfloor}]$ and $R_3 := [P_{\lfloor 2k/3 \rfloor + 1}, \dots, P_k]$. Note that in this case each R_i consists of one or two of $P_1, \dots, P_k$.

For $i \in \{1, 2, 3\}$, colour each vertex in R_i by i. Now, for each remaining vertex v in G, consider the path P_v from v to the root of T. Since r is on the outer-face of G^+, P_v contains at least one vertex of F. If the first vertex of P_v that belongs to F is in R_i then assign the colour i to v. In this way we obtain a 3-colouring of the vertices of G that satisfies the conditions of Sperner's Lemma. Therefore, by Sperner's Lemma there exists a triangular face $\tau = v_1 v_2 v_3$ of G whose vertices are coloured $1, 2, 3$ respectively.

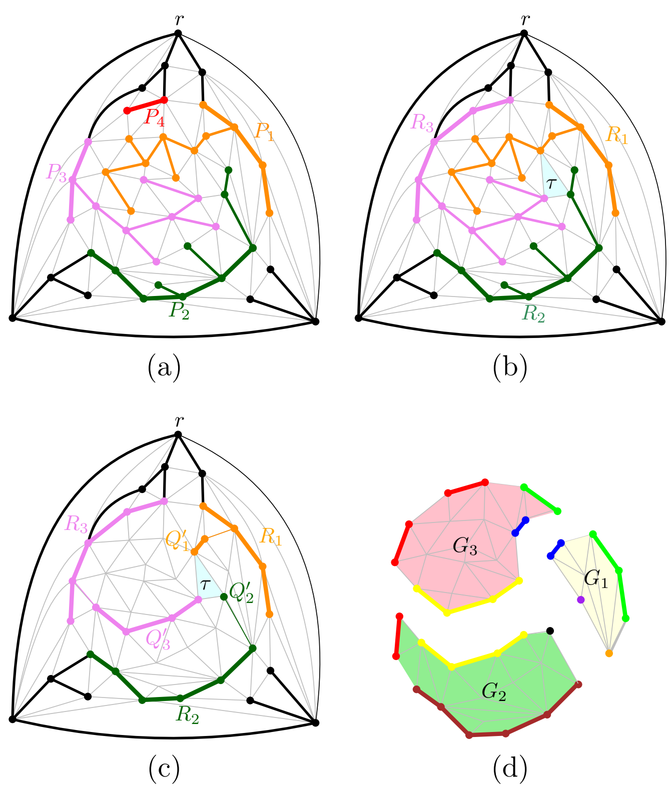

Figure 12: The inductive proof of Lemma 15: (a) the spanning tree T and the paths $P_1, \dots, P_4$; (b) the paths R_1, R_2, R_3, and the Sperner triangle τ; (c) the paths Q'_1, Q'_2 and Q'_3; (d) the near-triangulations G_1, G_2, and G_3, with the vertical paths of T on F_1, F_2, and F_3.

For each $i \in \{1,2,3\}$, let Q_i be the path in T from v_i to the first ancestor v'_i of v_i in T that is contained in F. Observe that Q_1, Q_2, and Q_3 are disjoint since Q_i consists only of vertices coloured i. Note that Q_i may consist of the single vertex $v_i = v'_i$. Let Q'_i be Q_i minus its final vertex v'_i. Imagine for a moment that cycle F is oriented clockwise, which defines an orientation of R_1, R_2 and R_3. Let R_i^- be the subpath of R_i that contains v'_i and all vertices that precede it, and let R_i^+ be the subpath of R_i that contains v'_i and all vertices that succeed it. Again, R_i^- and R_i^+ may be empty if v'_i is the first and/or last vertex of R_i.

Consider the subgraph of G that consists of the edges and vertices of F, the edges and vertices of τ, and the edges and vertices of $Q_1 \cup Q_2 \cup Q_3$. This graph has an outer-face, an inner face τ, and up to three more inner faces F_1, F_2, F_3 where $F_i = [Q'_i, R_i^+, R_{i+1}^-, Q'_{i+1}]$, where we use the convention that $Q_4 = Q_1$ and $R_4 = R_1$. Note that F_i may be *empty* in the sense that $[Q'_i, R_i^+, R_{i+1}^-, Q'_{i+1}]$ may consist of a single edge $v_i v_{i+1}$.

Consider any non-empty face $F_i = [Q'_i, R_i^+, R_{i+1}^-, Q'_{i+1}]$. Note that these four paths are pairwise disjoint, and thus F_i is a cycle. If Q'_i and Q'_{i+1} are non-empty, then each is a vertical path in T. Furthermore, each of R_i^- and R_{i+1}^+ consists of at most two vertical paths in T. Thus, F_i is the concatenation of at most six vertical paths in T. Let G_i be the near-triangulation consisting of all the edges and vertices of G^+ contained in F_i and the interior of F_i. Observe that G_i contains v_i and v_{i+1} but not the third vertex of τ. Therefore F_i satisfies the conditions of the lemma and has fewer than n vertices. So we may apply induction on F_i to obtain a partition $\mathcal{P}_i$ of G_i into vertical paths in T, such that $H_i := G_i/\mathcal{P}_i$ has a tree-decomposition $(B^i_x : x \in V(J_i))$ in which every bag has size at most 9, and some bag $B^i_{u_i}$ contains the vertices of H_i corresponding to the at most six vertical paths that form F_i. We do this for each $i \in \{1,2,3\}$ such that F_i is non-empty.

We now construct the desired partition $\mathcal{P}$ of G and tree-decomposition of H.

We start by defining $\mathcal{P}$. Initialise $\mathcal{P} := \{P_1, \dots, P_k\}$. Then add each Q'_i to $\mathcal{P}$, provided it is non-empty. Finally for $i \in \{1,2,3\}$, each path in $\mathcal{P}_i$ is either fully contained in F_i or it is an *internal path* with none of its vertices on F_i. Add all these internal paths of $\mathcal{P}_i$ to $\mathcal{P}$. By construction, $\mathcal{P}$ partitions $V(G)$ into vertical paths in T and it contains $P_1, \dots, P_k$.

The graph H obtained from G by contracting each path in $\mathcal{P}$ is planar since G is planar and $G[V(P)]$ is connected for each $P \in \mathcal{P}$.

Next we exhibit the desired tree-decomposition $(B_x : x \in V(J))$ of H. Let J be the tree obtained from the disjoint union of J_1, J_2 and J_3 by adding one new node u adjacent to u_1, u_2 and u_3. (Recall that u_i is the node of J_i for which the bag $B^i_{u_i}$ contains the vertices of H_i obtained by contracting the paths that form F_i.) For each node $x \in V(J_i)$, initialise $B_x := B^i_x$. Let the bag B_u contain all the vertices of H corresponding to $P_1, \dots, P_k, Q'_1, Q'_2, Q'_3$. It is helpful to think of J as being rooted at u. Since $k \leqslant 6$, $|B_u| \leqslant 9$.

The resulting structure, $(B_x : x \in V(J))$, is not yet a tree-decomposition of H since some bags may contain vertices of H_i that are not necessarily vertices of H (namely, vertices of H_i that are obtained by contracting paths in $\mathcal{P}_i$ that are on F.) We remedy that now. Recall that vertices of H_i, $i \in \{1,2,3\}$, correspond to contracted paths in $\mathcal{P}_i$. Each path $P \in \mathcal{P}_i$ that is in the cycle F is either a path P_j or a subpath of P_j for some $j \in \{1, \dots, k\}$. For each such path P, for $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to P by the vertex of H corresponding to P_j. This completes the description of $(B_x : x \in V(J))$. Clearly, $|B_x| \leqslant 9$ for every $x \in V(J)$. In remains to prove that $(B_x : x \in V(J))$ is indeed a tree-decomposition of H.

We first show that the above renaming of vertices does not cause any problems. In particular, it is possible that some pair of distinct vertices of H_i is replaced by a single vertex of H corresponding to some path P_j. However, by construction, this only happens for two vertices of H_i that correspond to two consecutive paths of $\mathcal{P}_i$ on F, thus these two vertices are adjacent in H_i. Consequently, the two subtrees of J_i whose corresponding bags contain these two vertices have at least one node in common and thus the set of nodes of J whose bags contain the vertex corresponding to P_j is a subtree of J. In fact, renaming these two vertices is equivalent to contracting the edge between them in H_i. Similarly, if there is an edge between a pair of vertices in H_i then some bag B^i_x contains both of these vertices and therefore some bag B_x (where $x \in V(J)$) contains the corresponding vertex or vertices of H.

Now we are ready to show that, for each vertex a of H, the set $\{x \in V(J) : a \in B_x\}$ forms a subtree of J. The only vertices of G that may appear in G_i and G_j for $i \neq j$ are those in $P_1, \ldots, P_k, Q_1, Q_2, Q_3$. The vertices of H obtained by contracting each of these paths are the only vertices of H that may appear in more than one of our tree-decompositions of G_1, G_2 and G_3. The bag B_u contains all of these vertices. If one such vertex a appears in the tree-decomposition of G_i for some $i \in \{1, 2, 3\}$, then the set of nodes of J_i whose bags contain a is a subtree of J_i by the above explanation on the effects of vertex replacing. The vertex a is in $B^i_{u_i}$ and is in B_u. Since u and u_i are adjacent in J, the set of nodes of J whose bags contain a is a subtree of J.

Finally we show that, for every edge ab of H, there is a bag B_x that contains a and b. If a and b are both obtained by contracting any of $P_1, \ldots, P_k, Q_1, Q_2, Q_3$, then a and b both appear in B_u. If a and b are both in H_i for some $i \in \{1, 2, 3\}$, then some bag B^i_x contains both a and b, by the above explanation on the effects of vertex replacing. The only possibility that remains is that a is obtained by contracting a path P_a in $G_i - V(F_i)$ and b is obtained by contracting a path P_b not in G_i. But in this case F_i separates P_a from P_b so the edge ab is not present in H. □

Theorem 12 and Observation 11 prove Theorem 10.

2.6 Solution to Problem 1 and other applications

Corollary 1. *[25] Problem 1 has an affirmative answer. In particular, every n-vertex planar graph G has a 3D grid drawing with $\mathcal{O}(n)$ volume.*

Proof. Theorem 12 and Observation 11 imply Theorem 10. Theorem 10, Lemma 9 and Lemma 8 imply that planar graphs have bounded track-number. Then Theorem 3 implies the claimed result. □

Similarly, the results presented in the previous sections solve the 27-year old problem by Heath, Leighton and Rosenberg's [42].

Corollary 2. *[25] Every planar graph has bounded queue number.*

Proof. Theorem 12 and Observation 11 imply Theorem 10. Theorem 10, Lemma 9 and Lemma 8 imply that planar graphs have bounded queue-number. □

Theorem 3 and Corollary 2 imply the following result.

Corollary 3. *[25] Every planar graph has bounded track-number.*

For more applications and generalizations of product structure theory see: [7, 9, 14, 19–21, 29]

3 Collinear sets – a tool for Problem 2

The goal of this section is to solve Problem 2 and in the process introduce a number of useful geometric and graph theoretic tools. We start with a very simple observation – yet one that raises two key questions about untangling of geometric planar graphs.

Lemma 16. *Let $\overline{H}$ be an untangling of some geometric planar graph $\overline{G}$. Let R be a set of vertices of $\overline{G}$ such that each vertex of R is on the* y*-axis in $\overline{H}$ and has the same* y*-coordinate in $\overline{H}$ as in $\overline{G}$. Then there exists an untangling $\overline{H'}$ of $\overline{G}$ in which the vertices in R are fixed.*

Proof. The proof uses the fact that it is possible to perturb the vertices of a crossing-free geometric graph without introducing crossings. More precisely, for any crossing-free geometric graph there exists a value $\varepsilon > 0$ such that each vertex can be moved a distance of at most ε, and the resulting geometric graph is also crossing-free. The maximum value ε for which this property holds is called the *tolerance* of the arrangement of segments. This concept, both for the geometric realization and the combinatorial meaning of the graphs was systematically studied in [1, 53].

Consider the untangling $\overline{H}$ of $\overline{G}$ and let $\varepsilon > 0$ be the tolerance of $\overline{H}$. Let X denote the maximum absolute value of an x-coordinate in $\overline{G}$ of a vertex in R. Let $\overline{H''}$ be the geometric graph obtained from $\overline{H}$ as follows. For each vertex $v \in R$ positioned at (x, y) in $\overline{G}$, move v from $(0, \mathsf{y})$ in $\overline{H}$ to $(\mathsf{x}\varepsilon/X, \mathsf{y})$ in $\overline{H''}$. The vertices not in R are unmoved. So each vertex moves a distance of at most ε, and $\overline{H''}$ is crossing-free. Scale $\overline{H''}$ by multiplying the x-coordinates of all vertices in $\overline{H''}$ by X/ε to obtain a crossing-free geometric graph $\overline{H'}$. Then every vertex of R has the same location in $\overline{H'}$ as it does in $\overline{G}$. Thus $\overline{H'}$ is an untangling of $\overline{G}$ that keeps the vertices of R fixed. □

A *straight-line crossing-free drawing* of a planar graph G is any crossing-free geometric graph isomorphic to G. The previous lemma hints at a possible relationship between Problem 2 and an existence of straight-line crossing-free drawings with many points on one line. Let's explore that direction. Suppose we are given a geometric planar graph $\overline{G}$ to untangle. Suppose that its underlying graph G has a straight-line crossing-free drawing with, vaguely speaking, lots of vertices on a line. Does Lemma 16 imply that $\overline{G}$ can be untangled while keeping lots of vertices fixed? Let's be more precise.

A set of vertices $S \subseteq V(G)$ in a planar graph G is a *collinear set* if G has a straight-line crossing-free drawing in which all vertices in S are mapped to a single line, see Figure 13. For ease of presentation we will always assume that that line is the y-axis. That can always be achieved by appropriate rotation of the drawing. Let $\bar{v}(G)$ denote the size of the largest collinear set of a planar graph G.

We are now ready to ask a more precise question. Does there exist an unbounded increasing function f such that for every geometric planar graph $\overline{G}$ (with the underlying graph G) can be untangled while keeping $f(\bar{v}(G))$ vertices fixed? While hinting at a possible relationship, Lemma 16 does not answer this question. The collinear set implicit in Lemma 16, the set R, must meet an extra condition (and a strong one at that) that each vertex in R has the same y-coordinate in a straight-line crossing-free drawing $\overline{H}$ of G as in $\overline{G}$.

That motivates the following notion. A set $R \subseteq V(G)$ is a *free collinear set* if there exists a total order $<_R$ of R (called a *good ordering*) such that, given any set of $|R|$ points on a line ℓ (assume again that the line is the y-axis), graph G has a straight-line crossing-free drawing where the vertices in R are mapped to the given points on ℓ and their order on ℓ matches the order $<_R$. Let $\tilde{v}(G)$ denote the size of the largest free collinear set of a planar graph G. Clearly every free collinear set is a collinear set, and thus $\tilde{v}(G) \leqslant \bar{v}(G)$, for every planar graph G. The other way around is not clear, in fact it is

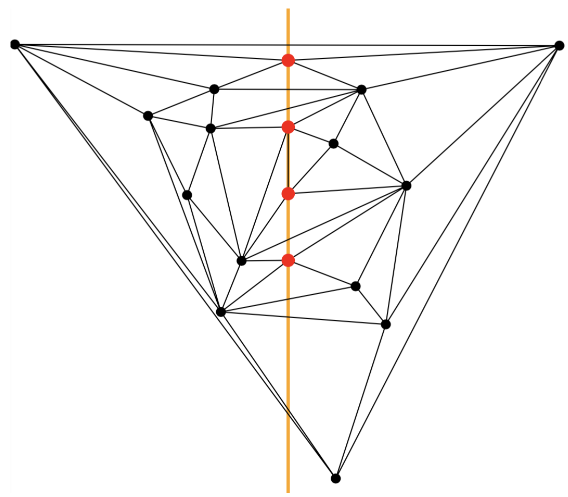

Figure 13: The 4 red vertices form a collinear set S.

not even clear if there exists an increasing unbounded function g such that for every planar graph G, $\tilde{v}(G) \geqslant g(\bar{v}(G))$.

Problem ($\star$). *Does there exists an increasing unbounded function g such that for every planar graph G, $\tilde{v}(G) \geqslant g(\bar{v}(G))$?*

Let's leave this key question aside for now and demonstrate first the usefulness of free collinear set for Problem 2 in the next lemma.

Lemma 17. [8] *Let $\overline{G}$ be a geometric planar graph and G its underlying graph. $\overline{G}$ can be untangled while keeping $\sqrt{\tilde{v}(G)}$ vertices fixed.*

Proof. We may assume (by appropriate rotation of the coordinate system) that no two vertices in $\overline{G}$ have the same y-coordinates. Let σ denote the total order of vertices of G by their y-coordinates in $\overline{G}$.

Let R be a largest free collinear set of G and let $<_R$ denote its good ordering. By the Erdős-Szekeres Theorem [32], there exists $S \subseteq R$ such that $|S| \geqslant \sqrt{|R|}$ (and thus $|S| \geqslant \sqrt{\tilde{v}(G)}$), and S is monotonically increasing or monotonically decreasing in $<_R$ and σ. By appropriate rotation of the coordinate system, we may assume that S is monotonically increasing in $<_R$ and σ. By the definition of free collinear set, G has a straight-line crossing-free drawing where vertices of S are placed on the y-axis and each vertex in S has the same y-coordinate in that drawing as in $\overline{G}$. Lemma 16 then implies that G can be untangled while keeping all the vertices in S fixed. That completes the proof given that the size of S is at least $\sqrt{\tilde{v}(G)}$. □

This lemma implies that if there exists $\epsilon > 0$ such that every planar graph G has free collinear set of size $\tilde{v}(G) \geqslant n^\epsilon$, then Problem 2 has an affirmative answer.

How would one go about proving that planar graphs have such large free collinear sets? Free collinear sets do not seem the easiest objects to understand so maybe we can start by trying to prove that planar graphs have large collinear sets. The following section and its main lemma sheds light on how to look for collinear sets. Keep in mind though that we are yet to determine if collinear sets are helpful for finding free collinear sets, or more precisely we are yet to determine if Problem ($\star$) has an affirmative answer.

3.1 From a geometric to a topological problem

In this section we show that the problem of determining the existence of a large collinear set in a planar graph, which is geometric by definition, can be transformed into a purely topological problem.

Given a planar drawing Γ of a planar graph G, we say that an open simple (i.e., non-self-intersecting) curve λ is *good* for Γ if, for each edge e of G, curve λ either entirely contains e or has at most one point in common with e (if λ passes through an endpoint of e, that counts as a common point). See Figure 14 (a) for an example of a good curve. Planar graphs may have many planar drawings/embeddings. (Recall that planar drawings are equivalent to planar embeddings. Follow this link for more on graph embeddings.) A good curve in any one of these drawings/embeddings is an object of our interest. In particular, we say that a planar graph has a good curve if some planar embedding of that graph does. Given a planar embedding of G and a good curve λ in that embedding, we denote by $R_{G,\lambda}$ the only unbounded region of the plane defined by that embedding of G and λ. Curve λ is *proper* if both of its endpoints are incident to $R_{G,\lambda}$. Again, see Figure 14 (a) for an example of a proper good curve. We have the following.

Theorem 18. *A planar graph G has a straight-line crossing-free drawing with x collinear vertices, if and only if G has a proper good curve that passes through x vertices of G. Equivalently, G has a collinear set of size x, that is $\bar{v}(G) = x$, if and only if G has a proper good curve that passes through x vertices of G.*

Proof. For the necessity, assume that G has a straight-line crossing-free drawing Γ with x vertices lying on a common line ℓ. We transform ℓ into a straight-line segment λ by cutting off two disjoint half-lines of ℓ in the outer face of G. This immediately implies that λ is proper. Further, λ passes through x vertices of G since ℓ does. Finally, if an edge e has two common points with λ, then λ entirely contains it, since λ is a straight-line segment and since e is a straight-line segment in Γ.

For the sufficiency, assume that G has a proper good curve λ passing through x of its vertices; see Fig. 14(a). Augment G by adding to it (refer to Fig. 14(b)): (i) a dummy vertex at each proper crossing between an edge and λ; (ii) two dummy vertices at the endpoints a and b of λ; (iii) an edge between any two consecutive vertices of G along λ, which now represents a path $(a, \dots, b)$ of G; (iv) two dummy vertices d_1 and d_2 in $R_{G,\lambda}$; and (v) edges in $R_{G,\lambda}$ connecting each of d_1 and d_2 with each of a and b so that cycles $C_1 = (d_1, a, \dots, b)$ and $C_2 = (d_2, a, \dots, b)$ are embedded in this counter-clockwise and clockwise direction in G, respectively. For $i = 1, 2$, let G_i be the subgraph of G induced by the vertices of C_i or inside it. Triangulate the internal faces of G_i with dummy vertices and edges, so that there are no edges between non-consecutive vertices of C_i; indeed, these edges do not exist in the original graph G, given that λ is good.

Represent C_1 as a convex polygon Q_1 whose all vertices, except for d_1, lie along a horizontal line ℓ, with a to the left of b and d_1 above ℓ; see Fig. 14(c). Graph G_1 is triconnected, as it contains no edge between any two non-consecutive vertices of its only non-triangular face. Thus, a straight-line crossing-free drawing of G_1 in which C_1 is represented by Q_1 exists [61]. Analogously, represent C_2 as a convex polygon Q_2 whose all vertices, except for d_2, lie at the same points as in Q_1, with d_2 below ℓ. Construct a straight-line crossing-free drawing of G_2 in which C_2 is represented by Q_2.

Removing the dummy vertices and edges results in a planar drawing Γ of the original graph G in which each edge e is a y-monotone curve; see Fig. 14(d). In particular, the fact that λ crosses at most once e ensures that e is either a straight-line segment or is composed of two straight-line segments that are one below and one above ℓ and that share an endpoint on ℓ. A straight-line crossing-free

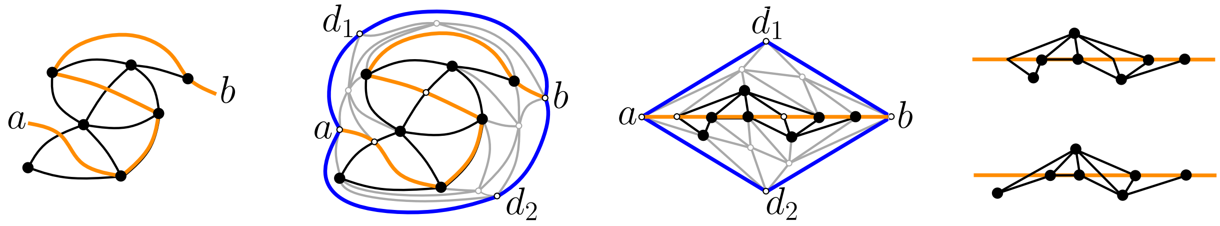

Figure 14: (a) A proper good curve λ (orange) for a planar graph G (black). (b) Augmentation of G with dummy vertices and edges. (c) A straight-line crossing-free drawing of the augmented graph G. (d) Planar polyline (top) and straight-line (bottom) crossing-drawings of the original graph G.

drawing Γ' of G in which the y-coordinate of each vertex is the same as in Γ always exists, as proved in [31, 50]. Since λ passes through x vertices of G, we have that x vertices of G lie along ℓ in Γ'. $\square$

3.2 From collinear sets to free collinear sets

Theorem 18 from the previous section armes us with a topological tool for finding collinear sets via proper good curves. Recall however that Lemma 17 states that we need free collinear sets to solve Problem 2. Thus before trying to use this topological tool, let's go back to the question if collinear sets (and thus proper good curves) are useful for finding free collinear sets. In other words, let's go back to Problem ($\star$). Without an affirmative answer to Problem ($\star$) this tool would not be helpful for us.

Surprisingly, not only is the answer to Problem ($\star$) affirmative, it is affirmative in the strongest possible sense (with equality), as stated in the next theorem. Before this theorem was proved, it was not even known if there is any function that bounds collinear sets by free collinear sets (that is, if Problem ($\star$) has an affirmative answer). The theorem also answers an open problem my Ravsky and Verbitsky [54].

Theorem 19. [22, 23] *Every collinear set is a free collinear set.*

The proof of this theorem can be found in [23]. The proof is fairly complex and lengthy. Its sketch may be given during the lectures, time permitting.

3.3 Proper good curves in (classes of) planar graphs

Now that we know that proper good curves with polynomially many vertices would answer Problem 2, let's study them in various classes of planar graphs. We start with the class of all planar graphs, as Problem 2 concerns those.

Theorem 20. [8] *Every n-vertex planar graph has a proper good curve with at least $\Omega(\sqrt{n})$ vertices.*

Proof setup. The proof will be presented during the lectures. It can also be found in [8]. The proof uses the notion of canonical orderings, a structure with many applications that we present here now.

It suffices to prove this theorem for edge-maximal geometric crossing-free graphs. Thus assume that G is edge-maximal.[7]

[7]A planar graph H is edge-maximal (also called, a *triangulation*), if for all $vw \notin E(H)$, the graph resulting from adding vw to H is not planar.

Let $\mathcal{E}$ be an embedded planar graph isomorphic to G. Each face of $\mathcal{E}$ is bounded by a 3-cycle. Canonical orderings of embedded edge-maximal planar graphs were introduced by de Fraysseix *et al.* [13]. They proved that $\mathcal{E}$ has a vertex ordering $\sigma = (v_1 := x, v_2 := y, v_3, \ldots, v_n := z)$, called a *canonical ordering*, with the following properties. Define G_i to be the embedded subgraph of $\mathcal{E}$ induced by $\{v_1, v_2, \ldots, v_i\}$. Let C_i be the subgraph of $\mathcal{E}$ induced by the edges on the boundary of the outer face of G_i. Then

- x, y and z are the vertices on the outer face of $\mathcal{E}$.
- For each $i \in \{3, 4, \ldots, n\}$, C_i is a cycle containing xy.
- For each $i \in \{3, 4, \ldots, n\}$, G_i is biconnected and *internally* 3-*connected*; that is, removing any two interior vertices of G_i does not disconnect it.
- For each $i \in \{3, 4, \ldots, n\}$, v_i is a vertex of C_i with at least two neighbours in C_{i-1}, and these neighbours are consecutive on C_{i-1}.

For example, the ordering in Figure 15(a) is a canonical ordering of the depicted embedded graph $\mathcal{E}$.

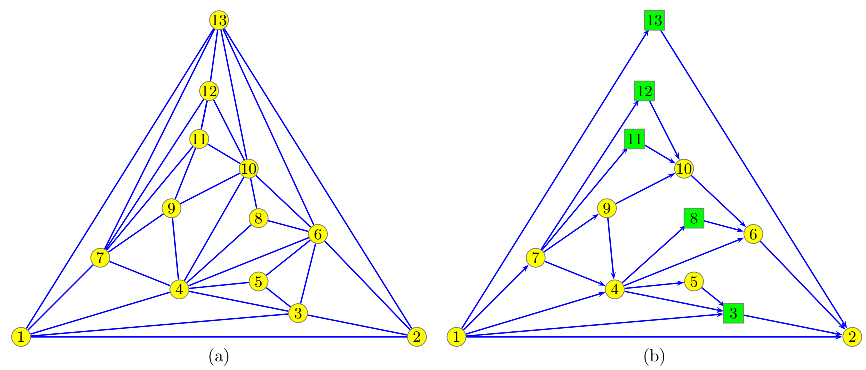

Figure 15: (a) Canonical ordering of $\mathcal{E}$, (b) Frame $\mathcal{F}$ of $\mathcal{E}$. Vertices forming a largest antichain in $<_{\mathcal{F}}$, that is the vertices in S, are depicted by squares.

We now introduce a new combinatorial structure that is critical to this theorem. A *frame* $\mathcal{F}$ of $\mathcal{E}$ is the oriented subgraph of $\mathcal{E}$ with vertex set $V(\mathcal{F}) := V(\mathcal{E})$, where:

- xy is in $E(\mathcal{F})$ and is oriented from x to y.
- For each $i \in \{3, 4, \ldots, n\}$ in the canonical ordering σ of $\mathcal{E}$, edges pv_i and v_ip' are in $E(\mathcal{F})$, where p and p' are the first and the last neighbour, respectively, of v_i along the path in C_{i-1} from x to y not containing edge xy. Edge pv_i is oriented from p to v_i, and edge v_ip' is oriented from v_i to p', as illustrated in Figure 15(b). We call p the *left predecessor* of v and p' the *right predecessor* of v.

The rest of the proof will be presented during the lectures. □

We now turn out attention to some subclasses of planar graphs.

Theorem 21. *Every n-vertex tree T has a proper good curve with at least $n/2$ vertices.*

Proof Sketch. Pick an arbitrary vertex of T to be the root of T. Construct a straight-line crossing-free drawing of T where the root is on the line $y = 0$, and where all the vertices at distance d from the root are on the line $y = -d$. Clearly the following two proper good curves exist in such a drawing of T: one that goes through all the vertices that lie on the lines $y = -d$ where d is even and the other proper good curve that goes through all the vertices that lie on the lines $y = -d$ where d is odd. One of those proper good curves contains at least $n/2$ vertices of T. That completes the proof. □

A graph is *outerplanar* if it has a planar drawing with all vertices on one face (e.g., the outer face).

Theorem 22. *Every n-vertex outerplanar graph has a proper good curve with at least $n/3$ vertices.*

Proof. Try to prove this. □

Theorem 23. [47] *Every n-vertex planar graph of treewidth at most* 3 *has a proper good curve with at least $\Omega(n)$ vertices.*

Proof. See [47] for the proof. □

Theorem 24. [47] *Every n-vertex triconnected cubic*[8] *planar graph has a proper good curve with at least $\Omega(n)$ vertices.*

Proof. See [47] for the proof. □

A natural question to consider is whether a linear bound is possible for all planar graphs. However that is not possible as observed first by Ravsky and Verbitsky [54].

Theorem 25. [54] *For infinitely many positive integers n, there exists an n-vertex planar graph such that every proper good curve has most $O(n^\sigma)$ vertices, where $\sigma < 0.986$.*

Proof sketch. Consider a planar triangulation G, on at least $n \geqslant 4$ vertices. A dual[9] of every such triangulation is a cubic planar graph. If G has a large collinear set (equivalently, a large proper good curve), then its dual graph has a cycle of proportional length. Since there are n-vertex triconnected cubic planar graphs whose longest cycle has length $O(n^\sigma)$ [38], it follows that there are n-vertex planar graphs in which the cardinality of every collinear set is $O(n^\sigma)$. Here σ is a known graph-theoretic constant called *shortness exponent*, for which the best known upper bound is $\sigma < 0.986$ [38]. □

The following theorem is proved using the connection between proper good curves of planar graphs and the length of the longest cycles in their duals.

Theorem 26. [26, 27] *Every n-vertex planar graph with maximum degree Δ has a proper good curve with at least $\Omega(n^{0.8}/\Delta^4)$ vertices.*

Proof. A sketch of this proof may be given during the lectures. See [27] for the proof. □

Theorem 20 and Theorem 26 suggest the following open problem.

[8] A graph G is triconnected if for every pair of distinct vertices in G, there are at least three internally disjoint paths between them. A graph is cubic if every vertex has degree exactly 3.

[9] Follow this link to read about duals of planar graphs.

Open Problem 1. *Every n vertex planar graph has proper good curve with at least $\Omega(n^\epsilon)$ vertices. By Theorem 20, $\epsilon \geqslant 1/2$ and by Theorem 25, $\epsilon < 0.986$. Can the lower bound 1/2 be improved? More strongly, can Δ in Theorem 26 be removed? What is the correct bound?*

3.4 Solution to Problem 2 and other applications

Corollary 4. *Problem 2 has an affirmative answer. In particular, every n-vertex geometric planar graph G can be untangled while keeping $\Omega(n^{1/4})$ vertex fixed.*

Proof. By Theorem 20, G has a proper good curve with at least $\Omega(\sqrt{n})$ vertices. By Theorem 18, that implies that G has a collinear set of the same size and thus of size at least $\Omega(\sqrt{n})$. By Theorem 19, G has a free collinear set of the same size, and thus of size at least s $\Omega(\sqrt{n})$. Finally, by Lemma 17, that implies that G can be untangled while keeping $\Omega(n^{1/4})$ vertices fixed. □

Similarly, other consequences of the results presented in the previous sections are:

Corollary 5. *Let G be an n-vertex geometric planar graph.*

(a) if G has bounded degree, then it can be untangled while keeping $\Omega(n^{0.4})$ vertex fixed.

(b) if G is a tree, or outerplanar or more generally has treewidth at most 3, then it can be untangled while keeping $\Omega(n^{1/2})$ vertex fixed. That bound is best possible.

(c) if G is a triconnected and cubic, it can be untangled while keeping $\Omega(n^{1/2})$ vertex fixed.

The proof that the bound in (b) is best possible can be found in [8]. The results in this section have a number of applications beyond untangling. Time permitting, they will be discussed during the lectures. See [18] for more examples.

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