Mathematics of Large Networks
Problem Sheet

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FIRST HOUR

1. *Graph Laplacians.*

   Consider an unweighted, undirected, simple network. Show that the smallest eigenvalue of the combinatorial graph Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is 0. How can one use the spectrum of the graph Laplacian to determine the number of components in the network? Do you have any ideas about how one might think about a graph that is "almost" separated into two disjoint components (and how one might measure how close the components are to being disconnected)?

2. *Modularity*

   (a) Apply modularity optimization techniques implemented in the library of your choice on some examples and visualise the results.

   (b) Write a function that takes a graph and its partition as an input and returns its modularity. Verify the values obtained in the previous exercise.

   (c) In the Louvain method, the efficiency of the algorithm partly resides in the fact that the variation of modularity Δ_{ij} obtained by moving a vertex i from its community to the community of one of its neighbors j can be calculated with only local information. In practice, the variation of modularity is calculated by removing i from its community $\Delta_{remove;i}$ (this is only done once) then inserting it into the community of j $\Delta_{insert;ij}$ for each neighbor j of i. The variation is therefore: $\Delta_{ij} = \Delta_{remove;i} + \Delta_{insert;ij}$. Derive analytically $\Delta_{remove;i}$ when removing node i from its community C_i.

   (d) Is it possible that the Louvain method produces communities that do not form connected components?

SECOND HOUR

1. *Dynamics, time-scales and Communities*

   (a) Ex.VII.1 : Write a code to simulate linear consensus dynamics on a network, and verify that the dynamics asymptotically converges towards the state $x_* = \mathbf{1}^\top \mathbf{x_0}/n$.

   (b) Ex.VII.2 : Write a code to reproduce the numerical results of the following figure.

2. *Random walks to reveal network structure*

   (a) Ex IX.1: Propose and justify a generalization of Markov stability in the case of directed networks.

   (b) Ex IX.2: Read "Comparing clusterings - an information based distance, M Meila, 2017", and implement numerically a method to compare different partitions. Compare the partitions obtained by maximising modularity and the Map Equation on the network of your choice.

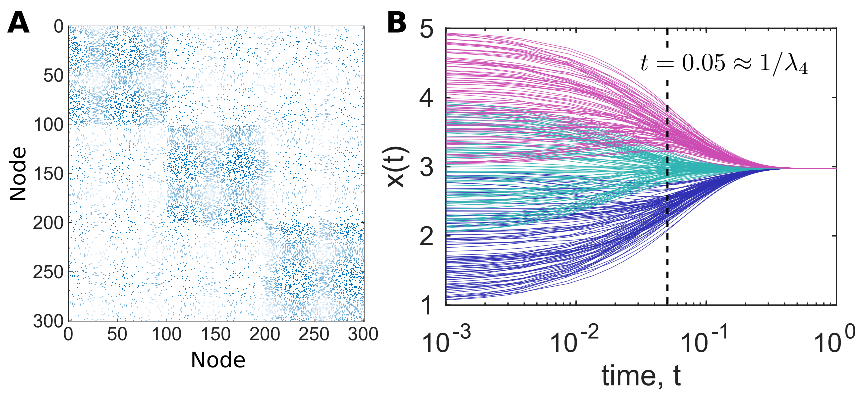

FIG. 1: **A** Adjacency matrix of a network with 3 groups **B** A consensus dynamics on this network displays a time-scale separation, as approximate consensus is reached within each group; then a consensus is reached between the groups.