Additive Combinatorics Methods in Fractal Geometry, Lecture 2

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Plan for the course

Lecture 1: Introduction to Additive Combinatorics.

Lecture 2: The Balog-Szemerédi-Gowers Theorem.

Lecture 3: Discretized Fractal Geometry.

Lecture 4: Sum-product and applications.

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1. Additive energy
2. The Balog-Szemerédi-Gowers Theorem
3. Convolutions

Recap: sumsets and Freiman's Theorem

- Recall: If A is a dense subset of a proper GAP, then $|A + A| \sim |A|$.
- By Freiman's Theorem, this is a characterization of sets with $|A + A| \sim |A|$.
- Sets A with $|A + A| \sim |A|$ are sometimes called sets with additive structure or approximate subgroups.
- We will now study a more analytical notion of additive structure: the additive energy.

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- We will now study a more analytical notion of additive structure: the additive energy.

Definition

The additive energy $E(A, B)$ between two sets A, B is

$$E(A,B) = \left|\{(x_1, x_2, y_1, y_2) \in A^2 \times B^2 : x_1 + y_1 = x_2 + y_2\}\right|$$

- Trivial lower bound: $|A||B| \leq E(A, B)$ since we always have the quadruples (x, x, y, y).
- Trivial upper bound: $E(A, B) \leq |A|^2|B|$, since once we have x_1, y_1, x_2, the value of y_2 is completely determined by $y_2 = x_1 + y_1 - x_2$.
- In particular, $|A|^2 \leq E(A, A) \leq |A|^3$.

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Additive structure through energy

We can think of sets A with $E(A, A) \sim |A|^3$ as sets with "additive structure". Examples:

- APs and GAPs.
- Dense subsets of APs and GAPs.
- Disjoint unions $A \cup B$ where $E(A, A) \sim |A|^3$ and B is arbitrary with $|B| = |A|$. Indeed, $E(A \cup B, A \cup B) \geq E(A, A)$.
- If $|B| = |A|$, where

$$E(A, A) \sim |A|^3,$$
$$|B + B| \sim |B|^2,$$

then $E(A \cup B, A \cup B) \sim |A \cup B|^3$ (because of A), but $|A \cup B + A \cup B| \sim |A \cup B|^2$ (because of B). This set has a lot of additive structure from the additive energy point of view, but none whatsoever from the sumset point of view.

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Two notions of additive structure

- Having small sumset and having large additive energy are indications of additive structure.
- But both the size of the sumset and the additive energy are increasing functions of A!
- So these notions appear to be extremely different from each other.
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\begin{aligned}
E(A,A) &= \left|\{(x_1,x_2,y_1,y_2)\in A^4 : x_1+y_1=x_2+y_2\}\right| \\
&= \sum_{z\in A+A} \left|\{(x_1,x_2,y_1,y_2)\in A^4 : x_1+y_1=x_2+y_2=z\}\right| \\
&= \sum_{z\in A+A} \left|\{(x,y)\in A^2 : x+y=z\}\right|^2 \\
&= \sum_{z\in A+A} \left|\{(x,y)\in A^2 : x=z-y\}\right|^2 \\
&= \sum_{z\in A+A} |A\cap(z-A)|^2 = \sum_{z\in A+A} N_A(z)^2,
\end{aligned}
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where

$$N_A(z) = |\{(x,y)\in A^2 : x+y=z\}| = |A\cap(z-A)|.$$

Small sumsets $\Rightarrow$ large energy

Lemma

$$E(A,A) \geq \frac{|A|^4}{|A+A|}.$$

Proof.

- We saw that, for $N_A(z) = |\{(x,y) \in A^2 : x+y=z\}|$,

$$E(A,A) = \sum_{z \in A+A} N_A(z)^2.$$

- But also, $|A|^2 = |A \times A| = \sum_{z \in A+A} N_A(z)$.
- Apply Cauchy-Schwarz! (or Jensen's inequality)

$$\left(|A|^2\right)^2 = \left(\sum_{z \in A+A} 1 \cdot N_A(z)\right)^2$$

$$\leq \left(\sum_{z \in A+A} 1^2\right)\left(\sum_{z \in A+A} N_A(z)^2\right) = |A+A| E(A,A).$$

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$$A \overset{G}{+} A := \{x + y : (x, y) \in G\}.$$

Remark

The full sumset $A + A$ corresponds to $G = A \times A$. More generally, $A' + A''$ where $A', A'' \subset A$ corresponds to G being a Cartesian product $A' \times A''$.

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$$A \overset{G}{+} A \leq 2K|A|.$$

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Proof.

This is the same Cauchy-Schwarz argument we saw for the full sumset:

$$|G|^2 = \left(\sum_{z \in A \overset{G}{+} A} 1 \cdot N_A(z) \right)^2 \leq \left(\sum_{z \in A \overset{G}{+} A} 1^2 \right) \left(\sum_{z \in A+A} N_A(z)^2 \right)$$

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Recap/Warmup: counting function

- Recall that

$$N_A(x) = |\{(a,b) \in A^2 : a+b = x\}| = |A \cap (x - A)|.$$

- We saw before that

$$|A|^2 = \sum_{x \in A+A} N_A(x),$$
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- The first equality generalizes as follows: if $S \subset A + A$, then

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$E(A,A) \ge |A|^3/K \implies \exists G \subset A \times A$ such that $|G| \ge |A|^2/2K$, and

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- $\sum_{x\in A+A} N_A(x)^2 = E(A,A) \ge |A|^3/K$.
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$$\sum_{x\in S} N_A(x)^2 = \sum_{x\in A+A} - \sum_{x\in A+A\setminus S} \ge \frac{|A|^3}{K} - \frac{|A|^3}{2K} = \frac{|A|^3}{2K}.$$

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Large additive energy $\Rightarrow$ small partial sumset

Lemma

$E(A,A) \geq |A|^3/K \Longrightarrow \exists G \subset A \times A$ *such that* $|G| \geq |A|^2/2K$, *and*

$$|A \overset{G}{+} A| \leq 2K|A|.$$

Proof.

- $\sum_{x \in A+A} N_A(x)^2 = E(A,A) \geq |A|^3/K$.
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Outline

- Additive energy is very natural for doing analysis. But it is easier to understand sets of small doubling (e.g. Freiman's Theorem).
- We have seen that small doubling implies large additive energy via Cauchy-Schwarz, but the reciprocal fails spectacularly.
- The examples of sets with additive energy $\sim |A|^3$ we have seen are of the form: a set with small doubling $\cup$ an arbitrary set of similar size. Are there any other examples?

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The Balog-Szemerédi-Gowers Theorem

Theorem (Balog-Szemerédi (1994), Gowers (1998), Schoen (2014))

There are constants $c, C > 0$ *such that the following holds. Suppose*

$$E(A, A) \geq \frac{|A|^3}{K}.$$

Then there exists $A' \subset A$ *such that:*

$$|A'| \geq \frac{c|A|}{K},$$
$$|A' + A'| \leq CK^4 |A'|.$$

In words: if A has “nearly maximal” additive energy, then it contains a “dense” subset A' with “nearly minimal” doubling.

Remarks and historical notes on BSG

- The proof of BSG is elementary and graph-theoretic.
- Balog and Szemerédi (1994) proved a non-quantitative form of the theorem.
- Gowers (1998) obtained polynomial bounds in K in his proof of a quantitative version of Szemerédi's Theorem for progressions of length 4.
- There is a very similar statement for two different sets A, B of similar size (for example, $B = -A$), but the bounds become meaningless if one set is much larger than the other. There is an asymmetric version of BSG that gives information if $\log|A|$ and $\log|B|$ are comparable.

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The following is a special case/corollary of the asymmetric version of the BSG theorem:

Theorem (Tao-Vu, based on ideas of Bourgain)

Given $\delta > 0$, there is $\varepsilon > 0$ such that the following holds for large enough N.
Let $A, B \subset \{1, \dots, N\}$ such that $E(A, B) \geq N^{-\varepsilon}|A||B|^2$.
Then there are sets $X, H \subset \{1, \dots, N\}$ such that:

- $|H + H| \leq N^{\delta}|H|$,
- $|A \cap (X + H)| \geq N^{-\delta}|A| \geq N^{-2\delta}|X||H|$,
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Corollary (of BSG and lemma)

If there is $G \subset A \times A$ such that

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then there is $A' \subset A$ such that

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The magic of BSG

- Cauchy-Schwarz tells us that small partial sumsets imply large additive energy. BSG allows us to reverse Cauchy-Schwarz (at the price of passing to a subset).
- The lemma we saw earlier tells us that large additive energy implies small partial sumsets. BSG allows us to replace the partial sumset by an honest sumset (again after passing to a subset). In other words, we can replace a "dense" subset G of $A \times A$ by a "dense" product set $A' \times A'$.
- Proving facts about sets with small sumsets is easier than proving facts about sets with large additive energy (e.g Freiman's Theorem). But in practice we often want a structural result about sets with large additive energy. BSG fills this gap.

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Outline

Definition

We work with finitely supported functions $f : Z \to \mathbb{R}$.

We define the L^p norms as $\|f\|_\infty = \max_x |f(x)|$ and

$$\|f\|_p^p = \sum_x f(x)^p.$$

The convolution of f and g is

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Additive energy as the L^2 norm of convolutions

Lemma

$$E(A,B) = \|\mathbf{1}_A * \mathbf{1}_B\|_2^2.$$

Proof.

Note that

$$\mathbf{1}_A * \mathbf{1}_B(z) = \sum_{(x,y):x+y=z} \mathbf{1}_A(x)\mathbf{1}_B(y)$$
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BSG as an inverse theorem for convolutions

- Young's inequality (convexity of $t \to t^p$) says that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

- In particular,

$$E(A, A)^{1/2} = \|1_A * 1_A\|_2 \leq \|1_A\|_1 \|1_A\|_2 = |A|^{3/2}.$$

- Thus, the inequality $E(A, A) \leq |A|^3$ is just a special case of Young's inequality.
- BSG can be seen as an inverse theorem for convolutions: if Young's inequality is "almost sharp" for $\|1_A * 1_A\|_2$, then A contains a dense subset with small doubling.
- Using pigeonholing and Hölder's inequality, one can extend this to more general functions and values of p.

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Conclusion

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