

Moduli spaces of algebraic varieties of general type

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Lecture 1: Lessons from history

Proof: Represent C as $\pi : C \rightarrow \mathbb{P}^1(x : y)$

degree = d , with simple ramification.

$R = 2g - 2 + 2d$ (now called Hurwitz formula)

varying the R -points: R -dim family.

overcount:

$L := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ varies in g -dim family (Abel, Jacobi)

picking $\pi^*x, \pi^*y \in H^0(C, L)$: $2(d + 1 - g) - 1$. (Riemann)

$$2g - 2 + 2d - g - 2(d + 1 - g) + 1 = 3g - 3.$$

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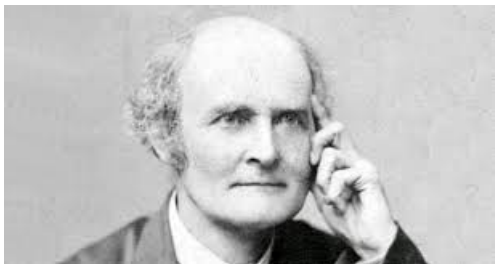
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$$2g - 2 + 2d - g - 2(d + 1 - g) + 1 = 3g - 3.$$

No claim that M_g is ...



Cayley (1862)

A new analytic representation of curves in space

- $C \subset \mathbb{P}^3 \mapsto \text{Cayley}(C) \subset \text{Grass}(\mathbb{P}^1, \mathbb{P}^3)$
 $\text{Cayley}(C) = \{\text{lines } L : L \cap C \neq \emptyset\}.$
- (Moduli of space curves) $\hookleftrightarrow$ (divisors on Grassmannian).
- Now Cayley form is called Chow form.

$$\mathrm{Univ}(\mathrm{Cayley}(C)) \subset \mathbb{P}^3 \times \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$$

Fibers of $\mathrm{Univ}(\mathrm{Cayley}(C)) \rightarrow \mathrm{Cayley}(C)$: lines.

Fibers of $\mathrm{Univ}(\mathrm{Cayley}(C)) \rightarrow \mathbb{P}^3$:

If $p \notin C$: projection of C from p .

If $p \in C$: all lines through p , hence $\cong \mathbb{P}^2$.

$\mathrm{Cayley}(C)$ 'knows' all projections of C .



Hurwitz (1891)

Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten

- Hurwitz space: branched covers of $\mathbb{CP}^1$,
- M_g is irreducible over $\mathbb{C}$.

(Char p : by Deligne-Mumford, 1969)



Klein (1897-1912 with Fricke)

Vorlesungen über die Theorie der automorphen Funktionen

- Riemann surfaces as quotients of the unit disc in $\mathbb{C}$,
- study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

Red Herring I

falsche Spur - fausse piste - falsa pista - félrevezető nyom

M_g = quotient of

$$\left\{ \begin{array}{l} \text{discrete, cocompact representations} \\ \pi_1(C) \rightarrow \mathrm{PGL}_2(\mathbb{R}) = \mathrm{Aut}(\text{unit disc}) \end{array} \right\} / \left\{ \begin{array}{l} \text{modulo} \\ \text{conjugation} \end{array} \right\}$$

This is a **real** manifold (of real dim $6g-6$)

Complex structure not natural, not considered much in
Fricke–Klein.

Forward looking Aside.

Weil knew that the

$H^{p,q} \subset H^{p+q}(X, \mathbb{C})$ vary **real** analytically with X .

Griffiths: the **filtration** varies **complex** analytically with X .



Severi (1915)

Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann

- return to algebraic theory: plane curves with nodes.
- M_g is unirational for $g \leq 10$.

No claim about the existence of M_g !

Theorem. For $g \leq 10$ there is a rational variety S_g parametrizing genus g curves such that all curves in an 'open, dense' subset of M_g appear in S_g .

Example: $S_3 \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ (smooth plane quartics).



Siegel (1935),

Über die analytische Theorie der quadratischen Formen

- A_g : analytic moduli of abelian varieties
- as quotient of the Siegel upper half plane.
- Reads like modern mathematics.

Red Herring II

Boundary points of A_g correspond to
lower dimensional abelian varieties.

Led to expectation:

Boundary points of M_g should correspond to
lower genus curves (possibly disconnected?).



Teichmüller (1944)

Veränderliche Riemannsche Flächen

- Teichmüller space T_g : Riemann surfaces
with marked generators of π_1 .
- Treats both complex structure and moduli functor.

Warning on compactifications

Many compactifications of moduli problems
look promising at first, but
give very badly behaved families.

Example: Genus 2 curves

- C : smooth, projective curve of genus 2
- C : smooth, compact Riemann surface of genus 2

Structure theorem. There is a unique $\tau : C \rightarrow \mathbb{P}^1$ of degree 2 ramified at 6 points.

Affine equation: $z^2 = f_6(x)$ (no multiple roots)

Corollary. M_2 , the set/space of all smooth, projective curves of genus 2 is

- $\{6 \text{ points in } \mathbb{P}^1\}/\mathrm{PGL}_2$, equivalently
- $(\mathrm{Sym}^6 \mathbb{P}^1 \setminus \text{diagonals})/\mathrm{PGL}_2$.

Compactifying M_2

Typical example: 4-fold root for $t = 0$:

$$f_6(x:y, t) = (x - ta_1y) \cdots (x - ta_4y)(x - a_5y)(x - a_6y)$$

Coordinate change $x = tx', y = y'$ and dividing by t^4 :

$$(x' - a_1y') \cdots (x' - a_4y')(tx' - a_5y')(tx' - a_6y')$$

which has only 2-fold root.

Lemma. Same trick achieves: at most triple root at $t = 0$.

Triple root case: $x^3(x - a_4y)(x - a_5y)(x - a_6y)$.

$x = tx', y = y'$ and dividing by $t^3 a_4 a_5 a_6$ we get

$$(x')^3 \left(\frac{t}{a_4} x' - y'\right) \left(\frac{t}{a_5} x' - y'\right) \left(\frac{t}{a_6} x' - y'\right).$$

For $t = 0$ this becomes

$(x')^3 (y')^3$: two triple roots.

GIT compactification $\bar{M}_2^{\text{GIT}}$

Points correspond to:

- : two triple roots (unique point) and
- : at most double roots.

Corresponding curves:

- : $z^2 = x^3(x-1)^3$ rational with 2 cusps.
- : at most double roots $z^2 = f_6(x)$.

Irreducible with at most nodes, except:

- $z^2 = x^2(x-1)^2(x+1)^2$. Contract one of the components:
rational with 1 triple point like the 3 coordinate axes.

End of old style story.

$\bar{M}_2^{\text{GIT}}$ is an unpleasant compactification.

- Stacky problem at $z^2 = x^2(x-1)^2(x+1)^2$.
- Local universal families:

At 2 cusp point $z^2 = x^3(x-1)^3$, deformations are

$$z^2 = (x^3 + ux + v)((x-1)^3 + s(x-1) + t).$$

Problem: $(u = v = 0)$ or $(s = t = 0)$ define disallowed curves.

Deligne–Mumford compactification $\bar{M}_2$

- at most double roots $z^2 = f_6(x)$: keep these
- $z^2 = x^2(x-1)^2(x+1)^2$: keep as is.
- $z^2 = x^3(x-1)^3$ **change to:**
double cover of pair of intersecting lines,
ramified at 3+3 pts plus the node:
= two elliptic curves meeting at a point.

Source of triple root problem: 3 **choices**

- contract one elliptic curve
- contract other elliptic curve
- blow up intersection point and contract both

Deligne–Mumford compactification $\bar{M}_g$

Stable curves:

Projective, connected, reduced curves C such that:

Local: at worst nodes: $(xy = 0)$ (locally analytically)

Global: ω_C is ample.

What is ω_C ?

– smooth curve: $\omega_C = \Omega_C = T_C^* = \mathcal{O}_C(K_C)$.

– for any plane curve, Poincaré residue map

$$\Re : \omega_{\mathbb{P}^2}(C)|_C \cong \omega_C$$

– if $C = \cup_i C_i$ and $P_i \subset C_i$ are the nodes then

$$\omega_C|_{C_i} = \omega_{C_i}(P_i).$$

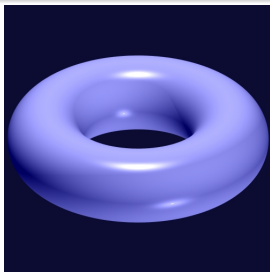
Red Herring III

Mumford and the others were too strong technically.
Made the 'linear' GIT work for canonical models, but
GIT breaks down at the boundary.

First hints: Mumford (1977) asymptotic stability of
singularities: seems pretty random class.

Definitive answer: Xiaowei Wang – Chenyang Xu (2012)
GIT compactification of the moduli of surfaces
forever depends on the pluricanonical embedding,
(both Chow and Hilbert versions).

Special homework



After plane and sphere,
this the 3rd best known algebraic surface.

What is it?

Lecture 2: Moduli of varieties in higher dimensions examples and definitions

Higher dimension, basic questions

What are the correct analogs of
smooth, projective curves of genus ≥ 2 ?

What are the correct analogs of stable curves?

Usually:

EASY: make it work for an **open** moduli space.

HARD: make it work for a **compact** moduli space.

Canonical models 1

Take any ω_X^m for $m \geq 1$.

Take any basis $s_0, \dots, s_{N(m)} \in H^0(X, \omega_X^m)$.

Get a map $\phi_m : X \dashrightarrow X_m \subset \mathbb{P}^{N(m)}$.

Theorem (Canonical models)

The closed images X_m are isomorphic to each other for $1 \parallel m$.

Get X^{can} , the *canonical model* of X .

- $\dim X = 2$: Castelnuovo, Enriques (+ Mumford)
- $\dim X = 3$: Mori (+ Kawamata, Kollár, Reid, Shokurov)
- $\dim X \geq 4$: Hacon–McKernan
(+ Birkar, Cascini, Corti, Shokurov)
(+ Fujino-Mori)

What is ω on a singular variety?

Recipe: (if X is normal)

Take smooth locus $X^0 \subset X$

$\omega_{X^0} = \Omega_{X^0}^n = (\det T_{X^0})^*$, then extend it to X .

Powers: $\omega_X^{[m]} :=$ extension of $\omega_{X^0}^m$.

- Hypersurfaces: $(g = 0) \subset \mathbb{C}^n$. Generator of ω :

$$(-1)^i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{\partial g / \partial x_i}$$

- Quotients: $\mathbb{C}^n / (\text{finite group } G)$. Generator of ω^m :

$$(dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$$

where $m = |G/G \cap \text{SL}_n|$.

Canonical models: internal definition

Canonical singularity: One can pull back pluricanonical forms: $p : Y \rightarrow X$ resolution, then we have

$$p^* \omega_X^{[m]} \rightarrow \omega_Y^{[m]}.$$

Canonical model: normal, projective,
– X has canonical singularities and
– ω_X is ample.

Compactification 2: Memento Mori

Lemma. B smooth curve, $B^0 = B \setminus \{0\}$
 $f^0 : Y^0 \rightarrow B^0$ a family of canonical models.

There is at most 1 extension to

$$\begin{array}{ccc} Y^0 & \subset & Y \\ f^0 \downarrow & & \downarrow f \\ B^0 & \subset & B \end{array}$$

such that

- Y has canonical singularities, and
- ω_Y (rather $\omega_{Y/B}$) is ample on every fiber.

Question: What about Y_0 ?

Converse problem

Example: $(xy = 0)$ is not canonical but
 $(xy + t^n = 0)$ is canonical $\forall n$.

Needed in general case: Assume

$0 \in D \subset X$ Cartier divisor, K_X $\mathbb{Q}$ -Cartier,

$X \setminus D$ has canonical sings, and

D has ????.

$\Rightarrow X$ has canonical sings.

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$\Rightarrow X$ has canonical sings.

Definition: ???? = semi-log-canonical.

What is semi-log-canonical?

What is a node?

$$C = (xy = 0) \subset \mathbb{C}^2.$$

generating section σ of ω_C

$$\sigma = \frac{dx}{x} \text{ on } x\text{-axis}, \quad \sigma = -\frac{dy}{y} \text{ on } y\text{-axis}.$$

Characterizations of nodes:

Using resolutions: $p : D \rightarrow C$ then $p^*\sigma$ has only simple poles.

Using local volume: Although the local volume is

$$\frac{i}{2\pi} \int_{|x| \leq 1} \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} = \infty,$$

it has only logarithmic growth:

$$\frac{i}{2\pi} \int_{|x| \leq 1} |x|^\epsilon \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} < \infty \quad \text{for } \epsilon > 0.$$

What is semi-log-canonical?

Take a resolution $f : Y \rightarrow X$. Write

$$K_Y = f^*K_X + J \text{ and } f^*D = D_Y + E.$$

- $J \geq 0$ iff X has canonical singularities and
- Mumford's semi-stable reduction: may assume that all coefficients in E equal 1.

Adjunction formula: $K_{D_Y} =$

$$(K_Y + D_Y)|_{D_Y} = (f^*(K_X + D) + J - E)|_{D_Y} = f^*K_D + (J - E)|_{D_Y}$$

Suggests: $J \geq 0 \iff (J - E)|_{D_Y} \geq -1.$

Almost what we want, but no information on exceptional divisors that are disjoint from D_Y .

Convexity of the coefficients of J settles the rest.

(Shokurov, Kollár, Kawakita, de Fernex-Kollár-Xu)

Detour: Reflexive powers

X normal (or satisfies Serre's S_2 condition)

L coherent, invertible outside a
codimension ≥ 2 subset $Z \subset X$.

Definition. $L^{[m]} :=$ reflexive hull of $L^{\otimes m}$.

Main case for us:

X has at worst nodes in codim 1, and

$L := \omega_X$.

Definition of semi-log-canonical

- X only nodes in codimension 1 (so ω_X makes sense),
- $\omega_X^{[m]}$ is locally free for some $m > 0$, and
- **Using resolutions:** $f : Y \rightarrow X$ with reduced exceptional divisor E , then we have pull-backs

$$f^*(\omega_X^{[r]}) \rightarrow \omega_Y^{[r]}(rE) \quad \forall r.$$

- **Using local volume:** σ^m : generating section of $\omega_X^{[m]}$.
Although the local volume $\int \sigma \wedge \bar{\sigma}$ can be ∞ , it has only logarithmic growth:

$$(i^?) \cdot \int |g|^\epsilon \cdot \sigma \wedge \bar{\sigma} < \infty$$

for every g vanishing on $\text{Sing } X$ and $\epsilon > 0$.

Moduli objects

Curve case.

- Interior: smooth, projective, ample K .
- Boundary: nodal, projective, ample K .

Surface case.

- Interior: Du Val (=ADE), projective, ample K .
- Boundary: semi-log-canonical, projective, ample K .

Higher dimensional case.

- Interior: canonical singularity, projective, ample K .
- Boundary: semi-log-canonical, projective, ample K .

Stable curve/surface/variety.

What are stable families?

Wrong answer:

Flat, projective morphisms with stable fibers.

Flatness not enough, example (i)

- Built on Franciosi-Pardini-Rollenske (2017)
- Uses: the cone over the degree 4 rational normal curve has $\mathbb{C}^2/\frac{1}{4}(1,1)$ singularity, and
2 types of smoothings:
 - to Veronese, with $(K^2) = 9$, and
 - to ruled surface with $(K^2) = 8$.

$(\mathbb{Z}/2)^2$ -covers

$S = S(f, g, h)$: composite of $\mathbb{P}^2(\sqrt{fg})$, $\mathbb{P}^2(\sqrt{gh})$, $\mathbb{P}^2(\sqrt{hf})$.

Generic local structure: $(f, g, h) = (x, y, 1)$.

composite of $\mathbb{C}^2[\sqrt{x}]$, $\mathbb{C}^2[\sqrt{y}]$: smooth

Special $(f, g, h) := (x, y, x - y)$: $\mathbb{C}_{uv}^2 / \frac{1}{4}(1, 1)$ with

$$x = (u^2 + v^2)^2, y = (u^2 - v^2)^2, x - y = (2uv)^2.$$

Flatness not enough, example (ii)

Pick $f, g, h \in \mathbb{C}[x, y, z]$, homog. of degrees 3, 3, 1.

$S = S(f, g, h)$: composite of $\mathbb{P}^2(\sqrt{fg}), \mathbb{P}^2(\sqrt{gh}), \mathbb{P}^2(\sqrt{hf})$.

$\mathbb{P}^2(\sqrt{fg})$ is a K3 (with 9 nodes), so

$K_S \sim C := \text{reduced preimage of } L := (h = 0)$.

Corollary. For general f, g, h , the surface S is smooth,
 K_S is ample and $(K_S^2) = 1$: Godeaux surface.

Flatness not enough, example (iii)

Special case: S_0 : when $f = g = h = 0$ has 2 points.

S_0 has 2 points $\mathbb{C}^2/\frac{1}{4}(1,1)$, both on C ,
and C is nodal at the 2 points.

Resolve: $S'_0 \rightarrow S_0$, with exceptional E_1, E_2 , and

$${}^{-4}_{E_1} = {}^{-1}_{C'} = {}^{-4}_{E_2}$$

We still have $K_{S'_0} \sim C'$.

Contract C' : get $S'_0 \rightarrow T_0$ and T_0 is a K3 surface!

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We still have $K_{S'_0} \sim C'$.

Contract C' : get $S'_0 \rightarrow T_0$ and T_0 is a K3 surface!

Lemma. Since S_0 has rational singularities, any flat deformation of S'_0 contracts to a flat deformation of S_0 .

Flatness not enough, example (iv)

Corollary. K_{S_0} is ample and S_0 has 2 kinds of flat smoothings:

$$\left\{ \begin{array}{l} \text{K3 surface} \\ \text{blown up once} \\ (K^2) = -1. \end{array} \right\} \rightsquigarrow S_0 \leftarrow \left\{ \begin{array}{l} \text{Godeaux surface} \\ K \text{ ample} \\ (K^2) = 1 \end{array} \right\}$$

The K3 surface can be **non-algebraic**.

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The K3 surface can be **non-algebraic**.

What distinguishes the two sides?

Bad deformation direction

Let $q : Y \rightarrow (0 \in B)$ represent

$$\left\{ \begin{array}{l} \text{K3 surface} \\ \text{blown up once} \end{array} \right\} \rightsquigarrow S_0$$

No multiple of $K_{Y/B}$ is Cartier, since $b \mapsto (K_{Y_b}^2)$ is not locally constant.

Good deformation direction

Let $p : X \rightarrow (0 \in B)$ represent

$$S_0 \leftarrow \left\{ \begin{array}{l} \text{Godeaux surface} \\ K \text{ ample, } (K^2) = 1 \end{array} \right\}.$$

- $b \mapsto (K_{X_b}^2)$ is locally constant, and
- $2K_{X/B}$ is Cartier: pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$.

Reflexive powers, relative case

$f : X \rightarrow S$ flat, fibers normal (or S_2)

L coherent, invertible outside $Z \subset X$ and

$X_s \cap Z$ has codimension ≥ 2 in $X_s \forall s \in S$.

Main technical question.

When are the $L^{[m]}$

- flat over S , and
- commute with further base changes $T \rightarrow S$?

Main case for us:

fibers at worst nodes in codim 1, and

$L := \omega_{X/S}$.

Detour: reflexive sheaves

X normal (or satisfies Serre's S_2 condition)

$Z \subset X$ has codimension ≥ 2

$j : U := X \setminus Z \hookrightarrow X$

F coherent sheaf on X , s.t. $F|_U$ is reflexive.

Equivalent

- (1) F is reflexive,
- (2) F satisfies Serre's S_2 condition
- (3) $F = j_*(F|_U)$,
- (4) $\text{depth}_Z F \geq 2$.

Detour: husks (= cover of an ear of corn)

X normal (or satisfies Serre's S_2 condition)

Definition. $\pi : F \rightarrow G$ map of coherent sheaves.

It is a **husk** of F if

G is reflexive, and

π isomorphism at codim 0 (= generic) points.

It is a **hull** of F if

G is reflexive, and

π isomorphism at codim ≤ 1 points.

Note: hull is unique, but can be many husks.

Detour: hulls and husks in families

$g : X \rightarrow S$ flat, projective, with normal (or S_2) fibers.

$\pi : F \rightarrow G$ map of coherent sheaves.

It is a relative **hull/husk** of F if

$\pi_s : F_s \rightarrow G_s$ is a hull/husk for every $s \in S$.

Detour: Hull and Husk I

$g : X \rightarrow S$ flat, projective, with normal (or S_2) fibers.

F coherent sheaf on X .

$h : T \rightarrow S$ gives $h_X : X \times_S T \rightarrow T$.

$\mathcal{H}ull(F)$ is the functor

$(h : T \rightarrow S) \mapsto \{\text{relative hulls of } h_X^* F\}$

$\mathcal{H}usk(F)$ is the functor

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Theorem (Kollár, 2008)

$g : X \rightarrow S$ flat, projective, with normal (or S_2) fibers.

F coherent sheaf on X .

Assume: $\exists Z \subset X$ such that

- $Z_s \subset X_s$ has codimension ≥ 2 for all $s \in S$, and
- $F|_{X \setminus Z}$ has reflexive fibers.

Then $\mathcal{H}usk(F)$ and $\mathcal{H}ull(F)$ are representable.

Detour: Hull and Husk II

Since hulls are unique:

Corollary

$\text{Hull}(F) \rightarrow S$ is a monomorphism.

Definition. $\tau : S' \rightarrow S$ is a monomorphism if for every X
 $\tau \circ : \text{Mor}(T, S') \rightarrow \text{Mor}(T, S)$ is injective.

Non-example 1: $C = \text{node}$, $C' = \text{normalization}$.

Example: $C = \text{node}$, $C' = \text{normalization} \setminus$ one preimage of node.

Non-example 2: $C = \text{cusp}$, $C' = \text{normalization}$.

Main Corollary for us

Theorem

Let $X \rightarrow S$ be a flat, proper morphisms with stable fibers.

There is a monomorphism $S' \rightarrow S$ such that

- all the $\omega_{X'/S'}^{[m]}$ are flat over S' and commute ...
- $S' \rightarrow S$ is universal with this property.

Proof: S' is the fiber product of all $\text{Hull}(\omega_{X/S}^{\otimes m}) \rightarrow S$.

Lecture 3: What are stable families?

Theorem (Kollár 2013)

Let $X \rightarrow S$ be a flat, proper morphisms with stable fibers, S reduced. Equivalent:

- 1 The volume of the fibers $(K_{X_s}^n)$ is locally constant.
- 2 The plurigenera $h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant $\forall m$.
- 3 $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.
- 4 The $\omega_{X_s}^{[m]}$ are the fibers of $\omega_{X/S}^{[m]}$.

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- 4 The $\omega_{X_s}^{[m]}$ are the fibers of $\omega_{X/S}^{[m]}$.

Aside: For ‘sufficiently divisible m ’ in (2) and (3) we need about the fibers only that:

S_2 (Serre's condition, eg. normal)

ω_{X_s} locally free in codim 1,

$\omega_{X_s}^{[m_s]}$ locally free and ample for some $m_s > 0$.

Clear:

(3) $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.

$\Rightarrow$

(4) The $\omega_{X_s}^{[m]}$ are the fibers of $\omega_{X/S}^{[m]}$.

(4) $\Rightarrow$ (2)

Assume (4): the $\omega_{X_s}^{[m]}$ form a flat family. $\Rightarrow$

(4') $\chi(X_s, \omega_{X_s}^{[m]})$ is locally constant.

Next: $h^0(X_s, \omega_{X_s}^{[m]}) = \chi(X_s, \omega_{X_s}^{[m]})$ if

either $m \gg 1$ (by Serre)

or $m \geq 2$ and stable fibers (by Ambro–Fujino vanishing)
(Kawamata–Viehweg not enough).

$m = 1$: Come back to this in Lecture 4.

(2) $\Rightarrow$ (3)

(2): $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.

If $\omega_{X/S}^{[M]}$ is locally free, then $\omega_{X/S}^{[m+rM]} \cong \omega_{X/S}^{[m]} \otimes (\omega_{X/S}^{[M]})^r$, so

Hilbert polynomial of $\omega_{X_s}^{[m]}$ is independent of s , so

(3): $\omega_{X/S}^{[m]}$ is flat and commutes with base change.

(2) $\Rightarrow$ (1)

(2): $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.

The $(K_{X_s}^n)$ are the leading terms, so

(1): $s \mapsto (K_{X_s}^n)$ is also locally constant.

(1) $\Rightarrow$ (2) slide 1

Theorem

Let $X \rightarrow S$ be flat, projective, with S_2 fibers; S reduced.
 L reflexive rank 1 sheaf, locally free in codim 1 on each fiber.
Assume that each $(L_s)^{**}$ is locally free and ample. Then

- ① $s \mapsto \text{volume}((L_s)^{**})$ is upper semicontinuous, and
- ② locally constant iff L is locally free.

Apply this to $\omega_{X/S}^{[m]}$ such that every $\omega_{X_s}^{[m]}$ is locally free.

Other m : Come back to this in Lecture 4.

Typical example (with divisors)

$$X = (xy - u^2 + t^2v^2 = 0) \subset \mathbb{P}^3 \times \mathbb{A}_t^1,$$

$$D := (x = u - tv = 0) + (y = u + tv = 0).$$

$t \neq 0$: X_t smooth, D_t Cartier and $(D_t^2) = 0$.

$t = 0$: X_0 cone, $D_0 = (u = 0)$ is Cartier, and $(D_0^2) = 2$.

Add hyperplane class:

$$(H_0 + D_0)^2 = 8, \quad (H_t + D_t)^2 = 6.$$

Note: D not Cartier at $x = y = u = t = 0$.

(1) $\Rightarrow$ (2) slide 2

$n = 2$ case: here cokernel of $L \rightarrow L_0^{**}$ is 0 dimensional, so

$$\chi(X_0, (L_0^{**})^m) \geq \chi(X_g, (L_g^{**})^m) = \chi(X_g, L_g^m).$$

Riemann-Roch:

$$\frac{1}{2}(L_0^{**} \cdot L_0^{**})m^2 + b_0m + \chi(X_0) \geq \frac{1}{2}(L_g \cdot L_g)m^2 + b_gm + \chi(X_g).$$

So $(L_0^{**} \cdot L_0^{**}) \geq (L_g \cdot L_g)$, and if $=$ then

$$b_0m \geq b_gm \quad \forall m \in \mathbb{Z}.$$

So $b_0 = b_g$.

(1) $\Rightarrow$ (2) slide 3

$n \geq 3$ induction (nontrivial) reduces to special case:
 L is locally free except at isolated points.

Local Grothendieck-Lefschetz: $x \in X_0 \subset X$ then

$\text{Pic}(X \setminus \{x\}) \hookrightarrow \text{Pic}(X_0 \setminus \{x\})$ if $\text{depth}_x X_0 \geq 3$.

Problem: we have only $\text{depth}_x X_0 \geq 2$.

Theorem. Still ok if $\text{depth}_x X_0 \geq 2$ and $\dim X_0 \geq 3$.

- slc case (K, 2013)
- normal case (Bhatt - de Jong, 2014)
- general case (K, 2016)
- see Stacks Tag 0FB2

Aside: Fulger - K - Lehmann, 2016

X normal, proper, D big $\mathbb{R}$ divisor, E effective $\mathbb{R}$ divisor.
Equivalent:

- $H^0(X, \mathcal{O}_{X \setminus m}(D - E)_{\perp}) = H^0(X, \mathcal{O}_{X \setminus m} D_{\perp}) \quad \forall m \geq 1.$
- $\text{volume}(D - E) = \text{volume}(D).$

We have proved:

Theorem

Let $X \rightarrow S$ be a flat, proper morphisms with stable fibers, S reduced. Equivalent:

- ① The volume of the fibers $(K_{X_s}^n)$ is locally constant.
- ② The plurigenera $h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant $\forall m$.
- ③ $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.
- ④ The $\omega_{X_s}^{[m]}$ are the fibers of $\omega_{X/S}^{[m]}$.

The correct moduli problem

Definition

A morphism $f : X \rightarrow S$ is **stable** iff

- 1 f is flat, projective with stable fibers, and
- 2 $\omega_{X/S}^{[m]}$ is flat and commutes with base changes $\forall m$.

Thesis

Stable morphisms are the correct higher dimensional analogs of flat families of stable curves of genus ≥ 2 .

- Warning: Serious problems in char p .

Terminology

- This is called KSB stable (= Kollár – Shepherd-Barron).
I am sure that this is the right notion.
- Later: pairs (X, Δ) where Δ is a $\mathbb{Q}$ or $\mathbb{R}$ -divisor.
There are several versions, they agree over reduced bases.
These are called KSBA stable
(= Kollár – Shepherd-Barron – Alexeev).
I am **not** sure that we have the right notion.
- The distinction is not systematic.

Moduli space of stable varieties

Main Theorem

The moduli functor of KSB-stable families of varieties has a

- *coarse moduli space*

(Kollár–Shepherd-Barron 1988, Kollár 2013)

- *connected components are proper*

(Alexeev 1993, Karu 2000, Hacon-McKernan-Xu 2018)

- *connected components are projective*

(Fujino 2013, Kovács-Patakfalvi 2017)

Detour for next lecture: cyclic covers 1

For $s \in H^0(X, L^{[m]})$ we have $\pi : X[\sqrt[m]{s}] \rightarrow X$ as

- $\text{Spec}_X(\mathcal{O}_X \oplus L^{[-1]} \oplus \dots \oplus L^{[1-m]})$, or as
- $(s=0) \subset \text{Spec}_X \oplus_{r \geq 0} L^{[r]}$.

Note that

- $\pi_* \omega_{X[\sqrt[m]{s}]} \cong \omega_X \oplus \omega_X[\otimes] L \oplus \dots \oplus \omega_X[\otimes] L^{[m-1]}.$

Thus, if $L = \omega_X$ then

- $\pi_* \omega_{X[\sqrt[m]{s}]} \cong \omega_X \oplus \omega_X^{[2]} \oplus \dots \oplus \omega_X^{[m]}.$

Detour for next lecture: cyclic covers 2

If L ample, then

- L^{-1} is direct summand of $\pi_* \mathcal{O}_{X[\sqrt[m]{S}]}$, so
- $H^i(X, L^{-1})$ is direct summand of $H^i(X[\sqrt[m]{S}], \mathcal{O}_{X[\sqrt[m]{S}]})$.

If $\omega_X = L$ ample, then

- $\omega_X^{[r]}$ are direct summands of $\pi_* \omega_{X[\sqrt[m]{S}]}$, so
- $H^i(X, \omega_X^{[r]})$ are direct summands of $H^i(X[\sqrt[m]{S}], \omega_{X[\sqrt[m]{S}]})$.

Detour: Examples of non-CM families

If X is CM (=Cohen-Macaulay) and proper, then $H^i(X, \mathcal{O}_X)$ and $H^{n-i}(X, \omega_X)$ are duals.

So in flat families $\pi : X \rightarrow S$ with CM fibers, the $h^i(X_s, \mathcal{O}_{X_s})$ are locally constant iff the $h^j(X_s, \omega_{X_s})$ are locally constant.

The next examples show what can happen for non-CM fibers.

Corollary. For KSB-stable families $\pi : X \rightarrow S$, the $R^i\pi_*\mathcal{O}_X$ and the $R^j\pi_*\omega_{X/S}$ need to be treated separately.

Examples: cones over $C \times D$, slide 1

Let C, D be a smooth, projective curves of genus ≥ 2 , and L, M ample line bundles of degrees d, e . Let

$X_{L,M} := \text{Spec } \bigoplus_{m \in \mathbb{Z}} H^0(C \times D, (L \boxtimes M)^m)$ be the cone over $C \times D$ with ample line bundle $L \boxtimes M$.

Then $\omega_{X_{L,M}}$ is the sheaf corresponding to $\bigoplus_{m \in \mathbb{Z}} H^0(C \times D, (\omega_C \boxtimes \omega_D) \otimes (L \boxtimes M)^m)$.

Examples. For $g - 1 < d \leq 2g - 2$ and suitable M

- 1 the $X_{L,M}$ form a flat family over $\text{Pic}^d(C)$, but the $\omega_{X_{L,M}}$ are not flat; or
- 2 the $X_{L,M}$ do not form a flat family over $\text{Pic}^d(C)$, but the $\omega_{X_{L,M}}$ are flat.

Examples: cones over $C \times D$, slide 2

Key property: $h^0(C, L^m)$ and $h^0(C, \omega_C \otimes L^{-m})$ vary with L only for $m = 1$.

So, the only summands that vary with L are

- $H^0(C, L) \otimes H^0(D, M)$ in $\mathcal{O}_{X_{L,M}}$, and
- $H^0(C, \omega_C \otimes L^{-1}) \otimes H^0(D, \omega_D \otimes M^{-1})$ in $\omega_{X_{L,M}}$.

Therefore:

- $X_{L,M}$ **not** flat over $\text{Pic}^d(C)$ iff $H^0(D, M) \neq 0$, and
- $\omega_{X_{L,M}}$ **not** flat over $\text{Pic}^d(C)$ iff $H^0(D, \omega_D \otimes M^{-1}) \neq 0$.

Lecture 4.1: Du Bois singularities

When to turn to Du Bois singularities?

Thesis

If you have singularities that are not rational or CM, but close to it, Du Bois singularities may give the answer.

Example. $X \rightarrow S$ flat. When is $\omega_{X/S}$ flat?

Classical answer: if fibers are CM (easy proof).

New answer: if fibers are Du Bois
(longer, roundabout proof).

When is ω flat?

X proper of dimension n , L ample. Then

- $\omega_X = \text{sheaf of } \bigoplus_m H^0(X, \omega_X \otimes L^m)$, and
- $H^0(X, \omega_X \otimes L^m)$ is dual to $H^n(X, L^{-m})$.

Corollary. $g : X \rightarrow S$ projective, relative dim n . Then $\omega_{X/S}$ is flat and commutes with base changes iff $R^n g_* L^{-m}$ is free for $m \gg 1$.

Principle:

- $\omega_X^\bullet$ is encoded in the $H^i(X, L^{-m})$ for L ample.

When is ω flat?

Let $\mathcal{S}$ be a class of singularities, closed under

- $X \mapsto X \times \mathbb{A}^1$, and
- general hyperplane sections $X \mapsto H \cap X$,
- so general cyclic covers with invertible L .

$\text{Flat}_n(\mathcal{S}) := \text{all } g : X \rightarrow B$

flat, projective, relative dim n , fibers in $\mathcal{S}$.

Corollary. For $\mathcal{S}$ equivalent:

- $R^n g_* L^{-m}$ is locally free for all ample L , and for all $(g : X \rightarrow B) \in \text{Flat}(\mathcal{S})$.
- $R^n g_* \mathcal{O}_X$ is locally free for all $(g : X \rightarrow B) \in \text{Flat}(\mathcal{S})$.

When is $H^i(X, \mathcal{O}_X)$ flat?

Cohomology and base change

Let A be Artinian with residue field k , and $g : X_A \rightarrow \operatorname{Spec} A$ flat, proper.

Equivalent:

- the $H^i(X_A, \mathcal{O}_{X_A})$ are free A -modules.
- $H^i(X_A, \mathcal{O}_{X_A}) \rightarrow H^i(X_k, \mathcal{O}_{X_k})$.

Illustration for $A = k[\epsilon]$:

$$\begin{array}{ccccc} H^i(X_k, \mathcal{O}_{X_k}) & \xrightarrow{\epsilon} & H^i(X_A, \mathcal{O}_{X_A}) & \rightarrow & H^i(X_k, \mathcal{O}_{X_k}) \\ H^{i+1}(X_k, \mathcal{O}_{X_k}) & \xrightarrow{\epsilon} & H^{i+1}(X_A, \mathcal{O}_{X_A}) & \rightarrow & H^{i+1}(X_k, \mathcal{O}_{X_k}) \end{array}$$

Du Bois singularities 1

Incorrect (?) global definition by properties:

- DB preserved by general hyperplane sections and cyclic covers.
- $H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathcal{O}_X)$ if X proper and DB.

Du Bois singularities 1

Incorrect (?) global definition by properties:

- DB preserved by general hyperplane sections and cyclic covers.
- $H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathcal{O}_X)$ if X proper and DB.

Principle:

- If need help, ask Sándor Kovács.

Du Bois singularities 2

Theorem (Du Bois–Jarraud, 1974)

$f : X \rightarrow S$ flat, proper, DB fibers.

Then the $R^i f_* \mathcal{O}_X$ are locally free.

Proof. Enough for A Artinian with residue field k .

Equivalent: $H^i(X_A, \mathcal{O}_{X_A}) \rightarrow H^i(X_k, \mathcal{O}_{X_k})$.

$$\begin{array}{ccc} H^i(X_A, \mathcal{O}_{X_A}) & \rightarrow & H^i(X_k, \mathcal{O}_{X_k}) \\ \uparrow & & \uparrow \\ H^i(X_A, \mathbb{C}) & \rightarrow & H^i(X_k, \mathbb{C}) \end{array}$$

Du Bois singularities 3

Theorem (Kollár-Kovács, 2010)

Semi-log-canonical implies Du Bois.

(More generally, (X, Δ) slc, then any union of log canonical centers is Du Bois. Kollár-Kovács, 2010, 2020).

Corollary. Let $g : X \rightarrow S$ be flat, fibers slc. Then $\omega_{X/S}$ is flat over S and commutes with base change.

Recall: A morphism $f : X \rightarrow S$ is **stable** iff

- ① f is flat, projective with stable fibers, and
- ② $\omega_{X/S}^{[m]}$ is flat and commutes with base changes $\forall m$.

A morphism $f : X \rightarrow S$ is **locally stable** iff

- ① f is flat, with slc fibers, and
- ② $\omega_{X/S}^{[m]}$ is flat and commutes with base changes $\forall m$.

KSB stability automatic in $\text{codim} \geq 3$

Theorem (Kollár-Kovács, 2024)

- $\pi : X \rightarrow B$ flat morphism, fibers slc, and
- locally KSB stable over $X^\circ \subset X$, where
- $X_b \setminus X^\circ$ has $\text{codim} \geq 3$ in X_b for all $b \in B$.

Then $\pi : X \rightarrow B$ is locally KSB stable.

Note: $X_b \setminus X^\circ$ has $\text{codim} \geq 2$ always.

Pair version: $\pi : (X, \sum a_i D_i) \rightarrow B$.

proof works if $a_i \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1\}$,
conjectured otherwise.

Localized K-K theorem 1

Theorem

- $\pi : X \rightarrow B$ flat morphism, $U \subset X$ open,
- B Artinian, X_k is slc,
- $X_k \setminus U$ has $\text{codim} \geq 3$,
- $\omega_{U_k}^{[m]}$ is free for some $m \geq 1$.

Then $\pi : X \rightarrow B$ is locally KSB stable.

Hope: $\omega_{U/B}^{[m]}$ is free and gives cyclic cover $\tilde{U} \rightarrow U$.

Localized K-K theorem 2

Can do: $\ker[\mathrm{Pic}(U) \rightarrow \mathrm{Pic}(U_k)]$ is a vectorspace, so there is an $L \in \ker[\mathrm{Pic}(U) \rightarrow \mathrm{Pic}(U_k)]$ such that $\omega_{U/B}^{[m]} \otimes L^m$ is free, so we get $(\tilde{U} \rightarrow U)$ quasi-étale. Extend to $(\tilde{X} \rightarrow X)$.

Warning: For $\tilde{X} \rightarrow B$, flatness/DB could fail over $X_k \setminus U$!

Luckily failure is **mild**^{*}, so

$\omega_{\tilde{X}/B}$ is flat and commutes ..., so

all the $\omega_{X/B}^{[m]}$ are flat and commute

(by end of Lecture 3).

What is mild*?

There is a $\tau_0 : X_0^* \rightarrow \tilde{X}_0$ such that

- X_0^* is slc,
- τ_0 is a homeomorphism (in Euclidean topology), and
- a scheme isomorphism outside a codim ≥ 3 subset.

Diagram chasing:

Flatness of $\omega_{X/B}$ needs

$$H^i(X_A, \mathcal{O}_A) \twoheadrightarrow H^i(X_k, \mathcal{O}_k) \text{ only for } i = n-1, n.$$

Lecture 4.2: Divisors in families

Divisors in families

Let $\pi : (X, \sum a_i D_i) \rightarrow S$ be KSBA stable.

Question. What about $\pi_i : D_i \rightarrow S$?

Example.[Hassett, 2001] Lines on quadrics:

General: smooth quadric $\subset \mathbb{P}^3$, $D_{\text{gen}} := M + \frac{1}{2}(L' + L'')$.

Special: quadric cone $\subset \mathbb{P}^3$, $D_0 := \ell + \frac{1}{2}(\ell' + \ell'')$.

Since L', L'' are disjoint, but ℓ', ℓ'' meet,
flatness **fails** with coefficient $= \frac{1}{2}$.

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Since L', L'' are disjoint, but ℓ', ℓ'' meet,
flatness **fails** with coefficient $= \frac{1}{2}$.

Theorem. [Kollár, 2014] If $a_i > \frac{1}{2}$ and S **reduced**, then
 $\pi_i : D_i \rightarrow S$ is flat.

Mumford divisors over reduced base

$\pi : X \rightarrow S$ flat, of relative dimension n .

$D \subset X$ a prime divisor.

- $D \rightarrow S$ has relative dimension $n-1$, and
- π is smooth at every generic point of D_s for every $s \in S$.

Theorem

$\pi : X \rightarrow S$ flat, projective, S reduced. There is a
 $\text{MDiv}(X/S) \subset \text{Hilb}_{n-1}(X/S)$
representing Mumford divisors.

Notion is close to

almost-Cartier divisors [Hartshorne, 1986]

Mumford divisors over Artinian base: Hopeless

Example. $X := \mathbb{CP}^2 \times \mathbb{C} \rightarrow \mathbb{C}$.

Every Mumford/Weil divisor is Cartier.

But for $X_n := \mathbb{CP}^2 \times \operatorname{Spec} \mathbb{C}[t]/(t^n) \rightarrow \operatorname{Spec} \mathbb{C}[t]/(t^n)$, the space of Mumford divisors is ∞ -dimensional for $n \geq 2$.

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Local version. The Picard group of the punctured spectrum of $\mathbb{C}[[x, y]][\epsilon]$ is

$$H_m^1(\mathbb{C}^2, \mathcal{O}) = \bigoplus_{i,j>0} x^{-i}y^{-j} \cdot \mathbb{C}.$$

$$cx^{-i}y^{-j} \leftrightarrow (x^{2i}, x^i y^j + c\epsilon, x^i \epsilon).$$

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$$cx^{-i}y^{-j} \leftrightarrow (x^{2i}, x^i y^j + c\epsilon, x^i \epsilon).$$

Note that: for $\mathbb{C}[[x, y, z]][\epsilon]$ the Picard group of the punctured spectrum is **trivial**.

Cayley and K-flatness: hoped-for definition

R complete, local ring,

$$D \subset \hat{\mathbb{A}}_R^N := \operatorname{Spec} R[[x_1, \dots, x_N]],$$

$D \rightarrow \operatorname{Spec} R$ of pure relative dimension n .

Cayley and K-flatness: hoped-for definition

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$$D \subset \hat{\mathbb{A}}_R^N := \operatorname{Spec} R[[x_1, \dots, x_N]],$$

$D \rightarrow \operatorname{Spec} R$ of pure relative dimension n .

Definition. D is **K-flat** over R iff

for every $\pi : \hat{\mathbb{A}}_R^N \rightarrow \hat{\mathbb{A}}_R^{n+1}$ such that $\pi|_D$ is finite,

$\pi(D) \subset \hat{\mathbb{A}}_R^{n+1}$ is flat over R (equiv: Cartier divisor).

Warning: non-linear π allowed

Cayley and K-flatness: hoped-for definition

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Warning: non-linear π allowed

But: could not make it work technically.

Cayley flatness

$D \subset \mathbb{P}_S^N$, such that
 $\pi : D \rightarrow S$ is pure n -dimensional,
flat at each generic point of each fiber.

Definition

D is Cayley flat over S iff the Cayley form
 $\text{Ca}(D) \subset \text{Grass}(\mathbb{P}^{N-n-1}, \mathbb{P}^N)$ is flat over S .

Cayley flatness

$D \subset \mathbb{P}_S^N$, such that
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Definition

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 $\text{Ca}(D) \subset \text{Grass}(\mathbb{P}^{N-n-1}, \mathbb{P}^N)$ is flat over S .

Weak but key technical step:

If $\dim D = 1$, one can approximate well enough
formal projections, by
linear projections of Veronese embeddings of D .

K-flatness

$\pi : X \rightarrow S$ projective, pure relative dim n ,
 $D \subset X$ a Mumford divisor, (S local, infinite residue field).

Definition. D is K-flat iff

$\pi(D) \subset \mathbb{P}_S^n$ is flat (hence Cartier) for
every finite S -morphism $X \rightarrow \mathbb{P}_S^n$.

Theorem (Kollár, 2023)

Representable by a scheme $\text{KDiv}(X/S)$.

Bertini for K-flatness

Theorem (Kollár, 2023)

D K -flat $\Leftrightarrow$ general curve section is K -flat.

Corollary

$\pi : (X, \Delta) \rightarrow S$. Assume that all fibers slc, and $\text{coeffs}(\Delta) \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1\}$. Then

$KSBA$ -stable $\Leftrightarrow$ general surface section is $KSBA$ -stable.

Properties of K-flatness

- K-flat $\Leftarrow$ flat
- K-flat $\Leftrightarrow$ flat if X/S is smooth.
- K-flat $\Leftrightarrow$ flat if D has normal fibers.
- K-flat $\Leftrightarrow$ Mumford if S is reduced.
- K-flat $\Leftrightarrow D_A$ K-flat for all Artinian $A \rightarrow S$.

Better than flatness:

- K-flatness preserved by linear equivalence,
- D_1, D_2 K-flat then so is $D_1 + D_2$,
- mD is K-flat iff D is K-flat.

Problems:

- Hard to compute.
- Deformation theory?