

Symplectic A'Campo spaces and two applications

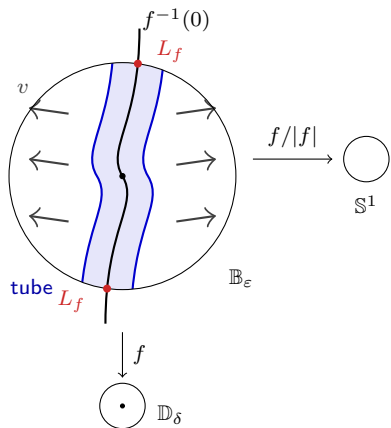
Lecture 1. From the Milnor fibration to the A'Campo space

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Lecture 1	A'Campo's monodromy and the A'Campo space
Lecture 2	symplectic monodromy at radius zero and equimultiplicity of μ -constant families
Complementary lecture	Floer homology, contact loci and the arc-Floer conjecture (Eduardo de Lorenzo Poza)
Lecture 3	Lagrangian torus fibrations for maximal Calabi–Yau degenerations

- $f: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ holomorphic, **isolated singularity**:
 $\mu(f) := \dim_{\mathbb{C}} \mathbb{C}\{z\}/(\partial_{z_0} f, \dots, \partial_{z_n} f) < \infty$, the Milnor algebra;
 $\mu(f) = \text{rank } H_n(F)$, an embedded topological invariant.
- **Milnor radius** $\varepsilon > 0$: $f^{-1}(0) \cap \partial \mathbb{B}_{\varepsilon'}$ for all $0 < \varepsilon' \leq \varepsilon$;
 $0 < \delta \ll \varepsilon$.
- **Tube**: $f: \mathbb{B}_{\varepsilon} \cap f^{-1}(\partial \mathbb{D}_{\delta}) \rightarrow \partial \mathbb{D}_{\delta}$, fibre
 $F := f^{-1}(\delta) \cap \mathbb{B}_{\varepsilon}$.
- **Sphere**: $\arg f: \mathbb{S}_{\varepsilon} \setminus L_f \rightarrow \mathbb{S}^1$, link $L_f := f^{-1}(0) \cap \mathbb{S}_{\varepsilon}$.
- Equivalent smooth fibrations [Milnor, Thm. 5.11]: a vector field v with $\langle v, x \rangle > 0$ and $\arg f$ constant along its flow inflates the tube onto $\mathbb{S}_{\varepsilon}$ minus a neighbourhood of L_f . The course works in the tube.
- $F \simeq \bigvee_{\mu(f)} \mathbb{S}^n$; $(\mathbb{B}_{\varepsilon}, f^{-1}(0) \cap \mathbb{B}_{\varepsilon}) \cong \text{Cone}(\mathbb{S}_{\varepsilon}, L_f)$
 (conical structure).
- **Monodromy**: lift one turn of $\partial \mathbb{D}_{\delta}$ to the tube: $\varphi: F \rightarrow F$
 diffeomorphism, $\varphi = \text{id}$ near ∂F , well defined up to isotopy.



tube $f^{-1}(\mathbb{D}_{\delta}) \cap \mathbb{B}_{\varepsilon}$ with its fibration f , sphere
 fibration $f/|f|$, flow of v inflating the tube onto $\mathbb{S}_{\varepsilon}$
 minus a neighbourhood of L_f

Zariski's question [Zariski 1971]

f, g with the same embedded topological type, i.e.
 $(\mathbb{C}^{n+1}, f^{-1}(0)) \cong_{\text{homeo}} (\mathbb{C}^{n+1}, g^{-1}(0))$ near 0. Is $\nu(f) = \nu(g)$?
 (ν = multiplicity)

- $\Lambda(\varphi^m) := \sum_{q \geq 0} (-1)^q \operatorname{tr}(\varphi_*^m | H_q(F; \mathbb{Q}))$, $m \geq 1$: depends only on the homotopy class of φ , hence only on the embedded topological type.

Lefschetz trace formula

If $\operatorname{Fix} \varphi$ is finite, $\Lambda(\varphi) = \sum_{x \in \operatorname{Fix} \varphi} \operatorname{ind}_x(\varphi)$, $\operatorname{ind}_x(\varphi)$ the local degree of $\operatorname{id} - \varphi$ at x .

So $\Lambda(\varphi^m)$ counts the periodic points of φ with multiplicity: the sequence $(\Lambda(\varphi^m))_{m \geq 1}$ is a **dynamical invariant** of the isotopy class of the monodromy.

- $\Lambda(\operatorname{id})$ (Recall Poincaré–Hopf Theorem).
- In passing: $\zeta_\varphi(t) = \exp \sum_{m \geq 1} \Lambda(\varphi^m) t^m / m$ is A'Campo's zeta function, equivalent to the characteristic polynomial of φ_* [A'Campo 1975]; historically central, not used today.
- Two ways to see the monodromy: by **morsification**, which gave the bouquet and the vanishing cycles; by **resolution**, which gives the pieces of F and the fixed points. They see different things.

Definition: special dynamics for φ^m

A diffeomorphism φ has **special dynamics for φ** if

$$\operatorname{Fix} \varphi = \bigsqcup_j B_j,$$

B_j compact codimension-zero submanifolds with boundary and corners, and near each B_j the map φ is the time-one flow of a vector field V_j vanishing exactly on B_j and tangent to the levels of a function in a neighborhood of B_j vanishing precisely at B_j .

Local Poincaré–Hopf

The fixed-point index of B_j is the index of V_j on a neighbourhood of B_j , which is $\chi(B_j)$ since $\dim B_j$ is even. Hence, for special dynamics,

$$\Lambda(\varphi^m) = \sum_j \chi(B_j).$$

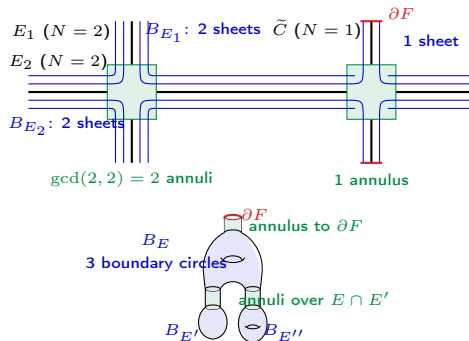
$\mu: (Y, D) \rightarrow (\mathbb{C}^2, 0)$ embedded resolution, $D = \sum_E N_E E$,
 $E^\circ := E \setminus \bigcup_{E' \neq E} E'$, $\chi(E^\circ) = 2 - \text{val}(E)$.

The pieces

- $B_E \rightarrow E^\circ$: the N_E -fold cyclic cover $\{u z^{N_E} = 1\}$ in a chart $f \circ \mu = u z^{N_E}$ [Denef–Loeser, §2.3], compactified at the punctures: N_E sheets along E ;
 $\#\partial B_E = \sum_{E'} \gcd(N_E, N_{E'})$,
 $\#\pi_0(B_E) = \gcd(N_E, \{N_{E'}\})$, $\chi(B_E) = N_E \chi(E^\circ)$.
- Over $E \cap E'$: $\gcd(N_E, N_{E'})$ annuli; one annulus from the strict transform to ∂F . $F = \bigcup_E B_E \cup \text{annuli}$.

A'Campo's theorem [1975]

One turn of the base turns the N_E sheets of $\{u z^{N_E} = \delta\}$ by $2\pi/N_E$. The monodromy can be chosen to be the rotation by $2\pi/N_E$ on B_E and, on the annuli over $E \cap E'$, $w_E + w_{E'} = 1$: $\theta_E \mapsto \theta_E + 2\pi w_E/N_E$, $\theta_{E'} \mapsto \theta_{E'} + 2\pi w_{E'}/N_{E'}$, the interpolation forced by continuity. For an *m-separating* resolution, $N_E + N_{E'} > m$ at double points, φ^m has special dynamics with $\text{Fix } \varphi^m = \bigsqcup_{N_E | m} B_E$.



A'Campo's formulae [1975]

$$\Lambda(\varphi^m) = \sum_{N_E | m} N_E \chi(E^\circ), \quad \zeta_\varphi(t) = \prod_E (1 - t^{N_E})^{-\chi(E^\circ)},$$

$$\mu = (-1)^n \left(-1 + \sum_E N_E \chi(E^\circ) \right),$$

for any embedded resolution, by local Poincaré–Hopf on an *m-separating* one; **same in every dimension**, with the Denef–Loeser covers.

Lefschetz numbers see ν from below

- $\Lambda(\varphi^k) = 0$ for $0 < k < \nu(f)$ [Deligne, letter to A'Campo].
- $\Lambda(\varphi) = 0$ if f is singular, $= 1$ if f is smooth [A'Campo 1973]:
Zariski's question when one germ is smooth.
- $\nu(f) \leq \min\{a \geq 1 : \Lambda(\varphi^a) \neq 0\}$ [A'Campo 1975, p. 245].
- Plane curves: $\Lambda(\varphi^\nu) = \nu(2 - \tau)$, τ = number of tangent directions.

What they miss

- The node xy : F an annulus, φ a Dehn twist, trivial on H_1 : $\Lambda(\varphi^m) = 0$ for all m , $\nu = 2$ invisible.
- Grima's curves [A'Campo 1975, p. 245]:
 $H_1 = \{(x^{14} - y^{11})(x^{21} - y^{44}) = 0\}$,
 $H_2 = \{(x^{22} - y^7)(x^{33} - y^{28}) = 0\}$: $\nu = 32$ and 35 ,
different topological types, *equivalent rational monodromies*. A'Campo: no hope from the rational monodromy.

Theorem [McLean, Geom. Topol. 2019]

F is a Liouville domain and φ an exact symplectomorphism. For each $m \geq 1$ there is a $\mathbb{Z}/2$ -graded $\mathbb{Z}/2$ -vector space

$$\mathrm{HF}_\bullet(\varphi^m, +) \quad (\text{fixed point Floer homology})$$

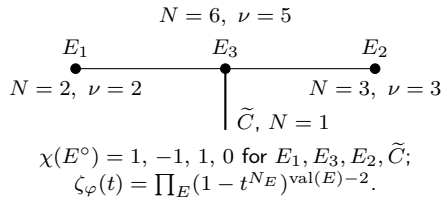
with $\chi(\mathrm{HF}_\bullet(\varphi^m, +)) = (-1)^n \Lambda(\varphi^m)$, and

$$\nu(f) = \min\{m \geq 1 : \mathrm{HF}_\bullet(\varphi^m, +) \neq 0\}$$

The grading also detects the log canonical threshold.

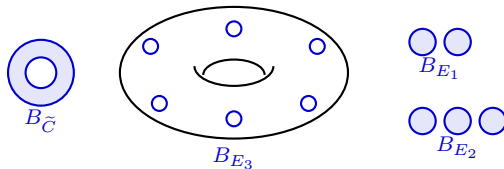
- A'Campo's hope is right one level up: the categorified Lefschetz number sees the multiplicity.
- McLean symplectizes the Milnor fibration *in the sphere*: contact structure on $\mathbb{S}_\varepsilon$, the link as a contact pair, Giroux's correspondence. Contact slide for experts later.
- The node again: $\Lambda \equiv 0$ but $\mathrm{HF}_\bullet(\varphi^2, +) \cong H_\bullet^{\mathrm{BM}}(\mathbb{C}^\times) \neq 0$ (Lecture 2).

Minimal embedded resolution: three exceptional curves and the strict transform; $\chi(E^\circ) = 2 - \text{val}(E)$; the resolution is m -separating for $m \in \{1, 2, 3, 6\}$.



The cusp

- $\Lambda(\varphi^m) = 0, 2, 3, -1$ for $m = 1, 2, 3, 6$; in general $\Lambda(\varphi^m) = 1 - 2 \cos \frac{m\pi}{3}$.
- $\zeta_\varphi(t) = \frac{1 - t^6}{(1 - t^2)(1 - t^3)} = \frac{1 - t + t^2}{1 - t}$,
 $\Delta(t) = t^2 - t + 1, \quad \mu = 2$.
- B_{E_1} : two discs; B_{E_2} : three discs; B_{E_3} : connected, $\chi = -6$, six boundary circles $\Rightarrow$ torus with six holes; $B_{\tilde{C}}$: annulus.
- Capping five holes of B_{E_3} with the discs: torus with one boundary circle = F .
- $\text{Fix } \varphi = B_{\tilde{C}}, \text{ Fix } \varphi^2 = B_{\tilde{C}} \sqcup B_{E_1}, \text{ Fix } \varphi^3 = B_{\tilde{C}} \sqcup B_{E_2},$
 $\text{Fix } \varphi^6 = \text{everything}.$



Theorem [F. de Bobadilla–Peřka, Ann. of Math. 2024]

$f_t: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, $t \in [0, 1]$, a continuous family of holomorphic germs with $\mu(f_t) < \infty$ constant. Then $\nu(f_t)$ is constant.

Corollary

A continuous family of isolated singularities of constant embedded topological type is equimultiple.

- Same statement for $f \in k[t][[z_0, \dots, z_n]]$, k any field of characteristic zero [Rem. 8.4]. No algebraic proof is known, not even for curves.
- Zariski's question, known cases: homeomorphism of class C^1 [Ephraim, Trotman]; plane curves [Zariski]; one germ smooth [A'Campo, Lê]; surfaces: quasihomogeneous [Xu–Yau], bilipschitz [FFS], $\nu = 2$ [Navarro Aznar]; $\mu \leq 2n + 1$ [Yau–Zuo]; bilipschitz right-left equivalence [Risler–Trotman]. General non-family case **remains open**.

Strategy

McLean's theorem is about a symplectization of the Milnor fibration *in the sphere*, through contact pairs and Giroux's correspondence. Over a fixed ball of f_0 there is no sphere fibration for f_t , so that route is replaced entirely by a symplectization of the Milnor fibration *in the tube*: a **smooth and symplectic enhancement of A'Campo's construction** in his Lefschetz-number papers. His radius-zero model becomes the boundary of a smooth space A over $\mathbb{C}_{\log}$ carrying a fiberwise symplectic form whose connection is his flow; the radial direction gives the isotopy to the tube. McLean's statement is recovered, none of his proof is used. This hour: the smooth part. Lecture 2: the symplectic part.

$$\begin{array}{ccc}
 A & \xrightarrow{\pi} & X \\
 f_A \downarrow & & \downarrow f \\
 \text{Diagram of } \mathbb{C}_{\log} & \xrightarrow{(r, \theta) \mapsto r\theta} & \text{Diagram of } \mathbb{C}
 \end{array}$$

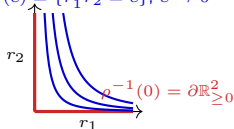
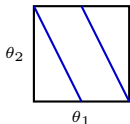
$\mathbb{C}_{\log} := [0, \infty) \times \mathbb{S}^1$

The model $f = z_1^{a_1} \cdots z_n^{a_n}$ in polar coordinates $z_i = r_i e^{i\theta_i}$ (passive coordinates: a factor $\mathbb{C}^m$)

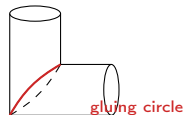
$$f = \underbrace{r_1^{a_1} \cdots r_n^{a_n}}_{\rho(r)} \cdot \exp\left(i \underbrace{(a_1\theta_1 + \cdots + a_n\theta_n)}_{\Theta(\theta)}\right), \quad f_{\log} = \rho \times \Theta: \underbrace{\mathbb{R}_{\geq 0}^n \times (\mathbb{S}^1)^n}_{X_{\log}} \longrightarrow \underbrace{\mathbb{R}_{\geq 0} \times \mathbb{S}^1}_{\mathbb{C}_{\log}}.$$

- **Angular part** $\Theta: (\mathbb{S}^1)^n \rightarrow \mathbb{S}^1, \theta \mapsto \sum_i a_i \theta_i$: a surjective homomorphism, hence a locally trivial fibration; fibre the subgroup $\{\sum a_i \theta_i = 0\}$, a union of $\gcd(a_1, \dots, a_n)$ tori $(\mathbb{S}^1)^{n-1}$; the monodromy acts by translation.
- **Radial part** $\rho: \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}, r \mapsto \prod r_i^{a_i}$. In $s_i = \log r_i$: on the open orthant a submersion with fibres the affine hyperplanes $\{\sum a_i s_i = \log c\} \cong \mathbb{R}^{n-1}$; over $c = 0$ the fibre is the **boundary of the orthant**, also $\cong \mathbb{R}^{n-1}$. Topologically a trivial fibration including $c = 0$: for $n = 2$, $(r_1, r_2) \mapsto (r_1^{a_1} r_2^{a_2}, r_1 - r_2)$ is a homeomorphism $\mathbb{R}_{\geq 0}^2 \rightarrow [0, \infty) \times \mathbb{R}$. Not a submersion along the boundary: the smooth structure breaks at radius zero.
- So $f_{\log}$ is a topologically locally trivial fibration over all of $\mathbb{C}_{\log}$, radius zero included. Its fibre over $(0, \theta)$ is $\partial \mathbb{R}_{\geq 0}^n \times \{\sum a_i \theta_i = \theta\}$: it splits in pieces $(X_I^\circ)_{\log}$, glued along the corners.
- In general $X_{\log} \rightarrow \mathbb{C}_{\log}$ is obtained by gluing these models over the adapted charts of the resolution.

$$\Theta^{-1}(0) = \{2\theta_1 + \theta_2 = 0\}, \text{ one circle} \quad \rho^{-1}(c) = \{r_1^2 r_2 = c\}, c \rightarrow 0$$



fibre over $(0, \theta)$: $\partial \mathbb{R}_{\geq 0}^2 \times \mathbb{S}^1$



$a = (2, 1)$: the two factors of $f_{\log}$, and the radius-zero fibre, two cylinders glued along one circle

$h: X \rightarrow \mathbb{C}^{n+1}$ a log resolution, $f = f_0 \circ h$, $D = \sum_{i=1}^N N_i D_i$ snc; strata X_I , X_I° ; dual complex $\Delta = \bigcup_{X_I \neq \emptyset} \Delta_I \subseteq \{w \in \mathbb{R}_{\geq 0}^N : \sum_i w_i = 1\}$.

- **Distance functions:** $\widehat{r}_i = \|\sigma_i\|$ for a section σ_i of $\mathcal{O}_X(D_i)$ cutting out D_i and a hermitian norm; in an adapted chart $\widehat{r}_i = |z_i| \cdot (\text{smooth positive})$. $\widehat{s}_i := \log \widehat{r}_i$.

- **Tropical functions:** $\widehat{t}_i := \frac{-1}{N_i \widehat{s}_i}$, $t := \frac{-1}{\log |f|}$,

$$\widehat{w}_i := \frac{t}{\widehat{t}_i} = \frac{N_i \log \widehat{r}_i}{\log |f|} \in [0, 1];$$

$$\sum_i \widehat{w}_i = 1 + t \log(\text{smooth positive}) \rightarrow 1 \text{ as } |f| \rightarrow 0.$$

- **The tropical map** $\mu := (\widehat{w}_1, \dots, \widehat{w}_N): X \setminus D \rightarrow \mathbb{R}^N$: as $|f| \rightarrow 0$ near X_I° , μ tends to the face Δ_I . *At radius zero μ lands in the dual complex, inside the unit simplex of $\mathbb{R}^N$.*

- **Definition** [Annals, Def. 3.3, (24); A'Campo 1975, p. 238]:
 $\Gamma := \overline{\text{graph}(\mu)} \subseteq X \times \mathbb{R}^N$, $A := \Gamma \times_X X_{\log}$, $\pi: A \rightarrow X$,
 $f_A := f_{\log} \circ (A \rightarrow X_{\log}) = (|f| \circ \pi, \theta): A \rightarrow \mathbb{C}_{\log}$.

$$\begin{array}{ccccc} A & \longrightarrow & \Gamma = \overline{\text{graph}(\mu)} & \hookrightarrow & X \times \mathbb{R}^N \\ \downarrow & & \downarrow p & & \\ X_{\log} & \longrightarrow & X & \xrightarrow{\mu} & \mathbb{R}^N \\ f_{\log} \downarrow & & \downarrow f & & \\ \mathbb{C}_{\log} & \longrightarrow & \mathbb{C} & & \end{array}$$

Reading the diagram

A is the fibre product over X of two blowups of X along D : the real oriented one $X_{\log}$, which supplies the **angles** θ_i , and the tropical one Γ , which supplies the **barycentric coordinates** w_i ; f_A records radius and angle of f .

A hybrid space

$A \setminus \partial A \cong X \setminus D$ and $\partial A := \pi^{-1}(D) = \bigsqcup_{I \neq \emptyset} A_I^\circ$, with

$$A_I^\circ \cong (X_I^\circ)_{\log} \times \Delta_I.$$

At radius zero three kinds of coordinates coexist on A_I° : **tropical** $(w_i)_{i \in I}$ on the simplex, **angular** $(\theta_i)_{i \in I}$ on the torus fibres of $(X_I^\circ)_{\log} \rightarrow X_I^\circ$, and the **natural** holomorphic coordinates of the stratum X_I° .

The dual complex becomes geometry

Used as a bookkeeping device of the resolution, the dual complex is here cut into its simplices Δ_I and glued into the boundary of A : the radius-zero fibre $f_A^{-1}(0, \theta)$ is fibred over it. This is A'Campo's construction of 1975: his simplicial model $\widetilde{X}_0$ and his $\widetilde{N} = \partial A$.

On top of $\hat{r}_i, \hat{s}_i, \hat{t}_i, t, \hat{w}_i$:

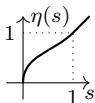
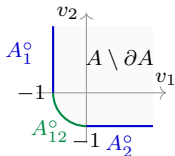
- **Rounding:** $\eta(s) = (1 - \log s)^{-1}$, all derivatives of η^{-1} vanish at 0; $g := \eta(t)$, $\hat{u}_i := \eta(\hat{w}_i)$; $\tilde{\Delta}_I := (\eta, \dots, \eta)(\Delta_I)$ the rounded simplex.
- **Hybrid coordinates:** $\hat{v}_i := \hat{t}_i - \hat{u}_i \in [-1, 1]$; on ∂A , $\hat{v}_i = -\hat{u}_i$. The boundary is $\{\min_i \hat{v}_i = -1\}$, rounded.
- **Angular forms:** $\hat{\alpha}_i := -d^c \hat{s}_i$, equal to $d\theta_i$ in an adapted chart up to a smooth form; $\theta = f/|f|$.
- Role of η : $r_i = \exp(-1/(N_i t_i))$ and $t_i = t/w_i$ must be flat at their zero loci for the charts to be smooth.

Charts [Annals §3.3–3.4; CPAM (15)–(18)]

Over an adapted chart with $S = \{i_1, \dots, i_k\}$, for $i \in S$, on $U_i = \{\hat{w}_i > \frac{1}{n+2}\}$:

$$\psi_i = (g, (\hat{v}_j)_{j \in S \setminus \{i\}}; (\theta_j)_{j \in S}, (z_j)_{j \notin S})$$

$$\psi_i: U_i \longrightarrow [0, \infty) \times \mathbb{R}^{k-1} \times (\mathbb{S}^1)^k \times \mathbb{C}^{n+1-k}.$$



Theorem [Annals, Prop. 3.33; CPAM, Prop. 3.3]

- The ψ_i form a C^1 atlas on A , independent of all choices.
- They form a C^∞ atlas; different distance functions $\hat{r}_i$ give C^1 -compatible smooth structures, equal on ∂A .
- $\pi: A \rightarrow X$ is smooth, a diffeomorphism $A \setminus \partial A \rightarrow X \setminus D$; $\hat{v}_i, \hat{w}_i, \hat{\alpha}_i$ are smooth on A .
- $(g, \theta): A \rightarrow \mathbb{C}_{\log}$ is a smooth proper submersion.

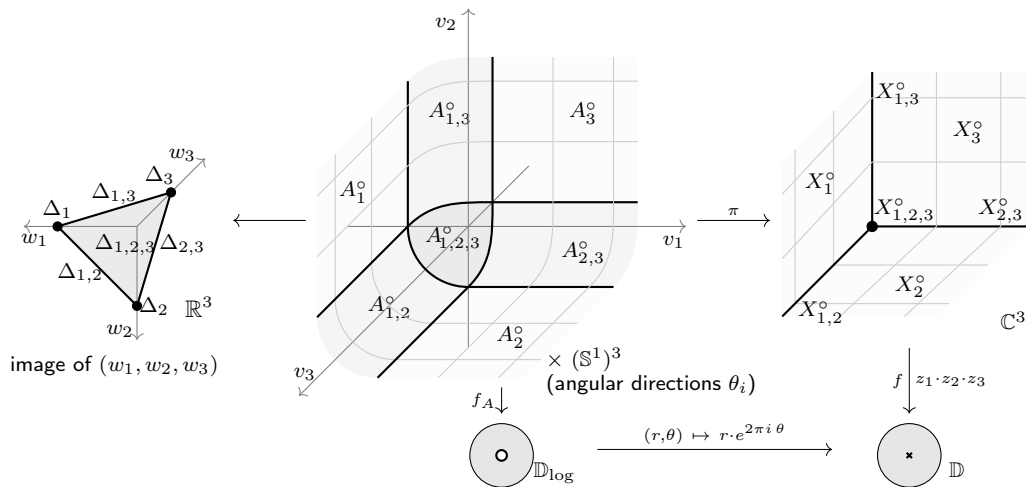
Ehresmann

(g, θ) proper submersion $\Rightarrow$ locally trivial fibration, $A \cong f_A^{-1}(\{0\} \times \mathbb{S}^1) \times [0, \delta)$ over $\mathbb{C}_{\log}$: the radius-zero fibration $\partial A \rightarrow \mathbb{S}^1$ is isomorphic to the Milnor fibration in the tube. The central fibre is replaced by A'Campo's model inside a locally trivial fibration.

Remark

A'Campo: Siebenmann's theorem for $\bar{P}: Z \rightarrow \bar{D}$ and the simplex-fibred map $\tilde{r}: \tilde{N} \rightarrow N$ (Lemma, p. 241); with the smooth structure, one application of Ehresmann.

- Choices: the distance functions $\hat{r}_i$ only [CPAM, Rem. 3.6].



$\partial A = \bigsqcup_I A_I^\circ$, $A_I^\circ \cong (X_I^\circ)_{\log} \times \tilde{\Delta}_I$; the radius-zero fibre is $\{\theta_1 + \theta_2 + \theta_3 = \theta\}$ [CPAM, Fig. 3]

A'Campo's flow [1975, p. 240; Annals (48)]

On $\partial A = \bigsqcup_I A_I^\circ$, with $A_I^\circ \cong (X_I^\circ)_{\log} \times \Delta_I$: the vector field

$$V = \sum_{i \in I} \frac{w_i}{N_i} \frac{\partial}{\partial \theta_i} \quad \text{satisfies} \quad d\theta(V) = \sum_{i \in I} N_i \frac{w_i}{N_i} = \sum_{i \in I} w_i = 1,$$

since $\theta = \sum_i N_i \theta_i$ on A_I° : it lifts the angular field of $\partial \mathbb{C}_{\log}$ and is continuous across the pieces. Its time- 2π map φ is a representative of the monodromy of $\partial A \rightarrow \mathbb{S}^1$, hence, by Ehresmann, of the Milnor monodromy.

- On $B_E = \{w_E = 1\} \cap F_\theta$: rotation by $2\pi/N_E$, the deck transformation of the Denef–Loeser cover. On the tubes over the strata: the interpolation. This is A'Campo's theorem of slide 4.

Proposition

If the resolution is m -separating, $N_E + N_{E'} > m$ whenever $E \cap E' \neq \emptyset$ [McLean 2019, Lemma 2.4], then φ has special dynamics for φ^m and $\text{Fix } \varphi^m = \bigsqcup_{N_E | m} B_E$.

To be proved in Lecture 2

There is a fiberwise symplectic form $\omega_A = \pi^* \omega_X + \varepsilon \omega^\sharp$ on A whose symplectic connection restricts on ∂A to V up to a reparametrization of the simplices, and whose radial lift is an isotopy of Liouville fibrations from radius zero to the Milnor fibration in the tube, over a fixed ball.

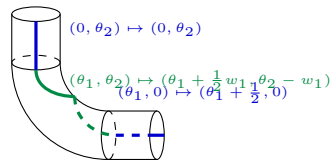
Proof at a double point $E \cap E'$, $w_E + w_{E'} = 1$: φ^m translates the torus over w by $(2\pi m w_E / N_E, 2\pi m w_{E'} / N_{E'})$, so it fixes a point over w iff $m w_E / N_E = p \in \mathbb{Z}$ and $m w_{E'} / N_{E'} = q \in \mathbb{Z}$. In the interior $p, q \geq 1$ and

$$1 = w_E + w_{E'} = \frac{p N_E + q N_{E'}}{m} \geq \frac{N_E + N_{E'}}{m} > 1.$$

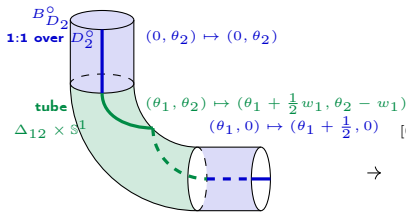
At $w_E = 1$ the shift is $(2\pi m / N_E, 0)$: zero iff $N_E \mid m$.

Near B_E , the field $2\pi m \left(\frac{w_E - 1}{N_E} \partial_{\theta_E} + \frac{w_{E'}}{N_{E'}} \partial_{\theta_{E'}} \right)$ has time-one flow φ^m and vanishes exactly on B_E : special dynamics. $\square$

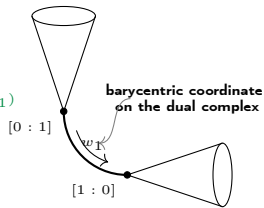
Higher dimension: the same on the tubes over the strata; the pairwise condition suffices.



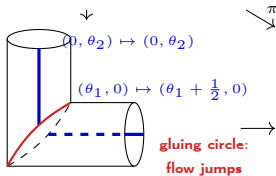
$f = z_1^2 z_2$ at radius zero, angles in full turns: the rotation about D_1 varies with w_1 along the tube



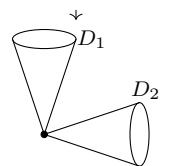
Step 3: fibre product $A = \Gamma \times_X X_{\log}$



Step 2: tropical blowup Γ



Step 1: Kato–Nakayama space $X_{\log}$
(polar coordinates extended to radius zero)

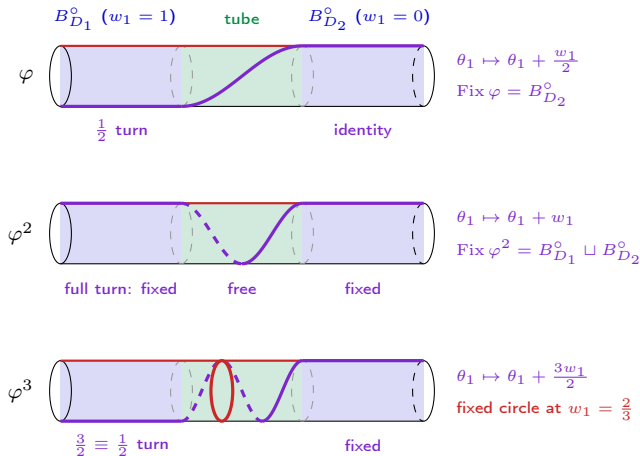


$X \supset D = 2D_1 + D_2$

The three steps are the three columns of (24). Follow the fibre over $(0, \theta)$, radius zero, angle θ :

- In $X_{\log}$: over $D_1^o \cong \mathbb{C}^*$ the circles $\{2\theta_1 = \theta - \arg z_2\}$, a connected double cover of D_1^o , an annulus; over D_2^o the circles $\{\theta_2 = \theta - 2 \arg z_1\}$, a simple cover, an annulus; over the origin one circle $\{2\theta_1 + \theta_2 = \theta\}$ of the torus. The two annuli are glued along that circle, and the naive rotation, by $\frac{1}{2}$ turn on one side and 0 on the other, jumps across it.
- In Γ : the origin is replaced by the edge Δ_{12} of the dual complex, coordinate $w_1 \in [0, 1]$; Δ_1, Δ_2 are the two rays.
- In A : three pieces. The Denef–Loeser double cover $B_{D_1}^o$, the simple cover $B_{D_2}^o$, and between them the tube $\Delta_{12} \times \{2\theta_1 + \theta_2 = \theta\}$, one annulus since $\gcd(2, 1) = 1$. The flow interpolates along the tube from $\frac{1}{2}$ turn to none: this is where the discontinuity of $X_{\log}$ is resolved.

angles in units of full turns; the monodromy is drawn on the radius-zero fibre



the fibre over $(0, \theta)$ is one annulus made of three; red: a reference arc γ along it; purple: its image, dashed on the back of the cylinder

Coordinates on the fibre: (w_1, θ_1) , with $\theta_2 = \theta - 2\theta_1$. A'Campo's flow

$V = \frac{w_1}{2} \partial_{\theta_1} + w_2 \partial_{\theta_2}$ gives, in turns,

$$\varphi^m: \theta_1 \mapsto \theta_1 + m \frac{w_1}{2}.$$

- $m = 1$: Fix $\varphi = B_{D_2}^\circ$. Lefschetz: $\chi(\text{annulus}) = 0$.
- $m = 2$: $N_1 + N_2 = 3 > 2$, the resolution is 2-separating; Fix $\varphi^2 = B_{D_1}^\circ \sqcup B_{D_2}^\circ$, on the tube $2 \cdot \frac{w_1}{2} = w_1 \in \mathbb{Z}$ only at the ends.
- $m = 3$: $N_1 + N_2 = 3 \not\equiv 3$, not 3-separating: $3 \cdot \frac{w_1}{2} \in \mathbb{Z}$ at $w_1 = \frac{2}{3}$ inside the tube, a fixed circle which is not a piece B_E . Blowing up the double point ($N = 3$) restores special dynamics.

Reading

Special dynamics is a property of the pair (iterate, resolution): the pieces are fixed or free, and the tubes are free exactly when the resolution separates.

Symplectic A'Campo spaces and two applications

Lecture 2. The symplectic A'Campo space and equimultiplicity of μ -constant families

Javier Fernández de Bobadilla

BCAM – Ikerbasque

Lecture 1	A'Campo's monodromy and the A'Campo space
Lecture 2	symplectic monodromy at radius zero and equimultiplicity of μ -constant families
Complementary lecture	Floer homology, contact loci and the arc-Floer conjecture (Eduardo de Lorenzo Poza)
Lecture 3	Lagrangian torus fibrations for maximal Calabi–Yau degenerations

Liouville domains

- **Liouville form** on M : $\lambda \in \Omega^1(M)$ with $d\lambda$ symplectic. **Liouville vector field** X_λ : $\iota_{X_\lambda} d\lambda = \lambda$; then $\mathcal{L}_{X_\lambda} d\lambda = d\lambda$.
- **Liouville domain** (M, λ) : M compact with boundary, X_λ pointing outwards along ∂M ; $(\partial M, \lambda|_{\partial M})$ is contact.
- Example: $\lambda_{\text{std}} = \frac{1}{2} \sum_j (x_j dy_j - y_j dx_j)$ on $\mathbb{C}^{n+1}$, $X_{\lambda_{\text{std}}}$ radial; $(F, \lambda_{\text{std}}|_F)$ for $F = f^{-1}(\delta) \cap \mathbb{B}_\varepsilon$, as $F \cap \mathbb{S}_\varepsilon$.

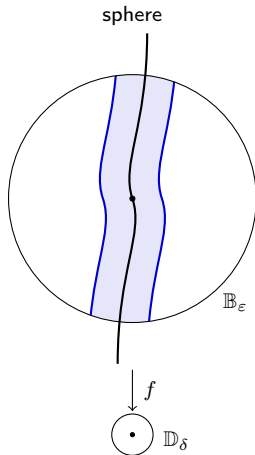
Abstract contact open books [McLean, Def. 3.12;

Annals, Def. 5.2]

- (M, λ, φ) : (M, λ) Liouville domain, $\varphi: M \rightarrow M$ diffeomorphism with $\varphi^* \lambda - \lambda = -da$, $a: M \rightarrow \mathbb{R}$ the action, and $\varphi = \text{id}$ near ∂M .
- **Isotopy**: a smooth family $(M_t, \lambda_t, \varphi_t)_{t \in [0,1]}$; iterates of isotopic books are isotopic.
- **Graded**: a trivialization of $(\Lambda_{\mathbb{C}}^n TM)^{\otimes 2}$ and a lift of $d\varphi$: Conley–Zehnder indices [McLean §3.1; Annals §5.4].

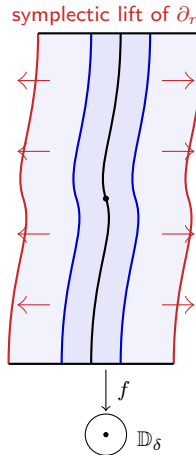
Liouville fibrations and symplectic monodromy [Annals §5.2–5.3]

- **Liouville fibration** (f, λ) : $f: M \rightarrow B$ a fibration, $(M_b, \lambda|_{M_b})$ a Liouville domain for every b .
- **Symplectic connection**: horizontal $= (\ker df)^\perp d\lambda$; the **symplectic lift** of a vector field on B ; parallel transport along a path is a symplectomorphism between the fibres.
- Over $B = \mathbb{S}^1$: a **symplectic monodromy** is a monodromy of the connection which is an exact symplectomorphism of the fibre, identity near its boundary. It need not exist; existence is proved in the cases used here [Annals §5] by choosing the domain: next slide.
- Two techniques: at positive radius the symplectic monodromy is built by **Moser-type arguments**, deforming a non-symplectic monodromy trivialization into an exact symplectomorphism equal to the identity near the boundary [Annals §5]; at radius zero it is read directly off the **symplectic connection** of ω_A .



fibres $f^{-1}(z) \cap \mathbb{B}_\varepsilon$:
volumes vary, no exact monodromy

versus



boundary swept out from ∂F_0 : fibres symplectomorphic,
constant volume, exact monodromy, identity near ∂F

The ball gives a fibration whose fibres have different volumes; the domain whose side boundary is swept out from ∂F_0 by the symplectic lift of the radial field is a Liouville fibration on which the symplectic monodromy exists [Annals, Setting 5.12, Exs. 5.20, 5.23]. No Milnor radius enters.

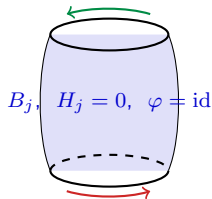
Definition [Annals, Def. 5.25]

An exact symplectomorphism φ of a Liouville domain (M, λ) , equal to the identity near ∂M or without fixed points near ∂M , has **special dynamics** if $\text{Fix } \varphi = \bigsqcup_j B_j$ with $B_j \subset \text{Int } M$ compact codimension-zero submanifolds with corners, and for each j there are a neighbourhood N_j of B_j and an autonomous Hamiltonian $H_j: N_j \rightarrow \mathbb{R}$ with

$$H_j^{-1}(0) = B_j = \text{Fix } \varphi|_{N_j}, \quad \varphi|_{N_j} = \psi_{H_j}^1.$$

- $\partial^+ B_j$, resp. $\partial^- B_j$: the boundary components of B_j along which $H_j > 0$, resp. $H_j < 0$, just outside B_j ; $\partial B_j = \partial^+ B_j \sqcup \partial^- B_j$.
- The action a , with $\varphi^* \lambda - \lambda = -da$, and the Conley–Zehnder index are constant on each B_j [Annals, Lemma 5.27].
- This is the symplectic form of the topological definition of Lecture 1: $V_j = X_{H_j}$ vanishes exactly on B_j and preserves the levels of the defining function H_j .

$\partial^+ B_j$: $H_j > 0$ outside



$\partial^- B_j$: $H_j < 0$ outside

the Hamiltonian flow rotates the boundary tori; the sense of rotation just outside B_j is the sign

$h: X \rightarrow \mathbb{C}^{n+1}$ an m -separating log resolution factoring through the blowup of 0; $\tilde{f} = f \circ h$, $D = \tilde{f}^{-1}(0) = \sum_E N_E E$.

- **Step 1.** $\tilde{\lambda} := h^* \lambda_{\text{std}}$; $d\tilde{\lambda} = h^* \omega_{\text{std}}$ is Kähler on $X \setminus D$, degenerate along the exceptional divisors.
- **Step 2.** $H = -\sum_{E \text{ exc}} b_E E$ ample, $b_E > 0$; s a section of $\mathcal{O}_X(bH)$ with poles of order bb_E along E ; $\lambda_{FS} := d^c \log \|s\|$, $d\lambda_{FS} = \omega_{FS}$ Kähler on X . $\lambda_X := \tilde{\lambda} + t_0 \lambda_{FS}$, $\omega_X := d\lambda_X$ Kähler on X .
In an adapted chart $\lambda_X = -\sum_{i \in S} t_0 b b_i d\theta_i + \beta$, β smooth on U_X : $\pi^* \lambda_X$ extends smoothly to A [Annals (116)].
- **Step 3.** A'Campo modification: with $\hat{s}_i = \log \hat{r}_i$, $\hat{r}_i$ the distance function to D_i of Lecture 1, $\hat{\alpha}_i := -d^c \hat{s}_i$ is the global angular form $d\theta_i$; $\lambda^\# := \sum_i \hat{v}_i \hat{\alpha}_i$ [CPAM (17),(20)],

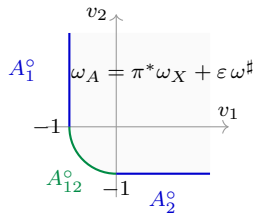
$$\lambda_A := \pi^* \lambda_X + \varepsilon \rho(|f|) \lambda^\#, \quad \omega_A := d\lambda_A,$$

$$\omega_A = \pi^* \omega_X + \varepsilon \omega^\# \text{ near } \partial A, \quad \omega_A = \pi^* \omega_X \text{ over } |z| \geq \delta'.$$

Proposition [Annals, Prop. 4.2; CPAM, Prop. 4.3]

For $\varepsilon > 0$ small, ω_A is fiberwise symplectic on all of A , ∂A included, and fiberwise Kähler at positive radius ($\lambda^\#$ is fiberwise d^c of a potential).

- Proof ideas: positivity of $\omega_A(\cdot, J\cdot)$ at positive radius by Gaussian diagonalization in an adapted chart; non-degeneracy at radius zero by computing the symplectic orthogonal to the fibres.
- On the interior of a maximal face, I with $\#I = n+1$: $\omega_A = \varepsilon \sum_{i \in I} dv_i \wedge d\theta_i$.
- Arrow (b) of the proof: $\lambda_t = \tilde{\lambda} + t\lambda_{FS}$, $t \in [0, t_0]$, is an isotopy of Liouville fibrations from the standard form to λ_X [Annals §7.2].



on A_{12}^o : $\varepsilon(dv_1 \wedge d\theta_1 + dv_2 \wedge d\theta_2)$

the domain: the central fibre with X_{λ_X} outward along its boundary, the rest swept out by the radial lift

The symplectic lift of ∂_θ at radius zero [Annals (48), (119)]

On $A_I^\circ \cap F_\theta$: coordinates $(v_i, \theta_i)_{i \in I}$, $\omega_A = \varepsilon \sum_{i \in I} dv_i \wedge d\theta_i$ on the open face, $v_i = -\eta(w_i)$, $\zeta := (\eta^{-1})'$. The ω_A -orthogonal lift of the angular field of $\partial \mathbb{C}_{\log}$ is

$$\left(\sum_{i \in I} N_i \zeta(u_i) \right)^{-1} \sum_{i \in I} \zeta(u_i) \frac{\partial}{\partial \theta_i},$$

hence $\varphi^m: \theta_j \mapsto \theta_j + 2\pi m \frac{w'_j}{N_j}$, $w'_j := \frac{N_j \zeta(u_j)}{\sum_{i \in I} N_i \zeta(u_i)}$.

$(w'_j)_{j \in I}$ is barycentric: **A'Campo's flow** V of Lecture 1 after a reparametrization of the simplices. Board: the computation of the orthogonal.

Definition: the symplectic monodromy at radius zero

$\varphi: F_\theta \rightarrow F_\theta$ the time- 2π map of this lift: an exact symplectomorphism for ω_A , identity near ∂F_θ , a symplectic monodromy of the Liouville fibration $\partial A \rightarrow \mathbb{S}^1$. The radial lift gives the isotopy to the tube, arrow (a) of the proof.

Dynamics, m -separating resolution

[Annals, Prop. 7.4]

Fix $\varphi^m = \bigsqcup_{N_E | m} B_E$.

- **Exceptional E** : B_E is a codimension-zero family in the symplectic sense with **all boundaries** ∂^+ , $\partial^- B_E = \emptyset$; $B_E \setminus \partial^+ B_E \cong B_E^\circ$; action constant on B_E .
- **Strict transform**: B_0 reaches ∂F_θ , where $\varphi^m = \text{id}$ and no Hamiltonian neighbourhood exists: φ^m has no special dynamics.
- Corollary: A'Campo's formula in every dimension.

Twisting

On a collar $C \cong \partial F \times [1 - \kappa, 1]$ with coordinate r , compose φ with the time- τ flow of $H = H(r)$, τ small: H **increasing** turns ∂F in the **positive** Reeb direction, **decreasing** in the **negative** one; $\check{\varphi}_{\pm} := \psi_{H_{\pm}}^{\tau} \circ \varphi$. No fixed points near ∂F , pieces B_E unchanged: $\check{\varphi}_{\pm}$ **has special dynamics**.

The strict-transform piece $B_0 \cong L \times [-1, 0]$

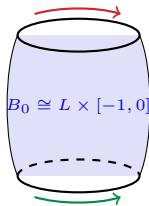
Inner end $L \times \{-1\}$: $\partial^+ B_0$ for both twists. Collar end $L \times \{0\}$: $\partial^- B_0$ for the positive twist, $\partial^+ B_0$ for the negative one [Annals, Rem. 5.26, §6.2]:

$$H_{\bullet}(B_0, \partial^+ B_0) = \begin{cases} 0 & \text{positive class,} \\ H_{\bullet-1}(L) \neq 0 & \text{negative class.} \end{cases}$$

- For τ small the isotopy class of $\check{\varphi}_{\pm}$ does not depend on τ , H : **two classes of symplectic monodromy with special dynamics** [Annals §6.2; McLean §4].

positive twist

collar end $L \times \{0\}$: ∂^-

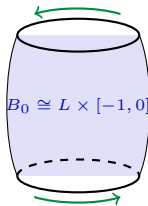


inner end $L \times \{-1\}$: ∂^+

$$H_{\bullet}(B_0, \partial^+ B_0) = 0$$

negative twist

collar end $L \times \{0\}$: ∂^+



inner end $L \times \{-1\}$: ∂^+

$$H_{\bullet}(B_0, \partial^+ B_0) = H_{\bullet-1}(L)$$

the lower circles turn the same way in both pictures, the upper ones turn opposite ways: this is the whole difference between the two classes, and only the positive one can give a multiplicity theorem

To every exact symplectomorphism φ of a graded Liouville domain (F, λ) , $\dim F = 2n$, with no fixed points on ∂F , is assigned a $\mathbb{Z}$ -graded $\mathbb{Z}/2$ -vector space $\mathrm{HF}_\bullet(\varphi)$. For a symplectic monodromy: $\mathrm{HF}_\bullet(\varphi, \pm) := \mathrm{HF}_\bullet(\check{\varphi}_\pm)$.

(A1) Invariance [Annals, Prop. 6.2; McLean (HF1)]

Isotopic φ 's, through exact symplectomorphisms without fixed points on the boundary, have isomorphic $\mathrm{HF}_\bullet$.

(A2) Euler characteristic [McLean §4]

$$\chi(\mathrm{HF}_\bullet(\varphi)) = (-1)^n \Lambda(\varphi).$$

(A3) The McLean spectral sequence, for φ with special dynamics [Annals, Prop. 6.3; McLean (HF3)]

If $\mathrm{Fix} \varphi = \bigsqcup_{B \in \mathcal{B}} B$ with each B satisfying $\partial^+ B = \emptyset$ or $\partial^- B = \emptyset$, and $\iota: \mathcal{B} \rightarrow \mathbb{Z}$ is strictly increasing in the action, then

$$E_{p,q}^1 = \bigoplus_{\iota(B)=p} H_{n+p+q+\mathrm{CZ}(B)}(B, \partial^+ B; \mathbb{Z}/2) \implies \mathrm{HF}_\bullet(\varphi).$$

- Three of McLean's properties; the fourth, dependence on the contact pair of the link, is not needed once the isotopy of Liouville fibrations is available.
- For a piece B_E over an exceptional divisor: $H_\bullet(B_E, \partial^+ B_E) = H_\bullet^{\mathrm{BM}}(B_E^\circ)$, Borel–Moore homology of the Denef–Loeser cover.
- Conley–Zehnder indices only shift degrees; not used today. The complementary lecture opens the box: twisted loops, action filtration, Morse–Bott.

f_t over the fixed ball $\mathbb{B}_{\varepsilon_0}$ of f_0 ; the case $t = 0$ included. h an m -separating resolution of f_t factoring through the blowup of 0, $D = \sum_E N_E E$, strict transform E_0 with $N_{E_0} = 1$. Symplectic monodromy of $f_t \circ h$ at radius zero, twisted into the positive or the negative class.

- Fixed points: $\text{Fix } \check{\varphi}_{\pm}^m = \bigsqcup_{N_E | m, E \text{ exc}} B_E \sqcup B_0$, all with special dynamics, $\partial^- B_E = \emptyset$ [Prop. 7.4].
- First page of (A3), one column per divisor since the action is constant on each piece:

$$E^1 = \bigoplus_{N_E | m, E \text{ exc}} H_{\bullet}^{\text{BM}}(B_E^{\circ}) \oplus H_{\bullet}(B_0, \partial^+ B_0).$$

- The exceptional terms are the same in both classes. **The only difference between + and − is the strict-transform term.**
- $B_0 \cong W := f_t^{-1}(0) \cap \mathbb{B}_{\varepsilon_0} \setminus \mathring{\mathbb{B}}_{\varepsilon_t}$, a cobordism from $L_{\text{in}} = f_t^{-1}(0) \cap \partial \mathbb{B}_{\varepsilon_t}$, the link of f_t , to $L_{\text{out}} = f_t^{-1}(0) \cap \partial \mathbb{B}_{\varepsilon_0}$:

$$H_{\bullet}(B_0, \partial^+ B_0) = \begin{cases} H_{\bullet}(W, L_{\text{in}}) & + \text{ class,} \\ H_{\bullet}(W, \partial W) \cong H^{2n-\bullet}(W) \neq 0 & - \text{ class.} \end{cases}$$

Consequence

The multiplicity argument needs the strict-transform term to vanish for $m < \nu$. In the negative class it never does: only the positive class can work.

When it vanishes

$H_{\bullet}(W, L_{\text{in}}) = 0$ exactly when W is a **homologically trivial cobordism**:

- f_0 : $W \cong L \times [-1, 0]$ by the conical structure theorem;
- f_t , $t \neq 0$: by Lê–Ramanujam.
- Then for $m < \nu$ the first page is zero, and for $m = \nu$ it has one nonzero column.

Positive class, radius zero, A'Campo form, $\mathbb{B}_{\varepsilon_0}$ a Milnor ball of f_0 ; resolution ν -separating, $\nu = \nu(f_0)$, through the blowup of 0:

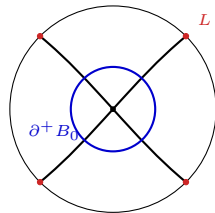
$E_1 \cong \mathbb{P}^n$ with $N_{E_1} = \nu$, every other exceptional $N_E > \nu$.

- $m < \nu$. Fix $\check{\varphi}_+^m = B_0$ and
 $H_\bullet(B_0, \partial^+ B_0) = H_\bullet(L \times [-1, 0], L \times \{-1\}) = 0$: $E^1 = 0$,
 $\text{HF}_\bullet(\varphi^m, +) = 0$.
- $m = \nu$. Fix $\check{\varphi}_+^\nu = B_0 \sqcup B_1$; the B_0 column is zero, the B_1 column is

$$H_\bullet^{\text{BM}}(B_1^\circ) \neq 0, \quad B_1^\circ = \{\text{in } f_0 = 1\} = F^{\text{in}},$$

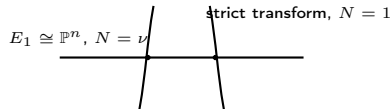
the ν -fold cover of $\mathbb{P}^n \setminus \{\text{in } f_0 = 0\}$: the Milnor fibre of the tangent cone, a nonempty complex manifold.

- So $\text{HF}_\bullet(\varphi^\nu, +) \cong H_{\bullet+c}^{\text{BM}}(F^{\text{in}}) \neq 0$: McLean's theorem recovered [Annals (125), Rem. 8.2–8.3].
- Cusp: B_1 two discs, $\text{HF}_\bullet(\varphi^2, +)$ of rank 2. Node: $\Lambda \equiv 0$ but $\text{HF}_\bullet(\varphi^2, +) \cong H_\bullet^{\text{BM}}(\mathbb{C}^\times) \neq 0$.



$$f_0^{-1}(0) \cap \mathbb{B}_{\varepsilon_0} \cong \text{Cone}(L)$$

$$B_0 \cong L \times [-1, 0], \quad H_\bullet(B_0, \partial^+ B_0) = 0$$



$$B_1^\circ: \nu\text{-fold cover of } \mathbb{P}^n \setminus \{\text{in } f_0 = 0\}$$

$F(x, t) = (f_t(x), t)$ a μ -constant family; ε_0 a Milnor radius of f_0 ; $\varepsilon_t < \varepsilon_0$ a Milnor radius of f_t .

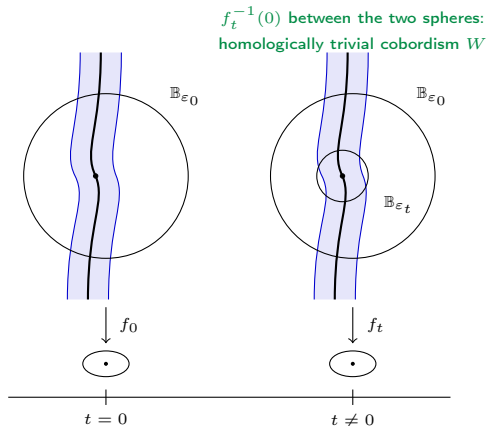
- Over $\mathbb{B}_{\varepsilon_0}$ the fibration of f_t exists, but $\mathbb{B}_{\varepsilon_0}$ is not a Milnor ball of f_t : $f_t^{-1}(0) \cap \mathbb{B}_{\varepsilon_0}$ is not a cone.
- The computation of the previous slides for f_t over $\mathbb{B}_{\varepsilon_0}$ is verbatim except for the strict-transform term, now

$$H_{\bullet}(W, L_{\text{in}}), \quad W = f_t^{-1}(0) \cap \mathbb{B}_{\varepsilon_0} \setminus \mathring{\mathbb{B}}_{\varepsilon_t}.$$

Theorem [Lê–Ramanujam 1976; Annals, Prop. 5.24]

For t small, W is a homologically trivial cobordism:
 $H_{\bullet}(W, L_{\text{in}}; \mathbb{Z}/2) = 0$.

- Hence, for every t :
 $\nu(f_t) = \min\{m \geq 1 : \text{HF}_{\bullet}(\varphi_t^m, +) \neq 0\}$, computed over the same ball $\mathbb{B}_{\varepsilon_0}$.
- The obstacle avoided: ε_t may shrink with t , and the cobordism between the links is topologically trivial for $n \geq 3$ but not known to be symplectically trivial [Annals §1.1]. Only its homology enters.



the family over the parameter line: both fibrations over the same ball

$\mathbb{B}_{\varepsilon_0}$; $f_0^{-1}(0)$ is a cone in it, $f_t^{-1}(0)$ only inside $\mathbb{B}_{\varepsilon_t}$

Theorem [F. de Bobadilla–Peřka, Ann. of Math. 2024]

$f_t: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, $t \in [0, 1]$, continuous family with $\mu(f_t) < \infty$ constant $\implies \nu(f_t)$ constant.

Six graded abstract contact open books over the fixed ball $\mathbb{B}_{\varepsilon_0}$, five isotopies; $\mathrm{HF}_\bullet(\cdot^m, +)$ is constant along the chain by (A1):

$$\begin{aligned} (F_0^A, \varphi_0^A)_{r=0} &\xrightarrow{(a)} (F_0^A, \varphi_0^A)_{r>0} \xrightarrow{(b)} (F_0^{\mathrm{std}}, \varphi_0^{\mathrm{std}}) \xrightarrow{(c)} (F_t^{\mathrm{std}}, \varphi_t^{\mathrm{std}}) \\ &\xrightarrow{(d)} (F_t^A, \varphi_t^A)_{r>0} \xrightarrow{(e)} (F_t^A, \varphi_t^A)_{r=0}. \end{aligned}$$

- **(a), (e)**: symplectic lift of the radial field of $\mathbb{C}_{\log}$, radius zero to positive radius, for f_0 and for f_t [Annals §7.2, Cor. 5.19].
- **(b), (d)**: the isotopy of forms $\lambda_s = \tilde{\lambda} + s\lambda_{FS}$, $s \in [0, t_0]$, read in opposite directions [§7.2].
- **(c)**: the Liouville fibration of f_t over the fixed ball of f_0 : the domain over the disc whose boundary is swept out from ∂F_0 by the symplectic lift of ∂_r , so that parallel transport never leaves it and the fibres have constant volume [Setting 5.12, Exs. 5.20, 5.23].
- Multiplicities computed at the two ends, at radius zero, with the A'Campo form. Continuous families reduce to holomorphic ones in t [Lemma 8.1].

The balls

- Form and A'Campo space: any domain with X_{λ_X} outward along the boundary of the central fibre, the rest swept out by the radial lift; no radius condition.
- f_0 : a Milnor ball of f_0 , used only for $H_\bullet(B_0, \partial^+ B_0) = 0$ by the conical structure.
- f_t : the same ball, not a Milnor ball of f_t ; $H_\bullet(B_0, \partial^+ B_0) = 0$ by Lê–Ramanujam.
- Over this ball there is no sphere fibration for f_t .

Zariski's question today

Known: homeomorphism of class C^1 [Ephraim, Trotman]; plane curves [Zariski]; one germ smooth [A'Campo, Lê]; surfaces: quasihomogeneous [Xu–Yau], bilipschitz [FFS], $\nu = 2$ [Navarro Aznar]; $\mu \leq 2n + 1$ [Yau–Zuo]; bilipschitz right-left equivalence [Risler–Trotman]; μ -constant families [FdB–Peřka]. Open in general.

Tube and sphere

- The course works with the Milnor fibration in the tube; McLean works in the sphere, through the contact structure on $\mathbb{S}_\varepsilon$ and Giroux's correspondence.
- The equivalence of the two symplectic Milnor fibrations is not written in the literature: an open point, not an obstacle. The tube version is what the A'Campo space provides.

Why the tube for families

- Over the Milnor radius of f_0 there is no sphere fibration for f_t .
- A sphere argument would need the symplectic triviality of a cobordism between contact pairs; the tube argument needs only the homological triviality of the Lê–Ramanujam cobordism.

Beyond

- Same statement for $f \in k[t][[z_0, \dots, z_n]]$, k any field of characteristic zero [Rem. 8.4]; no algebraic proof known.
- Topologically equivalent germs not joined by a μ -constant family [Ex. 8.6].
- The fixed point structure at radius zero is also what the arc-Floer conjecture computes on the arc side: complementary lecture.

The sphere side [McLean, Geom. Topol. 2019]

- Contact pair: $(L_f \subset \mathbb{S}_\varepsilon, \xi_{\text{std}}, \Phi_f)$, the link with the contact structure of the sphere and the trivialization Φ_f of its normal bundle by df [Ex. 3.8].
- The Milnor fibration $\arg f: \mathbb{S}_\varepsilon \setminus L_f \rightarrow \mathbb{S}^1$ is a contact open book supported by this pair [Caubel–Némethi–Popescu-Pampu].
- Giroux's correspondence: contact open books $\leftrightarrow$ abstract contact open books, up to isotopy [Giroux 2002; McLean, Thm. 3.15].
- (HF2): $\text{HF}_\bullet(\varphi^m, +)$ depends only on the graded contact pair.

The resolution side

- Model resolution: a log resolution with the Kähler form of an ample divisor supported on the exceptional locus, $\theta = -d^c \log \|s\|$ [Ex. 5.14].
- Its link, the boundary of a regular neighbourhood of the exceptional locus with the induced contact structure, is contactomorphic to the link of f : convexity of plurisubharmonic functions [Varchenko; CNPP, §3].
- Its monodromy has the fixed point structure of A'Campo's model, with the spectral sequence (A3).

Here

The A'Campo space realizes the same fixed point structure and supplies the isotopy of Liouville fibrations directly, which is what a family over a fixed ball requires; (HF2) and Giroux's correspondence are not used.

Symplectic A'Campo spaces and two applications

Lecture 3. Lagrangian torus fibrations for maximal Calabi–Yau degenerations

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BCAM – Ikerbasque

Lecture 1	A'Campo's monodromy and the A'Campo space
Lecture 2	symplectic monodromy at radius zero and equimultiplicity of μ -constant families
Complementary lecture	Floer homology, contact loci and the arc-Floer conjecture (Eduardo de Lorenzo Poza)
Lecture 3	Lagrangian torus fibrations for maximal Calabi–Yau degenerations

Definition

A **Calabi–Yau manifold** is a compact Kähler manifold X of dimension n with K_X trivial: there is a nowhere vanishing holomorphic n -form Ω , unique up to a constant. Here X is projective, with a Kähler form ω_{FS} from the polarization.

Calabi’s conjecture [1954], Yau’s theorem [1978]

Every Kähler class on X contains a unique Ricci-flat Kähler form ω_{CY} .

- Ricci-flatness is the **complex Monge–Ampère equation**: with $\text{Ric}(\omega) = -\iota \partial \bar{\partial} \log(\omega^n / \Omega \wedge \bar{\Omega})$,

$$\text{Ric}(\omega_{CY}) = 0 \iff \omega_{CY}^n = c \cdot \left(\frac{\iota}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega \wedge \bar{\Omega}, \quad c = \frac{\int_X \omega_{FS}^n}{\int_X \left(\frac{\iota}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega \wedge \bar{\Omega}}.$$

- By the $\partial\bar{\partial}$ -lemma, $\omega_{CY} = \omega_{FS} + dd^c \varphi_{CY}$ for a function φ_{CY} : **the Ricci-flat form is the given form modified by a potential**, and Yau’s theorem is the existence of φ_{CY} solving $(\omega_{FS} + dd^c \varphi_{CY})^n = c \Omega \wedge \bar{\Omega}$.
- The proof is hard analysis: the continuity method with a priori estimates up to $C^{2,\alpha}$ [Yau], later variational and pluripotential methods [Kłodziej; Berman–Boucksom–Guedj–Zeriahi; Eyssidieux–Guedj–Zeriahi]; each proof has its virtues and each has driven further developments. There is no closed formula for φ_{CY} .
- Today: modify the given Kähler form by an **explicit** potential, $\omega_q = \omega_X + dd^c \varphi_q$, on a degenerating family, and recover asymptotically what one expects from ω_{CY} .

- $f^\circ: X^\circ \rightarrow \mathbb{C}^*$ projective, fibres X_z Calabi–Yau n -folds. For simplicity, after a finite base change, a [semistable model](#) $f: X \rightarrow \mathbb{D}: X$ smooth, $X_0 = \sum_{i=1}^N D_i$ reduced snc.
- Dual complex Δ : a vertex per D_i , a face Δ_I per nonempty stratum $X_I = \bigcap_{i \in I} D_i$,
 $\Delta_I = \{w \in \mathbb{R}_{\geq 0}^N : \sum_i w_i = 1, w_i = 0 \text{ for } i \notin I\}$, with the euclidean metric of $\mathbb{R}^N$.
- **Weights:** Θ a generator of $K_{X/\mathbb{D}}$, $\Omega = \Theta/d \log f$ on X° ,
 $\nu_i := 1 + \text{ord}_{D_i} \Theta$, so $K_X + X_0 \sim \sum_i \nu_i D_i$: the log discrepancies of the pair (X, X_0) .
- **Essential components:** $\mathcal{S} = \{i : \nu_i \text{ minimal}\}$; [essential skeleton](#) $\Delta_{\mathcal{S}}$: the subcomplex of Δ spanned by the faces Δ_I with $I \subseteq \mathcal{S}$ [Kontsevich–Soibelman 2001; Mustașă–Nicaise 2015].

Berkovich characterization [MN 2015;

Nicaise–Xu 2016]

On the Berkovich analytification X_K^{an} of the generic fibre, $K = \mathbb{C}((z))$, the form Ω defines a [weight function](#) $\text{wt}_\Omega: X_K^{\text{an}} \rightarrow \mathbb{R} \cup \{\infty\}$, equal to ν_i at the divisorial point of D_i and affine on the faces of Δ , which embeds in X_K^{an} by monomial valuations. Then

$$\Delta_{\mathcal{S}} = \{\text{wt}_\Omega = \min\} \subseteq X_K^{\text{an}}.$$

- Hence $\Delta_{\mathcal{S}}$ depends only on (X_K, Ω) : independent of the model, as a piecewise affine space; the retraction $X_K^{\text{an}} \rightarrow \Delta$ of any snc model restricts to the identity on it.
- Compatible with base change $z = z'^k$: the weight function rescales, its minimum locus does not move.
- $\Delta_{\mathcal{S}}$ is a pseudomanifold with boundary [NX 2016], and $\dim \Delta_{\mathcal{S}} \leq n$.

Definition: three equivalent conditions [KS 2001;

NX 2016; Boucksom–Jonsson 2017]

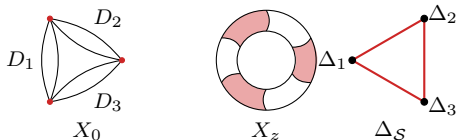
$f^\circ: X^\circ \rightarrow \mathbb{C}^*$ is a **maximal** Calabi–Yau degeneration if one of the following holds; then all do.

1. The monodromy T on $H^n(X_z)$ is maximally unipotent: $(T - 1)^{n+1} = 0 \neq (T - 1)^n$, one Jordan block of size $n + 1$.
2. The volume grows as fast as possible:
 $\int_{X_z} \Omega_z \wedge \bar{\Omega}_z \sim C |z|^{2a} |\log |z||^n$ as $z \rightarrow 0$.
3. $\dim \Delta_S = n$.

- Rescaling Ω by a power of z changes only the factor $|z|^{2a}$: **the exponent of $|\log |z||$ is the invariant**, and in general it equals $\dim \Delta_S$; the normalized volume converges to Lebesgue measure on the top-dimensional faces of Δ_S [Boucksom–Jonsson 2017].
- These are the large complex structure limits of mirror symmetry; the Ricci-flat metrics collapse to a base of half the real dimension exactly in this case [Li–Tosatti 2026].
- Faces: Δ_J **maximal**, $\#J = n + 1$, X_J a point; Δ_I **submaximal**, $\#I = n$, X_I a curve.

Examples: $z \sum_i x_i^{n+2} = \prod_i x_i$ in $\mathbb{P}^{n+1} \times \mathbb{C}$

$X_0 = \{\prod x_i = 0\}$, the $n + 2$ coordinate hyperplanes; every component essential; Δ_S the boundary of the $(n + 1)$ -simplex, a sphere $\mathbb{S}^n$.



quartic K3: $\partial\Delta^3 \cong \mathbb{S}^2$

quintic threefold: $\partial\Delta^4 \cong \mathbb{S}^3$

Hesse pencil: X_0 , a fibre X_z , $\Delta_S = \partial\Delta^2 \cong \mathbb{S}^1$; resolving the singular points does not change the skeleton

- The question: the asymptotic metric structure of $(X_z, \omega_{\text{CY}})$ as $z \rightarrow 0$, with the Riemannian metric $g_{\text{CY}} = \omega_{\text{CY}}(\cdot, J\cdot)$ rescaled to constant diameter, d_{CY} .
- Strominger–Yau–Zaslow [1996]: near a maximal degeneration the Calabi–Yau manifold should carry a [special Lagrangian torus fibration](#), the mirror being the dual fibration; fibres L with $\dim L = n$, $\omega_{\text{CY}}|_L = 0$ and $\text{Im}(\varpi \Omega)|_L = 0$ for a constant phase $\varpi \in \mathbb{S}^1$.

Conjecture [Kontsevich–Soibelman 2001, Conj. 1 and 2]

Let $f^\circ: X^\circ \rightarrow \mathbb{C}^*$ be a maximal Calabi–Yau degeneration.

1. There are $X_z^{\text{sm}} \subseteq X_z$ and a special Lagrangian torus fibration $\ell: X_z^{\text{sm}} \rightarrow B^\circ$ over an affine manifold B° with a metric d_B of the form below.
2. $(X_z, X_z \setminus X_z^{\text{sm}}; d_{\text{CY}}) \xrightarrow[\text{GH}]{z \rightarrow 0} (B, B \setminus B^\circ; d_B)$, with B compact and $\text{codim}_B(B \setminus B^\circ) \geq 2$.

Fine structure of the base

B° is an [affine manifold](#): charts with affine transition maps; d_B is a [Hessian metric](#), $g_B = \sum_{i,j} \frac{\partial^2 \varphi}{\partial w_i \partial w_j} dw_i dw_j$ in affine coordinates (w_i) , with φ convex and satisfying the [real Monge–Ampère equation](#) $\det \left(\frac{\partial^2 \varphi}{\partial w_i \partial w_j} \right) = \text{const}$, the collapsed form of the complex one. Expected: $B = \Delta_S$, $B^\circ = \Delta_S \setminus \Delta_S^{\geq 2}$, and $\text{vol}(X_z^{\text{sm}})/\text{vol}(X_z) \rightarrow 1$ [Li, survey 2022].

- Where the affine coordinates come from: for a Lagrangian torus fibration the action coordinates of Arnold–Liouville are integral affine coordinates on the base and the angle coordinates parametrize the fibres; in our construction they will be the w_i and the θ_i .

Known, 2000–2024

- K3 surfaces: hyperkähler rotation [Gross–Wilson 2000].
- Non-archimedean counterpart over $\Delta_S \setminus \Delta_S^{\geq 2}$ [Nicaise–Xu–Yu 2019]; the NA Calabi–Yau potential φ_0 [Boucksom–Favre–Jonsson 2015], C^0 limit of φ_{CY} [Li–Tosatti].
- Metric conjecture reduced to a comparison property of φ_0 [Li, Duke 2023]; Fermat families [Li, Acta 2022], classes of hypersurfaces [Li 2024; HJMM 2024].

The weak metric SYZ conjecture is now a theorem

Weak metric SYZ, the measure-theoretic form: for every $\delta > 0$ and $|z|$ small, a special Lagrangian T^n -fibration for $(X_z, \omega_{CY}, \Omega)$ on an open subset of Calabi–Yau measure $\geq 1 - \delta$. Valuable independence of a basis of $\bigoplus_l H^0(X_K, lL)$ gives the comparison property [Li, arXiv:2605.00516]; such bases exist for every polarised Calabi–Yau degeneration [Blum–Liu, arXiv:2604.27890].

What is achieved, sharply [Li 2026, Thm. 4.2, Cor. 4.3, Rem. 4.4]

- Base: an open dense U of full measure in the *union of the open maximal faces*, where φ_0 factors through the retraction and solves real Monge–Ampère.
- The fibration lives over a compact $U_\varepsilon \Subset U$ of measure $> 1 - \delta$, after removing the singular set of the real Monge–Ampère solution, closed of Hausdorff dimension $\leq n - 1$ [Mooney]: **codimension one, not two**, inside the maximal faces.
- Nothing is known about the topology of U : it could be disconnected; no clue about a codimension-two discriminant, or an enlargement of one as Joyce’s examples suggest [Joyce 2003].

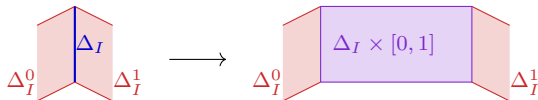
Today

Fake Calabi–Yau forms in place of ω_{CY} : a Lagrangian torus fibration over the **connected** expanded skeleton, discriminant in the codimension-two faces; the conjectured Gromov–Hausdorff limit; asymptotics on the generic region [FdB–Petka, CPAM 2026].

The model [CPAM, Prop. 2.1; cf. NXY]

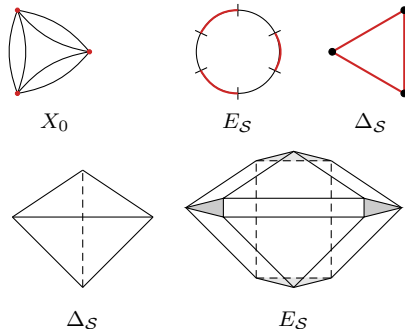
After Fujino's semistable MMP and a log resolution: essential components have $N_i = 1$ and $\nu_i = 1$, every submaximal essential face Δ_I lies in exactly two maximal faces Δ_I^0, Δ_I^1 , and $(X_I, X_I^\circ) \cong (\mathbb{P}^1, \mathbb{C}^*)$.

- E_S : replace each submaximal Δ_I by $\Delta_I \times [0, 1]$, glued to $\Delta_I \subseteq \Delta_I^j$ at $\Delta_I \times \{j\}$. A manifold with corners, connected, with interior homeomorphic to $\Delta_S \setminus \Delta_S^{\geq 2}$.
- $E_S^{\text{gen}} \subseteq E_S$: the open maximal faces, the **generic region**.



Hesse pencil [CPAM, Ex. 5.7]

Three vertices, three edges: E_S is a circle cut into six arcs, three for the lines D_i , three for the vertices $D_i \cap D_j$; the radius-zero fibre is a torus cut into six cylinders.

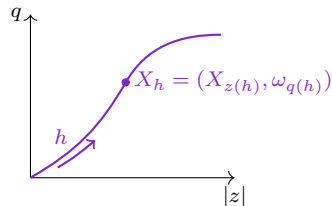


quartic K3: Δ_S a tetrahedron, E_S four triangles and six squares
[CPAM, Fig. 6]

$f: X \rightarrow \mathbb{D}_\delta$ the model above, ω_X any Kähler form on X . We construct **fake Calabi–Yau forms** $\omega_q = \omega_X + dd^c \varphi_q$, $q \in [0, 1]$, with $\omega_q|_{X_z}$ Kähler, and **fake Calabi–Yau metrics** $g_q = t \cdot \omega_q(\cdot, J\cdot)$, $t = -1/\log |z|$; move along an **admissible path** $h \mapsto (z(h), q(h))$, $|z(0)| = q(0) = 0$, $|z(h)|, q(h) > 0$ for $h > 0$; $X_h := (X_{z(h)}, \omega_{q(h)})$, d_h the metric of $g_{q(h)}$ rescaled to constant diameter, $\text{vol}_{\text{new}}^\Omega := t^n \cdot \text{const} \cdot \Omega \wedge \bar{\Omega}$.

Theorem [FdB–Peřka, CPAM 2026, Thms. 1.1, 1.2]

- Lagrangian fibration.** For $q > 0$: a Lagrangian torus fibration $\ell_h: X_h^{\text{sm}} \rightarrow E_S$, $X_h^{\text{sm}} \subseteq X_h$ of codimension zero with corners.
- Gromov–Hausdorff.** $(X_h, X_h \setminus X_h^{\text{sm}}; d_h) \xrightarrow[\text{GH}]{h \rightarrow 0} (\Delta_S, \Delta_S^{\geq 2}; d_{\text{std}})$, the euclidean metric of $\mathbb{R}^N$, for which real Monge–Ampère holds trivially.
- Volume filling.** $X_h^{\text{gen}} := \ell_h^{-1}(E_S^{\text{gen}})$: $\text{vol}_{\text{new}}^\Omega(X_h^{\text{gen}})/\text{vol}_{\text{new}}^\Omega(X_h) \rightarrow 1$.
- Asymptotic Ricci-flatness.** $\frac{1}{n!} \omega_{q(h)}^n = c_h \cdot \text{vol}_{\text{new}}^\Omega$ on X_h^{gen} with $c_h \rightarrow$ a positive constant.
- Asymptotic specialness.** Tame paths, e.g. $|z(h)| = h$: for $b \in E_S^{\text{gen}}$ and $L_h = \ell_h^{-1}(b)$, $\text{Im}(\varpi\Omega)|_{L_h}/\text{Re}(\varpi\Omega)|_{L_h} \rightarrow 0$, $\varpi \in \mathbb{S}^1$ independent of b ; the L_h asymptotically minimize volume in their class; Maslov class zero.



admissible path, [CPAM, Fig. 1a]

- $\omega_{q(h)}$ sits between ω_X and ω_{CY} : $(1 - \varepsilon)\omega_X + \varepsilon\omega_{q(h)}$ satisfies 1–5 for every $\varepsilon \in (0, 1]$.
- Off the generic region $\omega_{q(h)}^n$ does not approximate the Calabi–Yau volume [Rem. 7.4]: the metric behaviour is the conjectured one, the volume form is not.

A'Campo space of $f: X \rightarrow \mathbb{C}$, Lectures 1–2: $\widehat{s}_i = \log \widehat{r}_i$,
 $\widehat{t}_i = -1/(N_i \widehat{s}_i)$, $t = -1/\log |f|$, $\widehat{w}_i = t/\widehat{t}_i$, $\widehat{u}_i = \eta(\widehat{w}_i)$,
 $\widehat{v}_i = \widehat{t}_i - \widehat{u}_i$, angular forms $\widehat{\alpha}_i = -d^c \widehat{s}_i$.

A'Campo form, known

$\lambda^\# = \sum_{i=1}^N \widehat{v}_i \widehat{\alpha}_i$, $\omega^\# = d\lambda^\#$; fiberwise $\lambda^\# = d^c \varphi^\#$ [CPAM, Lemma 4.1]; fiberwise symplectic on all of A , radius zero included.

Semi-flat form, new

$\lambda^b = -\sum_{i \in \mathcal{S}} N_i \widehat{w}_i \widehat{\alpha}_i$, $\omega^b = d\lambda^b$. Board [CPAM, Lemma 4.2]: from $\widehat{w}_i = -N_i \widehat{s}_i t$,

$$d^c \widehat{w}_i = N_i t \widehat{\alpha}_i + t^{-1} \widehat{w}_i d^c t, \quad d^c t = -t^2 d\theta,$$

so with $\varphi^b = -\frac{1}{2t} \sum_{i \in \mathcal{S}} \widehat{w}_i^2$: $\lambda^b = d^c \varphi^b + \frac{1}{2} \sum_{i \in \mathcal{S}} \widehat{w}_i^2 d\theta$,
 fiberwise $\lambda^b = d^c \varphi^b$. On a maximal face, $\widehat{w}_i = w_i$, $\widehat{\alpha}_i = d\theta_i$:

$$\omega^b = -\sum_{i \in J} dw_i \wedge d\theta_i,$$

the flat form in action-angle coordinates. Fiberwise symplectic *only* near the open maximal faces [CPAM, Prop. 4.3(c)].

The family

$$\omega_q^\varepsilon := \pi^* \omega_X + \varepsilon(q\omega^\# + (1-q)\omega^b), \quad q \in [0, 1],$$

at positive radius = $\omega_X + dd^c \varphi_q^\varepsilon$: the fake Calabi–Yau forms.

Proposition [CPAM, Prop. 4.3]

For ε small and $|z| < \delta'$: Kähler on X_z for all $q \in [0, 1]$; fiberwise symplectic on every fibre of A , radius zero included, for $q \in (0, 1]$; for $q = 0$ fiberwise symplectic near the open maximal faces, and only there.

Why a path

We want two things: fiberwise symplectic forms, which $\omega^\#$ gives everywhere, and asymptotic flatness at radius zero, which ω^b gives near the maximal faces but which is not fiberwise symplectic elsewhere. The path $q \rightarrow 0$ trades the one for the other.

Over a maximal face Δ_J : tori on the nose

$J = n + 1$, X_J a point; $A_J^\circ \cong \Delta_J \times (\mathbb{S}^1)^{n+1}$, coordinates $(w_j; \theta_j)_{j \in J}$. **Fix the tropical coordinates w** : the torus (θ_j) , cut by $\sum_j N_j \theta_j = \theta$ into an n -torus. Lagrangian for every ω_q^ε : $\omega_q^\varepsilon(\cdot, \partial_{\theta_i}) = \varepsilon c_i d\hat{v}_i$, $c_i = q + (1 - q)N_i \zeta(u_i)$, vanishes on it [Lemma 4.5(f)]. $\ell_J = (w_j)$: $A_J^\circ \rightarrow \Delta_J$.

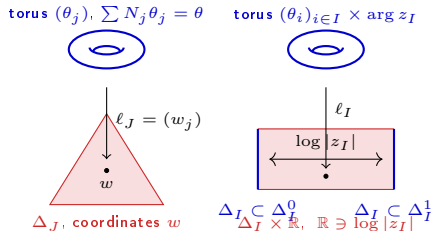
Over a submaximal face Δ_I : the angles plus one circle

$I = n$, X_I a curve; for the model of a maximal Calabi–Yau degeneration

$$X_I^\circ \cong \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^* \cong \mathbb{R} \times \mathbb{S}^1, \quad (\log |z_I|, \arg z_I).$$

$A_I^\circ \cong \Delta_I \times (X_I^\circ)_{\log}$, an $(\mathbb{S}^1)^n$ -bundle over $\mathbb{C}^*$ with angles $(\theta_i)_{i \in I}$. **Fix w and $\log |z_I|$** : the n angles times the circle $\arg z_I$, an $(n + 1)$ -torus cut by $\sum_i N_i \theta_i = \theta$ into an n -torus.

Lagrangian for $q > 0$. $\ell_I = (w, \log |z_I|)$: $A_I^\circ \rightarrow \Delta_I \times \mathbb{R}$.



Gluing [CPAM, Lemma 5.3, Prop. 5.6]

The $\mathbb{R}$ -direction of $\Delta_I \times \mathbb{R}$ is the direction transverse to Δ_I in the two adjacent maximal faces: at the ends, $-1/(N_1 \log |z_I|)$ and the face coordinate v_1 differ by a function flat to infinite order, so the maps glue to a smooth submersion; compactified, $\Delta_I \times \mathbb{R}$ is the square of E_S . Result: $\ell_S: A_{S, \theta}^{\text{sm}} \rightarrow E_S$, a Lagrangian torus fibration for $q > 0$.

Then transport by the symplectic connection to X_z : Theorem, item 1.

Fubini–Study ω_X

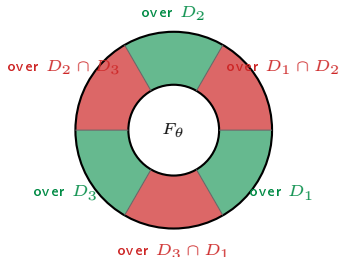
Volume of X_z stays on the divisors: $(X_z, \omega_X) \rightarrow (X_0, \omega_X)$ as $z \rightarrow 0$, the central fibre itself, a limit of full dimension $2n$. No collapse, no reason for Lagrangian tori to appear.

Calabi–Yau ω_{CY}

Normalized volume converges to Lebesgue measure on the maximal essential faces [Boucksom–Jonsson]; the metric collapses to the skeleton with its Monge–Ampère metric; special Lagrangian tori, but only over a full-measure open subset of the open faces.

Fake Calabi–Yau ω_q

$\omega_q = \omega_X + \varepsilon(q\omega^\sharp + (1-q)\omega^\flat)$ **steals a bit of volume from the divisors**, $O(\varepsilon^n)$, and puts it on the maximal faces. Rescaling by t kills the ω_X part; the stolen bit is all that survives: $g_{q,\text{new}} \rightarrow (1-q)\varepsilon \sum_i (d\hat{w}_i)^2$.



- red: the maximal faces Δ_{ij} , where the Calabi–Yau volume sits
- green: the divisors D_i , where the Fubini–Study volume sits
- ↓ fake Calabi–Yau: a bit of volume moves from green to red
- the Hesse pencil: the radius-zero fibre is a torus in six bands

With just that bit: Lagrangian tori, the conjectured Gromov–Hausdorff limit, volume filling, asymptotic Ricci-flatness and specialness on the generic region. The price: ω_q instead of ω_{CY} , Lagrangian instead of special Lagrangian away from the limit.

Chart at X_J , $f = z_1 \cdots z_{n+1}$, $w_j = -t \log |z_j|$, $\theta_j = \arg z_j$, $\sum_j w_j = 1$, $t = -1/\log |z|$.

The metric: how the tori collapse

$\varphi^b = -\frac{1}{2t} \sum_j w_j^2$, so $\omega^b = -\sum_j dw_j \wedge d\theta_j$; since $J\partial_{w_j} = -t^{-1}\partial_{\theta_j}$,

$$t \cdot \omega^b(\cdot, J\cdot) = \sum_j ((dw_j)^2 + t^2(d\theta_j)^2).$$

Hence on the face, for $q = 0$,

$$g_{0,\text{new}} = \varepsilon \sum_j (dw_j)^2 + \varepsilon t^2 \sum_j (d\theta_j)^2 + t \pi^* \omega_X(\cdot, J\cdot).$$

Base directions: length $\sqrt{\varepsilon}$, fixed. Tori: diameter of order $\sqrt{\varepsilon}t = \sqrt{\varepsilon}/|\log |z||$, **collapsing logarithmically slowly**. The ω_X part is $O(t)$ and dies. Limit: the face with the euclidean metric $\varepsilon \sum (dw_j)^2$: **Theorem, item 2**. For $q > 0$ the corrections are $O(q)$, and $q \rightarrow 0$ along the path.

The holomorphic volume form: specialness

Recall: L is **special Lagrangian** if $\omega|_L = 0$ and $\text{Im}(\varpi\Omega)|_L = 0$ for a constant $\varpi \in \mathbb{S}^1$; then $\text{Re}(\varpi\Omega)$ calibrates L , which minimizes volume in its homology class [Harvey–Lawson]. On the face,

$$\Omega = c \bigwedge_{j \neq i} d \log z_j = c \bigwedge_{j \neq i} (-t^{-1} dw_j + i d\theta_j),$$

so on a torus $L = \{w = \text{const}\}$, where $dw_j|_L = 0$,

$$\Omega|_L = c i^n d\theta_1 \wedge \cdots \wedge d\theta_n.$$

Special on the nose, with phase $\varpi = \pm \bar{c} i^{-n}/|c|$, the same for all tori of the face; $|c| = |c_0|$ is the same on all maximal faces by the residue theorem on the curves $X_I \cong \mathbb{P}^1$. At positive radius the transported tori have $\Omega|_{L_h}$ with an $O(h)$ correction, controlled along tame paths: **asymptotic specialness, item 5**; Maslov class zero.

- Volume: $\Omega \wedge \bar{\Omega} \propto |c|^2 t^{-n} \bigwedge_j dw_j \wedge d\theta_j$:
 $\text{vol}_{\text{new}}^\Omega = t^n \text{vol}^\Omega$ is finite on the faces, zero off them:
items 3, 4.