PLAN FOR PROBLEM SESSIONS (E. DINEZZA'S LECTURES)

1. Day 1

Exercise 1.1 ($\partial\bar{\partial}$-lemma)**.** Let X be a compact Kähler manifold and let $\alpha \in \mathcal{A}^{p,q}(X)$, $\mathrm{d}\alpha = 0$. Show that TFAE:

(i) α is d-exact;
(ii) α is ∂-exact;
(iii) α is $\bar{\partial}$-exact;
(iv) α is $\partial\bar{\partial}$-exact.

Exercise 1.2. Let X be an n-dimensional Fano manifold and $\omega \in c_1(X)$. Denote by $\omega_\varphi := \omega + \mathrm{dd^c}\varphi$. Show that TFAE:

(i) ω_φ is a Kähler–Einstein metric;
(ii) ω_φ is a Kähler metric with constant scalar curvature;
(iii) φ solves
$$(\omega + \mathrm{dd^c}\varphi)^n = e^{-\varphi+h}\omega^n$$
for some smooth $h : X \to \mathbb{R}$ such that $\mathrm{Ric}(\omega) = \omega + \mathrm{dd^c}h$.

Exercise 1.3. Let $\Omega \subset \mathbb{C}$ be a domain. Show the following statements:

(i) If $(u_j)_j \subset \mathrm{SH}(\Omega)$ and $u_j \searrow u \not\equiv -\infty$, then $u \in \mathrm{SH}(\Omega)$.
(ii) If $u \in \mathrm{SH}(\Omega)$, for all $r > 0$, $\Omega_r := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > r\}$, there exists $u_j \in \mathrm{SH}(\Omega_r) \cap \mathcal{C}^\infty(\Omega_r)$ such that $u_j \searrow u$.
(iii) If $u \in \mathrm{SH}(\Omega)$, then $u \in L^1_{\mathrm{loc}}(\Omega)$, i.e. $\forall K \Subset \Omega$, $\int_K |u| \mathrm{d}A < +\infty$.
(iv) If $(u_j)_j \subset \mathrm{SH}(\Omega)$ is locally uniformly bounded from above and $u_j \nearrow u$, then $u^* \in \mathrm{SH}(\Omega)$ and $u = u^*$ almost everywhere. Here $u^*(z) = \limsup_{w \to z} u(w)$ is the upper semi-continuous regularization.

Audiences can try psh functions. Remark similar properties for functions in $\mathrm{PSH}(X, \omega)$.

2. Day 2

Exercise 2.1. Show the following statements:

(i) If $\varphi, \psi \in \mathrm{PSH}(X, \omega)$, then $\log(e^\varphi + e^\psi)$, $\max\{\varphi, \psi\} \in \mathrm{PSH}(X, \omega)$.
(ii) If $\varphi \in \mathrm{PSH}(X, \omega)$, $\chi \in \mathcal{C}^2$ such that $\chi'' \geq 0$ and $0 \leq \chi' \leq 1$, then $\chi \circ \varphi \in \mathrm{PSH}(X, \omega)$.

Exercise 2.2. Compute explicit radial examples of $\mathcal{E}^p$ functions on $(\mathbb{P}^n, \omega_{\mathrm{FS}})$:

(i) Example of the form $-(-\log \|z\|)^\alpha$ for $\alpha \in (0, 1)$ and $1 \leq p \leq \frac{n(1-\alpha)}{\alpha}$;
(ii) Example of the form $-\log(-\log \|z\|)$ for all $p \geq 1$.

Exercise 2.3. Let (X, ω) be a compact Kähler manifold and let $\varphi \in \mathrm{PSH}(X, \omega)$ with $\varphi \leq -1$.

(i) Show that $\varphi_\alpha := -(-\varphi)^\alpha \in \mathcal{E}^p(X, \omega)$ for $\alpha \in (0, 1)$ and $1 \leq p \leq \frac{1-\alpha}{\alpha}$.
(ii) Show that $\psi := -\log(-\varphi) \in \mathcal{E}^p(X, \omega)$ for all $p \geq 1$.

3. Day 3

Exercise 3.1 (Following Guedj 2014 note, section 4)**.** Explain some properties of geodesics in the toric setting.

4. Day 4

Exercise 4.1 (Following Darvas 2019 note, page 76-78)**.** In the Fano setting, explain (hamiltonian) holomorphic vector fields induce geodesics.