On the Number of Multidimensional Integer Partitions

Multidimensional Partitions and Lower Sets

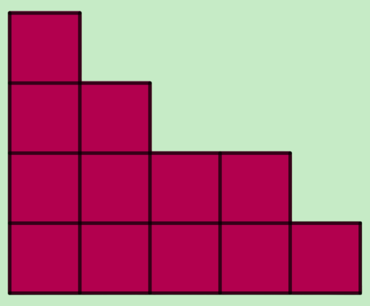
For a given d, we call a set $Q \subset \mathbb{Z}_+^d$ a ***lower set*** if for any $q = (q_1, \ldots, q_d) \in \mathbb{Z}_+^d$ the condition $q \in Q$ implies $q' = (q_1', \ldots, q_d') \in Q$ for all $q' \in \mathbb{Z}_+^d$ with $q_i' \le q_i$, $1 \le i \le d$. Importantly, there is a one-to-one correspondence between d-dimensional lower sets of cardinality n and $(d-1)$-dimensional partitions of n, that is, representations of the form

$$n = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \ldots \sum_{i_{d-1}=1}^{\infty} n_{i_1 i_2 \ldots i_{d-1}}, \quad n_{i_1 i_2 \ldots i_{d-1}} \in \mathbb{Z}_+,$$

where $n_{i_1 i_2 \ldots i_{d-1}} \ge n_{j_1 j_2 \ldots j_{d-1}}$ if $j_k \ge i_k$ for all $k = 1, 2, \ldots, d-1$. By $p_d(n)$ we denote the number of lower sets in $\mathbb{Z}_+^d$ containing n points.

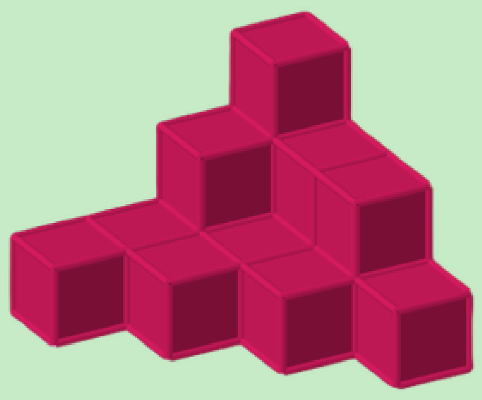
- **Integer partitions ($d = 2$).** The celebrated result by Hardy and Ramanujan (1917) revealed the asymptotics of $p_2(n)$, while the first steps in this problem go back to Leibniz (1674) and the generating function for $p_2(n)$ was established by Euler (1753):

$$p_2(n) \sim \frac{e^{\sqrt{\frac{2n}{3}}\pi}}{4\sqrt{3}n}, \qquad \prod_{k=1}^{\infty} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p_2(n) x^n.$$



- **Plane partitions ($d = 3$).** In this case the generated function (MacMahon, 1912) and the asymptotics (Wright, 1931) are also known:

$$p_3(n) \sim \frac{(2\zeta(3))^{\frac{7}{36}} e^{\zeta'(-1)}}{\sqrt{2\pi} n^{\frac{25}{36}}} e^{3(\zeta(3))^{\frac{1}{3}} 2^{-\frac{2}{3}} n^{\frac{2}{3}}}, \qquad \prod_{k=1}^{\infty} (1 - x^k)^{-k} = \sum_{n=0}^{\infty} p_3(n) x^n.$$



- **General case.** The general estimate for $p_d(n)$ was given by Bhatia, Prasad, and Arora (1997):

$$C_1(d) \le \frac{\log p_d(n)}{n^{1-\frac{1}{d}}} \le C_2(d).$$

- **Explicit constants** were suggested by Dai, Prymak, Shadrin, Tikhonov, and Temlyakov (2021):

$$C_2(d) = \pi \sqrt{\frac{2}{3}} d^{\log d} \qquad \text{and} \qquad C_1(d) = 0.9 \frac{d}{(d!)^{\frac{1}{d}}} \log 2 \quad \text{for } n > 55^d.$$

- **Absolute constants.** We show that both constants above do not depend on the dimension d provided that the number n of elements in the lower set is large enough in terms of d (K.O., 2022):

$$1 < \frac{\log p_d(n)}{n^{1-\frac{1}{d}}} < 7200 \quad \text{for } n \ge (30d)^{2d^2}.$$

Available Subsets of a Lower Set

The crucial role in our analysis is given to the "top subsets" of lower sets. Namely, we call a subset $\widetilde{Q}$ of a given lower set Q ***available*** if for any $q' \in \widetilde{Q}$ there is no $q \in Q \setminus \{q'\}$ such that $q_i \ge q_i'$ for all $i = 1, 2, \ldots, d$. Denoting by $M(Q)$ the maximal available subset of Q, we have:

- For any $d \ge 2$ and $n \ge d^{6d \log d}$,

$$T(n) := \max_{\text{lower sets } Q:\ |Q|=n} |M(Q)| \le \prod_{k=1}^{d-1} \left(1 + \frac{1}{k^2}\right) n^{1-\frac{1}{d}} < \frac{\sinh \pi}{\pi} n^{1-\frac{1}{d}}.$$

Note that this estimate is up to an absolute constant sharp, which can be seen by considering the sets Q_k defined by $x \in Q_k \Leftrightarrow x_1 + \ldots + x_d \le k$.

Such an estimate allows us to obtain the number of lower subsets of a lower set according to the following result:

- For the number $C(Q, k, d)$ of all lower subsets Q', $|Q'| \ge n - k$, of a d-dimensional lower set Q, $|Q| = n$, there holds

$$C(Q, k, d) < \left(\max \left\{ 8, \frac{4eT(n)}{k} \right\} \right)^k.$$

K. Oganesyan, *Bounds for the number of multidimensional partitions*, Eur. J. Comb., 120 (2024), 103982.

High-dimensional Lower Sets

What can we say if n is not necessarily much greater than d?

- Cohen, Migliorati, and Nobile (2017)

$$p_d(n) \le 2^{dn} \quad \text{and} \quad p_d(n) \le d^{n-1}(n-1)!$$

- Dai, Prymak, Shadrin, Tikhonov, and Temlyakov (2021)

$$\frac{d^{n-1}}{(n-1)!} < \binom{d+n-2}{n-1} \le p_d(n) \le d^{n-1}.$$

- K.O. (2022)
  - If $d > n^3/2$, then
  $$1 \le \frac{p_d(n)}{\binom{d+n-2}{d-1}} < \frac{1}{1 - \frac{n^3}{2d}}.$$
  - If $dn^{-2} \to \infty$ as $n \to \infty$, then
  $$\log p_d(n) = (n-1)(\log d - \log n + 1) + o(n).$$
  - If d satisfies $cn^2 \le d \le Cn^2$ for some constants c and C,
  $$\log p_d(n) = n \log n + O(n).$$
  - If $dn^{-2} \to 0$ and $\log d \ge \log n + o(\log n)$ as $n \to \infty$,
  $$\log p_d(n) = n \log n + o(n \log n).$$

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