Additive Combinatorics Methods in Fractal Geometry, Lecture 4

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Plan for the course

Lecture 1: Introduction to Additive Combinatorics.

Lecture 2: The Balog-Szemerédi-Gowers Theorem.

Lecture 3: Discretized Fractal Geometry

Lecture 4: Sum-product and applications.

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Heuristic Principle

If A is a finite subset of a Abelian ring Z, then one of the following must hold:

1. *A is "dense" in a finite sub-ring of Z.*
2. *$A + A$ is "much larger" than A.*
3. *$A.A$ is "much larger" than A.*

Conjecture

If $A \subset \mathbb{R}$ is finite, then

$$\max\left\{|A+A|, |A\cdot A|\right\} \gg_{\varepsilon} |A|^{2-\varepsilon}.$$

Remark

The best known current exponent (in place of 2) is slightly larger than $4/3$.

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No direct sum-product for Hausdorff dimension

Lemma

For any $s \in (0,1)$*, there exists a compact set* $A \subset \mathbb{R}$ *such that*

$$\dim_{\mathrm{H}}(A+A) = \dim_{\mathrm{H}}(A \cdot A) = \dim_{\mathrm{H}}(A) = s.$$

Proof.

Idea: construct A so that it looks like an arithmetic progression at some scales, and like a geometric progression at a complementary set of scales. □

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Example

Let $A = [1, 1 + \sqrt{\delta}]$. Then

$$A + A = [1, 1 + 2\sqrt{\delta}],$$
$$A \cdot A = [1, 1 + 2\sqrt{\delta} + \delta]$$

so

$$|A + A|_\delta \sim |A \cdot A|_\delta \sim |A|_\delta \sim \delta^{-1/2}.$$

What is going on?

- For Hausdorff dimension, the issue is that different behaviour can happen at different scales (this depends on the existence of infinitely many scales).
- For the discretized version, the issue is that the set may be too concentrated in an interval.
- Both issues can be addressed by modifying the problem in a suitable way, leading to an extremely rich theory.

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Definition

A set $A \subset B^d(0,1)$ is called a (δ, s, K)-set if for every $r \in [\delta, 1]$ and every $x \in \mathbb{R}^d$, we have

$$|A \cap B(x,r)|_\delta \leq K \cdot r^s \cdot |A|_\delta.$$

- (δ, s)-sets with $s \in (0, d)$ are not concentrated in small balls. This strongly excludes any set that looks like $[1, 1+\sqrt{\delta}]$ (take $r = \sqrt{\delta}$).
- Taking $r = \delta$ shows that $|A| \geq K^{-1}\delta^{-s}$.
- This definition is a variant of the notion of (δ, s)-sets introduced by N. Katz and T. Tao in 2001. It is inspired by Frostman's Lemma.
- One can think of (δ, s)-set as sets of "Hausdorff dimension s at scale δ"

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Conjecture (Katz-Tao 2001)

Let $A \subset [0,1]$ be a (δ, s, K) set with $0 < s < 1$. Then there is a constant $\varepsilon = \varepsilon(s) > 0$ such that

$$\max\left\{|A+A|_\delta, |A \cdot A|_\delta\right\} \geq C(K) \cdot \delta^{-s-\varepsilon}.$$

Theorem (Bourgain 2003)

The Katz-Tao conjecture is true.

The discretized sum-product problem

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The Erdős-Volkmann sub-ring problem

Theorem (Erdős-Volkmann 1966)

For any $0 < s < 1$ *there is a Borel additive subgroup* $A \subset \mathbb{R}$ *with* $\dim_H(A) = s$. *The same is true for multiplicative subgroup.*

Conjecture (Erdős-Volkmann 1966)

If $A \subset \mathbb{R}$ *is a Borel sub-ring* ($A + A = A$ *and* $A \cdot A = A$)*, then* $\dim_H(A) \in \{0, 1\}$.

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Remarks on discretized sum-product

- In applications, one often needs the conclusion to hold under much weaker non-concentration assumptions on A. This was proved by Bourgain-Gamburd (2009).
- Bourgain's original proof was extremely complicated. It was simplified by Borgain in 2010, using the inverse theorem for sumsets we saw last time, but it remained very involved.
- A much simpler and quantitative proof was obtained by L. Guth-N. Katz-J. Zahl (2021). It still uses additive combinatorics.
- Very recently, very strong bounds were obtained by Y. Fu-K. Ren, T. Orponen-P.S., A. Mathe-W. O-Regan and K. Ren-H. Wang. Most of these results ultimately use the original non-quantitative version!

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Current best bound on discretized sum-product

Theorem (Y. Fu-K. Ren 2021)

Let $A \subset [0,1]$ be a (δ, s, K) set with $s \in [2/3, 1)$. Then

$$\max\{|A+A|_\delta, |A\cdot A|_\delta\} \geq C(K,\varepsilon)\cdot \delta^{-\frac{s+1}{2}+\varepsilon}.$$

This is sharp.

Theorem (K. Ren-H. Wang 2023)

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Definition

Let μ be a finite Borel measure on $\mathbb{R}^d$. The Fourier transform of μ is the function

$$\widehat{\mu}(t) = \int e^{-2\pi i x \cdot t} \, d\mu(x).$$

- The decay of $|\widehat{\mu}(t)|$ as $|t| \to \infty$ gives significant information about μ.
- We say that μ has power Fourier decay if $|\widehat{\mu}(t)| \leq C \cdot |t|^{-\alpha}$ for some $\alpha > 0$ and $C \geq 1$.
- Measures with power Fourier decay have many nice properties; for example, μ almost all points are normal to all bases.

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Let $\mu_1, \dots, \mu_k$ be finite Borel measures on $[1, 2]$. The multiplicative convolution of the μ_i is the push forward of $\mu_1 \times \cdots \times \mu_k$ via the map $(x_1, \dots, x_k) \mapsto x_1 \cdots x_k$. It is denoted by

$$\mu_1 \boxtimes \cdots \boxtimes \mu_k.$$

More explicitly,

$$\int f(x)\, d(\mu_1 \boxtimes \cdots \boxtimes \mu_k)(x) = \int f(x_1 \cdots x_k)\, d\mu_1(x_1) \cdots d\mu_k(x_k).$$

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Theorem (Bourgain 2010)

For every $s \in (0,1)$ there is $k = k(s) \in \mathbb{N}$ such that the following holds.
Let $\mu_1, \ldots, \mu_k$ be finite Borel measures on $[1,2]$ satisfying the Frostman condition

$$\mu_i(B(x,r)) \leq C\, r^s.$$

Then, $\mu_1 \boxtimes \cdots \boxtimes \mu_k$ has power Fourier decay.

Remarks on Bourgain's Theorem

- Bourgain's Theorem is a (non-trivial) consequence of the discretized sum-product theorem. The idea is that the Fourier transform has "additive structure", and multiplicative convolution introduces "multiplicative structure"; the combination of both produces decay.
- Bourgain conjectured that $ks > 1$ is enough. This was proved by N. de Saxcé-T. Orponen-P.S. (2023).
- Our quantitative result has the following corollary: let $A_1, \ldots, A_k$ be Borel sets on $\mathbb{R}$ such that

$$\sum_{j=1}^{k} \dim_{\mathrm{H}}(A_j) > 1.$$

Then, the additive subgroup generated by $A_1 \cdots A_k$ is $\mathbb{R}$.

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Fourier decay of dynamically defined measures

- J. Bourgain-S. Dyatlov used the Fourier decay of multiplicative convolutions to prove power decay for an important class of dynamically defined measures (Patterson-Sullivan measures on limits sets of Schottky groups).
- The method was extended and adapted by many other authors. For example, T. Sahlsten-C. Stevens proved that self-conformal measures for nolinear analytic iterated function systems have power Fourier decay.
- There are many other methods to prove Fourier decay of dynamicall defined measures. Some use additive combinatorics but not the sum-product theorem. Some do not use additive combinatorics at all.
- Nonlinearity is crucial in most of these results. The problem of power Fourier decay for self-similar measures is extremely hard.

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Outline

Definition

Given a set $A \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, the radial projection of A from x is the set

$$\pi_x(A) = \left\{ \frac{y - x}{|y - x|} : y \in A \setminus \{x\} \right\}.$$

- The radial projection $\pi_x(A)$ is a subset of the unit sphere S^{d-1}. It is the set of directions in which the points of A are seen from x.
- Radial projections generalize orthogonal projections: if x is a point at infinite in direction v, then $\pi_x(A)$ is the orthogonal projection of A onto the hyperplane orthogonal to v. Alternatively, one can convery π_x to an orthogonal projection by means of a projective transformation.

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Problem

Let $A \subset \mathbb{R}^2$ be a Borel set. What is

$$\sup_{x \in A} \dim_H \pi_x(A) \, ?$$

- If A is contained in a line, then $\pi_x(A)$ is a singleton. So one needs to assume that A is not contained in a line.
- The map π_x is Lipschitz outside a small neighbourhood of x, so $\dim_H \pi_x(A) \leq \dim_H A$.

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Let $A \subset \mathbb{R}^2$ be a Borel set. What is

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Theorem (T. Orponen 2015)

Let A be a Borel set which is not contained in a line. Then

$$\sup_{x \in A} \dim_H \pi_x(A) \geq \frac{1}{2} \dim_H A.$$

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If there was a Borel sub-ring of the reals X of dimension $s/2$, then the bound would be sharp: take $A = X \times X$. Then $\dim_H(A) \geq s$. But up to an arctan *change of coordinates, if $x = (x_1, x_2) \in A$, then*

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This suggests a connection between radial projections and sum-product.

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Let $X \subset \mathbb{R}$ be a Borel set. Then

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Remarks on sharp radial projections

- The proof of the sharp bound ultimately depends on an iterative application of the discretized projection theorem, which is a close cousin of the discretized sum-product theorem.
- The Theorem can be seen as a far reaching generalization of Kaufman's projection theorem, Orponen's radial projection theorem, and the discretized sum-product theorem.
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Outline

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The set of lines in $\mathbb{R}^2$ is a 2-dimensional manifold, so there is a notion of Hausdorff dimension for sets of lines.

Alternatively, we can identify the line $y = ax + b$ with $(a, b) \in \mathbb{R}^2$ and consider the set of (non-vertical) lines as a subset of $\mathbb{R}^2$.

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A set $F \subset \mathbb{R}^2$ is called an (s, t)-Furstenberg set if there exists a set of lines $\mathcal{L}$ with $\dim_{\mathrm{H}}(\mathcal{L}) = t$ such that for every line $L \in \mathcal{L}$, the intersection $F \cap L$ has dimension at least s.

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Problem

Let F be an (s,t)-Furstenberg set in $\mathbb{R}^2$. What is the smallest possible dimension of F?

1. The problem arose from work of Furstenberg in the 1960s, but was first formulated in print by T. Wolff in an influential survey from 1999.
2. The problem is again related to sum-product (for example, if X is a sub-ring of dimension $1/2$, then $X \times X$ is a “small” $(1, 1/2)$-Furstenberg set).
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Theorem (T. Orponen-P.S. (2023) + K. Ren-H. Wang (2023))

Let F be an (s,t)-Furstenberg set in $\mathbb{R}^2$. Then

$$\dim_{\mathrm{H}} F \geq \min\left\{s+t, \frac{3s+t}{2}, s+1\right\}.$$

This is sharp.

Logic of the proof

Sum-product $\Longrightarrow$ ε-improved Furstenberg set estimates
$\Longrightarrow$ radial projections
$\Longrightarrow$ asymmetric sum-product
$\Longrightarrow$ projections of regular sets
$\Longrightarrow$ Furstenberg sets with regular line sets
$\Longrightarrow$ general Furstenberg sets

and

High-low method $\Longrightarrow$ Furstenberg for semi-well spaced lines
$\Longrightarrow$ general Furstenberg sets

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