Additive Combinatorics Methods in Fractal Geometry, Lecture 3

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Lecture 1: Introduction to Additive Combinatorics.
Lecture 2: The Balog-Szemerédi-Gowers Theorem.
Lecture 3: Discretized Fractal Geometry
Lecture 4: Sum-product and applications.

Plan for the course

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Outline

1. Introduction to Discretized Fractal Geometry
2. Discretized inverse sumset theorems
3. Inverse theorem for convolutions and applications
4. On fractal Szemerédi's Theorem

- Plünnecke-Ruzsa Inequalities: if $|A+B| \leq K|A|$, then $|nB| \leq K^n|A|$.
- Balog-Szemerédi-Gowers Theorem: if $E(A,A) \geq |A|^3/K$, then there is a $A' \subset A$ such that $|A'| \geq c|A|/K$ and $|A'+A'| \leq CK^4|A'|$.
- Freiman's Theorem: if $|A+A| \leq K|A|$, then there is a GAP P such that $A \subset P$, where $|P|/|A|$ and rank(P) are bounded in terms of K.
- Szemerédi's Theorem: every set of integers of positive density contains arbitrarily long arithmetic progressions.

Recap

- Plünnecke-Ruzsa Inequalities: if $|A + B| \leq K|A|$, then $|nB| \leq K^n|A|$.
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From Combinatorics to Fractal Geometry

- Additive Combinatorics studies the additive structure of finite sets in Abelian groups.
- Fractal Geometry studies properties of (Borel) sets and measures in metric spaces.
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Given a set $A \subset \mathbb{R}^d$, we can discretize it at resolution $\delta > 0$ in several ways:

- $A \to \delta$-neighborhood of A.
- $A \to$ maximal δ-separated subset of A.
- $A \to$minimal cover of A by δ-balls.
- $A \to$ union of δ-dyadic cubes that intersect A.
- $A \to$ corners of δ-dyadic cubes of side length δ that intersect A.

Definition

$$A^{(m)} = \bigcup \{2^{-m}(j_1, \ldots, j_d) : A \cap I_m(j_1, \ldots, j_d) \neq 0\},$$

where $I_m(j_1, \ldots, j_d)$ is the dyadic cube

$$I_m(j_1, \ldots, j_d) = [2^{-m}j_1, 2^{-m}(j_1 + 1)) \times \cdots \times [2^{-m}j_d, 2^{-m}(j_d + 1)).$$

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Definition

Given $A \subset \mathbb{R}^d$, and a dyadic $\delta > 0$, the δ-covering number $|A|_\delta$ (or box counting number) is the number of dyadic cubes of side length δ that intersect A.

- By definition, the box counting (Minkowski) dimension of A is the growth rate of $|A|_\delta$ as $\delta \to 0$:

$$\dim_B(A) = \lim_{\delta \to 0} \frac{\log |A|_\delta}{\log(1/\delta)}.$$

- For Hausdorff dimension, the situation is more subtle, but one can still relate the Hausdorff dimension of A to the growth rate of $|A'|_\delta$ for certain sets A'.

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Hausdorff dimension via covering numbers

Lemma

Let μ be a measure supported on $A \subset \mathbb{R}^d$. If

$$\mu(A') \geq m^{-2} \quad \Longrightarrow \quad |A'|_{2^{-m}} \geq 2^{sm},$$

then $\dim_H(A) \geq s$.

Proof.

Let $\{B(x_j, r_j)\}$ be a covering of A with r_j small. Let

$$A_m = \bigcup \{B(x_j, r_j) : 2^{-m} < r_j \leq 2^{1-m}\}.$$

By dyadic pigeonholing, there is m with $\mu(A_m) \geq m^{-2}$. Then $|A_m|_{2^{-m}} \geq 2^{sm}$, and it follows that

$$\sum_j r_j^s \gtrsim 2^{sm} \cdot 2^{-sm} = 1 > 0.$$

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Remark

In some cases (but definitely not in all cases!) one can extend results from the discrete to the discretized realm simply by considering a finite approximation such as $A^{(m)}$ of the objects involved.

Theorem (Plünnecke Inequalities, 1969)

Suppose $|A + A| \leq K|A|$. *Then* $|nA| \leq K^n|A|$.

More generally, if $|A + B| \leq K|A|$, *then* $|nB| \leq K^n|A|$.

Corollary

Let $A \subset \mathbb{R}^d$ *be bounded. Suppose* $|A + A|_\delta \leq K|A|_\delta$. *Then* $|nA|_\delta \leq C_n \cdot K^n \cdot |A|_\delta$.

More generally, if $|A + B|_\delta \leq K|A|_\delta$, *then* $|nB|_\delta \leq C_n \cdot K^n \cdot |A|_\delta$.

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Apply the discrete version to $A^{(m)}, B^{(m)}$ (+ some calculations). □

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Discretized additive energy

Definition

Let $A, B \subset \mathbb{R}^d$ be bounded sets and $\delta = 2^{-m}$. We define the δ-discretized additive energy $E_\delta(A, B)$ of A and B by

$$E_\delta(A, B) = E(A^{(m)}, B^{(m)}).$$

Lemma

$$E_\delta(A, B) \sim \left|(x_1, x_2, y_1, y_2) \in A^2 \times B^2 : |x_1 + x_2 - y_1 - y_2| \leq \delta\right|_\delta .$$

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Corollary (of BSG)

There are constants $c, C > 0$ such that the following holds. Suppose $A \subset \mathbb{R}^d$ is bounded, and

$$E_\delta(A, A) \geq \frac{|A|_\delta^3}{K}.$$

Then there exists $A' \subset A$ such that:

$$|A'|_\delta \geq \frac{c|A|_\delta}{K},$$
$$|A' + A'|_\delta \leq CK^4 |A'|_\delta.$$

Outline

Discretized Freiman?

- Recall that Freiman's Theorem says that if $|A+A| \leq K|A|$, then A is densely contained in a GAP of bounded rank.
- Freiman's Theorem also extends in a straightforward way to δ-covering numbers.
- However, in applications to fractal geometry, we often have $|A|_\delta \sim \delta^{-s}$ for some $s > 0$, and we are interested in the case where $K = \delta^{-\varepsilon} = |A|_\delta^{-\varepsilon/s}$.
- Even the best quantitative known version of Freiman's Theorem does not give any information whatsoever when K grows polynomially in the size of A.
- Bourgain proved a "local" version of Freiman's Theorem in this "fractal setting" in the real line. Extensions and variants were later given by M. Hochman, P.S. and P. Varju.

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Tree representation of sets

- Let $\Delta = 2^{-k}$ be a fixed dyadic scale.
- Given $A \subset [0, 1)^d$, we can represent it as a tree $T_\Delta(A)$ whose vertices of level s are the dyadic cubes of side length Δ^s that intersect A. The root of the tree is the unit cube $[0, 1)^d$.
- In most cases, we only care about A at some resolution 2^{-km}. In this case, the tree has m levels.

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Definition

We say that a set A is Δ-uniform at scale Δ^m if the associated tree $T_\Delta(A)$ has the property that each vertex of level s has the same number of R_s of children.

In other words, for every dyadic cube I of size Δ^s, we have

$$|A \cap I|_{\Delta^{s+1}} = R_s.$$

We call R_s the branching numbers of A.

The following uniformization lemma is extremely useful.

Lemma

Let $A \subset [0,1)^d$ be a set and let $\Delta \in 2^{-\mathbb{N}}$ be a fixed scale. For every $m \in \mathbb{N}$, there exists a subset $A' \subset A$ such that:

1. *A' is Δ-uniform at scale Δ^m.*
2. $|A'|_{\Delta^m} \geq (2\log(1/\Delta))^{-\Delta m}|A|_{\Delta^m}$.

Remark

In many cases, the factor $(2\log(1/\Delta))^{-\Delta m}$ is harmless (because $\Delta m \ll m$). So the lemma says that that arbitrary sets contain "dense" uniform subsets.

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Theorem (Bourgain, 2010)

Given $\varepsilon > 0$, there exist $\eta > 0$ and $\Delta \in 2^{-\mathbb{N}}$ such that the following holds for all large $m \in \mathbb{N}$.
Let $A \subset [0,1)$ be a set such that, writing $\delta = \Delta^m$,

$$|A + A|_\delta \le \delta^{-\eta}|A|_\delta.$$

Then there exists a Δ-uniform set $A' \subset A$ at scale δ such that:

1. $|A'|_\delta \ge \delta^\varepsilon |A|_\delta$.
2. *Let $(R_s)_{s=0}^{m-1}$ be the branching numbers of A'. Then, for all $s \in \{0,\dots,m-1\}$, either $R_s = 0$ or $R_s \ge |\Delta|^{-(1-\varepsilon)}$.*

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Remarks on Bourgain's Theorem

- The theorem can be summarized as saying: if A has small doubling in an exponential sense, then (after passing to a "dense" uniform subset) locally it looks either like a point or like an interval.
- The proof is very short (but very clever); it uses Plünnecke-Ruzsa Inequalities.
- The quantitative dependency of η on ε can be made explicit but is very poor.
- Since intervals ressemble arithmetic progressions, we can also say that A locally ressembles an AP.

Remarks on Bourgain's Theorem

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Inverse sumset theorem in higher dimension

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Let $A \subset [0,1)^d$ be a set such that, writing $\delta = \Delta^m$,

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Then there exists a Δ-uniform set $A' \subset A$ at scale δ such that:

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- Recall that Balog-Szemerédi-Gowers can be seen as "reducing" inverse theorems for convolutions to inverse theorems for sumsets.
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There exists a set $E \subset (1/2, 1)$ of zero Hausdorff dimension such that ν_λ is absolutely continuous with a density in L^p for all finite p, for all $\lambda \in (1/2, 1) \setminus E$.

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- Dimension+structure: even though dimension is not enough, perhaps some additional structure (randomness, Fourier decay, etc) may be enough to guarantee progressions.
- Other notions of size: besides dimension, there are other notions of “size” for fractals which are better connected to the existence of patterns: thickness, winning sets, etc.
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