Mathematics of Large Networks: Summer School Abstracts

30th May – 3rd June 2022

Contents

Chapter 1

Mini courses

1.1 Ginestra Bianconi

Title: **Higher-order networks: An introduction to simplicial complexes**

Abstract: Higher-order networks encode interactions between two or more nodes and allow us to describe complex systems going beyond the assumption of having exclusively pairwise interactions. New results have shown that higher-order networks can sustain higher-order dynamical processes that take advantage of the topology and geometry of the higher-order networks. Most relevantly these processes cannot be accounted by considering only the pairwise interactions of higher-order networks. For these reasons higher-order networks constitute a very hot topic in Network Science with applications spanning from brain networks to social collaboration networks.

In this course we will cover the main aspects of this exciting new research subject covering:

1. the main structural combinatorial, topological and geometrical aspects of higher-order networks;

2. important equilibrium (maximum entropy) and non-equilibrium (growing) models for higher-order networks;

3. the interplay between higher-order structure and dynamics.

The module will be based on the recent book:

Bianconi, Ginestra. *Higher order network: An Introduction to Simplicial Complexes.* Cambridge University Press, 2021.

1.2 Remco van der Hofstad

Title: **Local and global structure of random graphs and complex networks**

Abstract: Empirical findings have shown that many real-world networks share fascinating features. Indeed, many real-world networks are small worlds, in the sense that typical distances are much smaller than the size of the network, and are scale-free, in the sense that there is a high variability in the number of direct connections of the elements in the network.

Spurred by these empirical findings, many models have been proposed for such networks. In this lecture series, we discuss empirical findings of real-world networks, and describe some of the random graph models proposed for them, such as the classical Erdös-Rényi random graph, as well as the more relevant configuration model, generalized random graphs and preferential attachment models.

We discuss local convergence in random graphs, and its relation to branching process approximations in, or the locally tree-like nature of, random graphs. While local convergence is related to the local structure around typical vertices in random graphs, it also has indirect implications for many global quantities such as the giant component and small-world properties of random graphs. For example, we show how the statement that the 'giant component is almost local' can be made precise, and how it can be related to the small-world nature of random graphs.

Outline of the lecture series:

Lecture 1: Real-world networks and random graphs
Lecture 2: Local convergence of random graphs
Lecture 3: The giant in random graphs is almost local

This lecture series is based on joint work with, amongst others: Gerard Hooghiemstra, Shankar Bhamidi, Júlia Komjáthy, Piet Van Mieghem, Henri van den Esker, and Dmitri Znamenski.

1.3 Renaud Lambiotte

Title: **Modularity and Random Walks on Networks**

Abstract: In this short tutorial, we will explore the interplays between community structure and Dynamics on Networks. Starting from general principles in dimensionality reduction and, in particular, time-scale separation, we will then investigate how the graph structure affects the spectral properties of the transition matrix. As a next step, we will overview different approaches based on random walks for community detection.

1.4 Gergely Palla

Title: **Hyperbolic networks: models, embedding and community structure**

Abstract: Hyperbolic network models are centred around the idea of placing nodes at random in a metric space with hyperbolic geometry, and connecting node pairs according to a probability that is decreasing as a function of the hyperbolic distance. A fascinating feature of these models is that they inherently reproduce the most important universal properties of real networks: the graphs generated this way are small-world, highly clustered and scale-free. The success of hyperbolic models provides a strong motivation for the inverse problem given by hyperbolic embedding as well, where the task is to find an optimal arrangement of the nodes in the hyperbolic space based on a given input network structure. The resulting layout can be beneficial from various aspects, including greedy routing and link prediction problems. A recently discovered further interesting property of hyperbolic networks is that they can display a very pronounced community structure as well.

During the lectures, we shall discuss the above-mentioned aspects of hyperbolic network models and hyperbolic network geometry. The first part of the lectures will focus on the most well known hyperbolic models such as the popularity-similarity optimisation model and the random hyperbolic graph. This will be followed by an overview of different hyperbolic embedding techniques, together with the quality measures that can be applied on the resulting layouts for comparing different methods on a quantitative base. For closing, the community structure of hyperbolic networks will be discussed.

1.5 Kavita Ramanan

Title: **Dynamics on Sparse and Heterogeneous Networks**

Abstract: We will outline some of the challenges that arise in the study of dynamics on sparse and heterogeneous networks and describe recently developed theory that allows one to provide approximations to empirical measure and marginal network dynamics that are provably accurate in a suitable asymptotic regime. We will also provide illustrative examples of insights that can be gained from these approximations, and discuss several open problems.

Chapter 2

Contributed talks

2.1 Luisa Andreis

Title: **Phase transitions and large deviations in inhomogeneous random graphs**

Abstract: Inhomogeneous random graphs are a natural generalization of the well-known Erdős-Rényi random graph, where vertices are characterized by a type and edges are present independently according to the type of the vertices that they are connecting. In the sparse regime, these graphs undergo a phase transition in terms of the emergence of a giant component exactly as the classical Erdős-Rényi model. We will present an alternative approach, via large deviations, to prove this phase transition. This allows a comparison with the gelation phase transition in coagulation processes and with phase transition of condensation type emerging in several systems of interacting components.

This is based on joint work with Wolfgang König (WIAS and TU Berlin), Heide Langhammer (WIAS) and Robert Patterson (WIAS).

2.2 Elsa Andres

Title: **Distinguishing simple and complex contagion processes on complex networks**

Abstract: There exists two main models to study spreading on complex networks: (i) the simple contagion for which the nodes are infected under a certain probability after a contact with a contaminated node and (ii) the complex contagion for which a node is infected if a certain proportion of its neighbors is contaminated. The past works have analyzed contagions on the macroscopic level, assuming everything in known of the propagation, but nothing as been done at the microscopic level. On that scale, we aim to classify infected nodes between the simple and the complex contagions knowing the time-line of the contagion of its ego-network. We use the likelihood as a baseline model for the classification which constrains us to assume the processes are Markovian. The machine learning method allows us to have a better accuracy in the classification as it ables to use non-Markovian features.

2.3 Daniela Aguirre Guerrero

Title: **Distributed rewiring model for complex networking**

Abstract: This work studies some of the forces that help a network to evolve to the point where structural properties are settled. We propose a distributed rewiring model that creates networks with features like those found in complex networks. Using a discrete event simulator, we get each node (of an initial graph) to discover the shortcuts that may connect it with regions away from its local environment. Based on this partial knowledge, each node can rewire some of its links, which allows modifying the topology of the initial graph to achieve new structural properties. Although each node acts in a distributed way and seeking to reduce only the trajectories of its packets, it is observed a decrease of diameter and an increase in clustering coefficient in the global structure compared to the initial graph. Furthermore, it can be found different final structures depending on slight changes in the local rewiring rules.

2.4 Bharath Roy Choudhury

Title: **Genealogy of records for a class of random walks**

Abstract: Consider the trajectory $w(i)$ of a random walk with i.i.d. increments taking their values in the set of integers larger than -2 and with mean 0. For each time i, draw an edge from $(i, w(i))$ to $(j, w(j))$ where j is the smallest time larger than i such that $w(j) \geq w(i)$. This defines the record random graph of the random walk. We show that this random graph is an infinite one-ended unimodular random tree, and more precisely a unimodular Eternal Galton-Watson Tree. We analyze the distribution of the random walk seen from the n-th record location of $(0, w(0))$ and the limiting value of this distribution as n tends to infinity. We show that the latter can be described in terms of the Kesten Tree associated with this Galton-Watson Tree. We also establish various invariance properties of the trajectory of the random walk based on invariance properties of the Eternal Galton-Watson Tree.

2.5 Umberto De Ambroggio

Title: **Unusually large components in near-critical Erdos-Renyi graphs via ballot theorems**

Abstract: I will talk about largest components in (near-critical) Erdos-Renyi random graphs. More specifically, I will briefly introduce a purely probabilistic methodology to derive precise bounds for the probability of observing an unusually large maximal component when the edge probability equals 1/n, the so-called critical regime. Compared to the approach based on martingales introduced in 2010 by A. Nachmias and Y. Peres, our method provides the correct asymptotic order for the above mentioned probability.

2.6 Fernando Diaz-Diaz

Title: **Generalized diffusion in graphs/networks**

Abstract: Normal and anomalous diffusion are ubiquitous in many complex systems. In this talk, I will define a time and space generalized diffusion equation (GDE), which uses fractional-time derivatives and transformed d-path Laplacian operators on graphs/networks. I will find analytically the solution of this equation and prove that it covers the regimes of normal, sub- and superdiffusion as a function of the two parameters of the model. I will also extend the GDE to consider a system with temporal alternancy of normal and anomalous diffusion, which is able to sucessfully model the diffusion of proteins along a DNA chain. Finally, I will briefly comment how a subdiffusive-superdiffusive alternant regime allows the diffusive particle to explore more slowly small regions of the chain with a faster global exploration.

2.7 Sima Farokhnejad

Title: **A Data-driven Approach to Cattle Epidemic Modelling under Uncertainty**

Abstract: Cattle movement is an intrinsic part of animal husbandry (i.e., breeding, maintenance, slaughter of livestock). There are an estimated 1 billion cattle heads in the world used for the production of meat, milk, leather, among other products, and consumed by billions of people. The pressures of delivering animal products to individuals and the efficiency it requires, lead to a stress in the system both in the number of heads kept and traded, and in the number of possible contacts between these heads. Under these conditions, contact tracing and avoidance is an essential part of modern agriculture because highly contagious diseases such as brucellosis and foot-and-mouth disease can spread through contact, leading to heavy economic costs. Many countries track their cattle with electronic tags (e.g. Australia, Canada) which leads to a highly precise monitoring capability. Unfortunately, several of the largest producers in the world (e.g. Brazil, Mexico, USA), do not mandate such use, and some do not even mandate the tracking of animal movement (e.g. Mexico, USA). Added to this, the lack of tracking capabilities enable people to take advantage of the system by engaging in unregulated cattle trade. The consequence is that official movement data contain uncertainty in the number of cattle movements as well as the number of actual trades.

This work focuses on understanding uncertainty in cattle movement tracking and its relation to epidemic modelling.

2.8 Tejas Iyer

Title: **Preferential Attachment Trees with Neighbourhood Influence**

Abstract: Motivated by the structure of complex networks such as the internet, we consider a growing model of preferential attachment trees with neighbourhood influence, where vertices arrive one at a time, are equipped with independent weights, and connect to existing vertices with probability proportional to their fitness function: a function of their own weight and the weights of their neighbours. In this model we prove almost sure limiting statements for the proportion of vertices with a given degree having weight belonging to a given measurable set, and the proportion of edges in the tree with endpoint belonging to a measurable set. We show that under certain conditions, the latter quantity demonstrates a condensation phenomenon, in which a positive proportion of edges in the network accumulate among those of weight that confers maximal reinforcement of fitness. Finally, we prove that in this model the degree distribution behaves like a power law - a ubiquitous feature of many real-world complex networks. Joint work with Nikolaos Fountoulakis.

2.9 Leah Keating

Title: **Modelling complex contagion on clustered networks using multi-type branching processes**

Abstract: Online social networks such as Twitter, Facebook, Instagram and TikTok serve as media for the spread of information between their users. We are interested in developing models for this information diffusion to gain a greater understanding of its drivers. Some models for the spread of online behaviour and information assume that the information behaves similarly to a virus, where infection is equally likely after each exposure, these dynamics are known as a simple contagion. In a simple contagion, the exposures are independent of each other. However, online adoption of some behaviour and content has been empirically observed to be more likely after multiple exposures from their network neighbours [1-2], the exposures are not independent of each other, we refer to this as a complex contagion. Analytically tractable descriptions of complex contagions have been developed for continuous-time dynamics. These extend mean-field and pair approximation methods to account for clustering in the network topologies [3]; however, no such analogous treatments for discrete-time cascade processes exist using branching processes. We describe a novel definition of complex contagion adoption dynamics and show how to construct multi-type branching processes which account for clustering on networks [4]. We achieve this by tracking the evolution of a cascade via different classes of clique motifs which contain different numbers of active, inactive and removed nodes. This description allows for accurate analytical calculation of cascade sizes, determination of critical behaviour and we also describe how the branching process description allows us, using probability generating functions, to derive full distributions of cascade sizes and other quantities of interest from the model.

[1] D. Centola, The spread of behavior in an online social network experiment, Science 329, 1194 (2010).
[2] D. M. Romero, B. Meeder, and J. Kleinberg, Differences in the mechanics of information diffusion across topics: idioms, political hashtags, and complex contagion on twitter, in Proceedings of the 20th international conference on World wide web (2011) pp. 695–704.
[3] D. J. P. O'Sullivan, G. J. O'Keeffe, P. G. Fennell, and J. P. Gleeson, Mathematical modeling of complex contagion on clustered networks, Frontiers in Physics 3,10.3389/fphy.2015.00071 (2015).
[4] Keating, L.A., Gleeson, J.P. and O'Sullivan, D. J.P. Multitype branching process method for modeling complex contagion on clustered networks. Physical Review E, 105(3), 034306 (2022).

2.10 Vitalii Konarovskyi

Title: **Stochastic block model in a new critical regime**

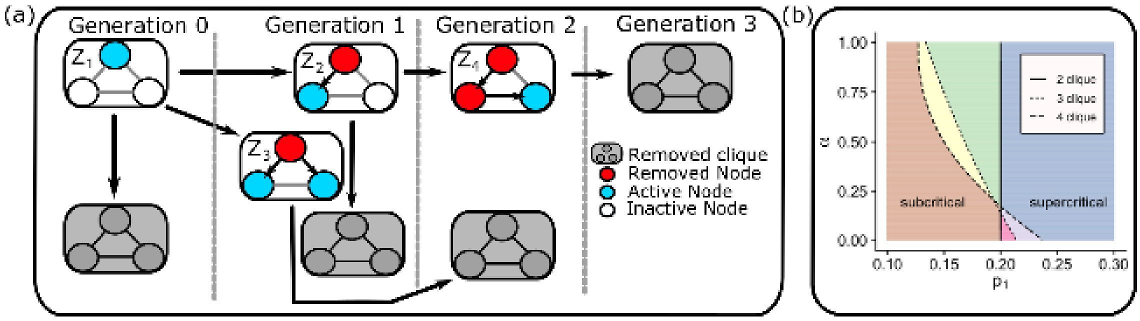

Figure 2.1: (a) The possible motifs in a network composed with 3-cliques. Active nodes are shown in red, inactive nodes are white and removed nodes are grey. The arrows represent the direction of the diffusion. (b) Critical behaviour for k-regular clique network we consider.

Abstract: In the talk, we will discuss a novel phase transition for the classical stochastic block model. We will show that the scaling limit of the component sizes of the stochastic block model in the corresponding near-critical regime can be described by a two-parameter stochastic process arising as standard Aldous' multiplicative coalescents with interaction. This is joint work with Vlada Limic.

2.11 Abbas Rizi

Title: **Modeling pharmaceutical and non-pharmaceutical interventions for epidemics**

Abstract: A big lesson from the Covid-19 pandemic was that health behaviors can shape the dynamics of epidemics in different ways. In this talk, by introducing a mathematical framework, we discuss how vaccination and contract tracing can change the course of a pandemic when there are various types and levels of heterogeneities in contact/transmission networks. Specifically, we show how the herd immunity threshold and the expected epidemic size depend on homophily with respect to vaccine adoption.

2.12 Erin Russell

Title: **Playing with Fire: The Necessary Evil of Self-organised Criticality**

Abstract: Consider a mean field Erdős-Rényi random digraph process on n vertices. Let each possible directed edge arrive with rate $n/2$. Without opposition, this process is guaranteed to result in the n-complete graph. Hence we introduce a Poisson rain of "lightning strikes" to each vertex with rate $\lambda(n)$ which propagates across its out-graph, burning all edges of an affected vertex. Subsequently, the system continues to fluctuate to no end in a battle between creation and destruction.