LECTURES ON DISORDERED MODELS - EXERCISE ON UNIQUENESS OF THE GROUND STATE IN THE TWO-DIMENSIONAL RANDOM-FIELD ISING MODEL

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1. General notation

1.1. **Lattices.** We consider the lattice $\mathbb{Z}^d$ in dimension $d \geq 1$. Given two vertices $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if they are nearest-neighbours in $\mathbb{Z}^d$.

Given an integer $L \geq 0$, we consider the box $\Lambda_L := \{-L, \ldots, L\}^d \subseteq \mathbb{Z}^d$. Denote by $\partial \Lambda_L := \Lambda_{L+1} \setminus \Lambda_L$ its external vertex boundary and by $|\Lambda_L|$ its cardinality (i.e., $|\Lambda_L| = (2L+1)^d$).

Given a measurable set $A \subseteq \mathbb{R}$, we denote its Lebesgue measure by $\mathrm{Leb}(A)$.

1.2. **Ground state of the disordered Ising model.** We introduce the following notation for the *configurations* of the Ising model in the box Λ_L with + and - boundary conditions, respectively,

$$\mathcal{S}_L^+ := \left\{ \sigma : \mathbb{Z}^d \to \{-1, 1\} \text{ with } \sigma_v = 1 \text{ for } v \notin \Lambda_L \right\},$$
$$\mathcal{S}_L^- := \left\{ \sigma : \mathbb{Z}^d \to \{-1, 1\} \text{ with } \sigma_v = -1 \text{ for } v \notin \Lambda_L \right\}.$$

An *external field* is a function $h : \mathbb{Z}^d \to \mathbb{R}$. We will later take this function to be random, in which case we will denote it by ζ. Given a vertex $y \in \mathbb{Z}^d$ and an external field $h : \mathbb{Z}^d \to \mathbb{R}$, we denote by $\tau_y h : \mathbb{Z}^d \to \mathbb{R}$ the shifted field defined by $(\tau_y h)_x := h(x+y)$.

For each external field $h : \mathbb{Z}^d \to \mathbb{R}$, we define the *energy of the finite-volume ground states* of the Ising model with + and - boundary conditions and external field h by

$$F_L^+(h) := \sup_{\sigma \in \mathcal{S}_L^+} \left(\sum_{\substack{x \sim y \\ \{x,y\} \cap \Lambda_L \neq \emptyset}} \sigma_x \sigma_y + \sum_{x \in \Lambda_L} h_x \sigma_x \right) \quad \text{and} \quad F_L^-(h) := \sup_{\sigma \in \mathcal{S}_L^-} \left(\sum_{\substack{x \sim y \\ \{x,y\} \cap \Lambda_L \neq \emptyset}} \sigma_x \sigma_y + \sum_{x \in \Lambda_L} h_x \sigma_x \right)$$

and denote the energy difference by

$$F_L(h) := F_L^+(h) - F_L^-(h).$$

Note that, for almost every value of the field h on Λ_L, there are *unique* maximisers in the definitions of $F_L^+(h)$ and $F_L^-(h)$. We denote them by $\sigma_L^+(h)$ and $\sigma_L^-(h)$, respectively (the finite-volume ground states).

2. The Imry-Ma phenomenon

2.1. **Preliminaries: An analysis lemma.** For each pair of Lipschitz, convex functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$, we introduce the set (of points with δ-diverging derivatives)

$$\mathrm{Div}(F_1, F_2, \delta) := \{ t \in \mathbb{R} : F_1 \text{ and } F_2 \text{ are differentiable at } t \text{ and } |F_1'(t) - F_2'(t)| > \delta \}.$$

Exercise 1. *Show that there exists a constant $C > 0$ such that for each pair of convex and 1-Lipschitz functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ satisfying $|F_1 - F_2| \leq 1$ and each $\delta > 0$, one has the upper bound,*

$$\mathrm{Leb}\left(\mathrm{Div}(F_1, F_2, \delta)\right) \leq \frac{C}{\delta^2}. \tag{2.1}$$

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2.2. The Imry–Ma phenomenon.

Exercise 2. *In this guided exercise we explain how Exercise 1 may be used to deduce the uniqueness of the ground state in the two-dimensional random-field Ising model. The exercise is loosely based on [1], where a quantitative bound is achieved using an additional fractal (Mandelbrot) percolation.*

Fix $\lambda \in (0, \infty)$. Let $(\zeta_x)_{x \in \mathbb{Z}^2}$ be independent Gaussian random variables with expectation 0 and variance λ^2.

(1) (Convexity, differentiability and deterministic bound) Show the following properties of F_L^+, F_L^-, F_L:
   (i) The functions $h \mapsto F_L^+(h)$ and $h \mapsto F_L^-(h)$ are convex.
   (ii) The functions $h \mapsto F_L^+(h)$ and $h \mapsto F_L^-(h)$ are differentiable almost everywhere and for every $x \in \Lambda_L$ and almost every value of h on Λ_L,

$$\frac{\partial F_L^+}{\partial h_x}(h) = \sigma_{L,x}^+(h) \quad \text{and} \quad \frac{\partial F_L^-}{\partial h_x}(h) = \sigma_{L,x}^-(h).$$

   (iii) For any external field $h : \Lambda_L \to \mathbb{R}$,

$$|F_L(h)| \le 2\,|\partial \Lambda_L|\,.$$

(2) (Extremal boundary conditions) Show that, for almost every ζ and every $x \in \mathbb{Z}^d$,

$$\begin{cases} \sigma_{L,x}^-(\zeta) \le \sigma_{L,x}^+(\zeta), \\ \sigma_{L+1,x}^-(\zeta) \ge \sigma_{L,x}^-(\zeta) \\ \sigma_{L+1,x}^+(\zeta) \le \sigma_{L,x}^+(\zeta). \end{cases}$$

(3) (Convergence and translation covariance) For almost every ζ, deduce that for every $x \in \Lambda_L$,

$$\begin{cases} \sigma_{L,x}^-(\zeta) \underset{L\to\infty}{\longrightarrow} \sigma_x^-(\zeta), \\ \sigma_{L,x}^+(\zeta) \underset{L\to\infty}{\longrightarrow} \sigma_x^+(\zeta), \end{cases}$$

(where σ^-, σ^+ are defined as the limiting configurations) and, for every $y \in \mathbb{Z}^d$,

$$\sigma_y^-(\zeta) = \sigma_0^-(\tau_y \zeta) \quad \text{and} \quad \sigma_y^+(\zeta) = \sigma_0^+(\tau_y \zeta).$$

(4) (Magnetisation from energy) Let $1_{\Lambda_L} : \mathbb{Z}^d \to \{0,1\}$ be the indicator function of Λ_L. We set

$$\frac{\partial F_L}{\partial \hat{h}_L}(h) := \lim_{\delta \to 0} \frac{F_L(h + \delta 1_{\Lambda_L}) - F_L(h)}{\delta}.$$

Show that the following identity holds almost surely,

$$\frac{\partial F_L}{\partial \hat{h}_L}(\zeta) = \sum_{x \in \Lambda_L} \left(\sigma_{L,x}^+(\zeta) - \sigma_{L,x}^-(\zeta)\right).$$

(5) (Main bound: high density of uniqueness points) Assume that the dimension is $d = 2$.
Deduce, using Exercise 1, that, for any $\delta > 0$,

$$\liminf_{L \to \infty} \mathbb{P}\left[\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \left(\sigma_{L,x}^+(\zeta) - \sigma_{L,x}^-(\zeta)\right) < \delta\right] > 0.$$

(6) (Uniqueness of the ground state) Still assume that the dimension is $d = 2$.
Deduce from the previous questions and the ergodic theorem that, for almost every ζ,

$$\sigma^-(\zeta) = \sigma^+(\zeta).$$

Hints:

- Questions 2 and 3: for $y \in \mathbb{Z}^d$, denote by $\sigma_{y+\Lambda_L}^+$ and $\sigma_{y+\Lambda_L}^-$ the finite-volume ground states of the Ising model in the box $(y + \Lambda_L)$ with + and - boundary conditions, respectively. Show that if $(y + \Lambda_L) \subseteq (y' + \Lambda_{L'})$ then $\sigma_{y+\Lambda_L}^+ \ge \sigma_{y'+\Lambda_{L'}}^+$ and $\sigma_{y+\Lambda_L}^- \le \sigma_{y'+\Lambda_{L'}}^-$.
- Question 5: We may use the following property of the Gaussian variables: if we denote by

$$\hat{\zeta}_L := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \zeta_x \quad \text{and} \quad \zeta_L^{\perp} := \zeta - \hat{\zeta}_L,$$

then the random variable $\hat{\zeta}_L$ and the random vector $\zeta_L^{\perp}$ are independent. Then fix a realization of $\zeta_L^{\perp}$ and apply a suitably rescaled version of Exercise 1 with the functions

$$\hat{\zeta}_L \to F_L^{+}(\hat{\zeta}_L, \zeta_L^{\perp}) \quad \text{and} \quad \hat{\zeta}_L \to F_L^{-}(\hat{\zeta}_L, \zeta_L^{\perp}).$$

One also needs the fact that the standard Gaussian distribution has full support on $\mathbb{R}$.

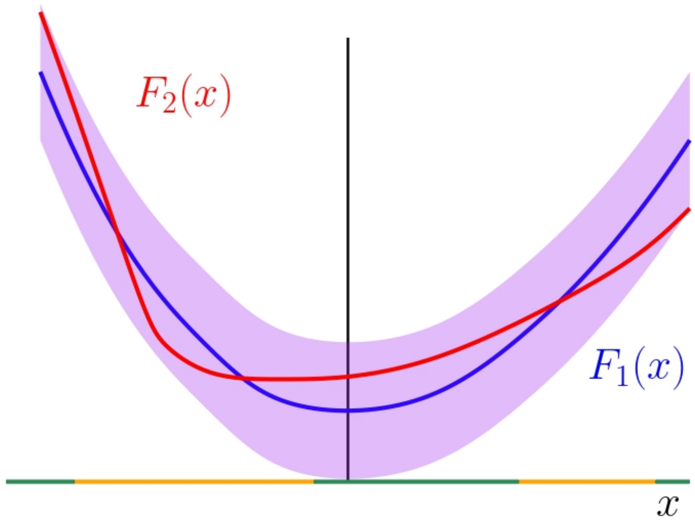

FIGURE 1

3. SOLUTION

3.1. **Solution of Exercise 1.** For simplicity, let us assume that the functions F_1 and F_2 are in the space $C^1(\mathbb{R})$.

We fix $\delta > 0$ and observe that if a point $t \in \mathbb{R}$ belongs to the set $\mathrm{Div}(F_1, F_2, \delta)$, then we have:

(1) Either the inequality $F_1'(t) - F_2'(t) > \delta$ holds;
(2) Or the inequality $F_2'(t) - F_1'(t) > \delta$ holds.

Let us first assume that the inequality (1) is satisfied; we claim that it implies the estimate

$$F_2'\left(t + \frac{4}{\delta}\right) \geq F_2'(t) + \frac{\delta}{2}. \tag{3.1}$$

To prove (3.1), note that the assumption $\sup_{s\in\mathbb{R}} |F_1(s) - F_2(s)| \leq 1$ implies, for any $s \in \mathbb{R}$,

$$F_1(s) - 1 \leq F_2(s) \leq F_1(s) + 1. \tag{3.2}$$

Using the inequality $F_1'(t) - F_2'(t) > \delta$ and the convexity of the map F_2, we see that, for any $s > t$,

$$F_1(s) \geq F_1(t) + F_1'(t)(s-t) \geq F_2(t) - 1 + (F_2'(t) + \delta)(s-t). \tag{3.3}$$

A combination of the estimates (3.2) and (3.3) yields

$$\frac{F_2(s) - F_2(t)}{s-t} > F_2'(t) + \delta - \frac{2}{s-t}.$$

Choosing the value $s = t + 4/\delta$ in the previous inequality and using the convexity of F_2 shows

$$F_2'\left(t + \frac{4}{\delta}\right) \geq \frac{F_2\left(t + \frac{4}{\delta}\right) - F_2(t)}{4/\delta} > F_2'(t) + \delta - \frac{\delta}{2} \geq F_2'(t) + \frac{\delta}{2}.$$

The proof of the claim (3.1) is complete. In the case when the inequality (2) is satisfied, a similar argument yields the estimate

$$F_2'\left(t - \frac{4}{\delta}\right) \leq F_2'(t) - \frac{\delta}{2}. \tag{3.4}$$

A combination of (3.1) and (3.4), and the assumption that F_2 is convex (which implies that its derivative is increasing) shows that, for any point $t \in \mathrm{Div}(F_1, F_2, \delta)$,

$$F_2'\left(t + \frac{4}{\delta}\right) \geq F_2'\left(t - \frac{4}{\delta}\right) + \frac{\delta}{2}. \tag{3.5}$$

Using that the map F_2 is convex and 1-Lipschitz, we see that, for any triplet of real numbers $t_-, t, t_+ \in \mathbb{R}$ satisfying $t_- < t < t_+$,

$$-1 \leq F_2'(t_-) \leq F_2'(t) \leq F_2'(t_+) \leq 1. \tag{3.6}$$

The estimates (3.5) and (3.6) imply that there cannot exist a family $t_1, \ldots, t_{\lfloor \frac{4}{\delta} \rfloor + 1}$ of $\left(\lfloor \frac{4}{\delta} \rfloor + 1 \right)$-points satisfying the following properties:

(1) For any pair of distinct integers $i, j \in \{1, \ldots, \lfloor 4/\delta \rfloor + 1\}$, one has $|t_i - t_j| > \frac{8}{\delta}$;
(2) For any integer $i \in \{1, \ldots, \lfloor 4/\delta \rfloor + 1\}$, the point t_i belongs to the set $\mathrm{Div}(F_1, F_2, \delta)$.

This property implies that the set $\mathrm{Div}(F_1, F_2, \delta)$ is included in the union of (at most) $\lfloor \frac{4}{\delta} \rfloor$ intervals of length $16/\delta$ which implies the upper bound

$$\mathrm{Leb}\left(\mathrm{Div}(F_1, F_2, \delta)\right) \leq \frac{C}{\delta^2}.$$

This is (2.1).

3.2. **Solution of Exercise 2.**

3.2.1. *Question 1.* (i) The maps $h \mapsto F_L^+(h)$ and $h \mapsto F_L^-(h)$ are suprema of affine functions (in h). They are thus convex.

(ii) We only prove the formula for F_L^+. Since $h \mapsto F_L^+(h)$ is the supremum of a finite number of affine functions in h, the following properties hold for almost every $h : \Lambda_L \to \mathbb{R}$:

- The maximum $\sigma_{L,x}^+(h)$ is uniquely defined;
- There exists a neighborhood V_h of h such that, for any $h' \in V_h$,

$$F_L^+(h') = \sum_{\substack{x,y \in \Lambda_{L+1} \\ x \sim y}} \sigma_{L,x}^+(h) \sigma_{L,y}^+(h) + \sum_{x \in \Lambda_L} h_x' \sigma_{L,x}^+(h).$$

  The result is then obtained by noting that the right-hand of the previous display is an affine function in h' and by differentiating both sides at the value $h' = h$.

(iii) To prove the inequality $|F_L(h)| \leq 2\,|\partial \Lambda_L|$, we consider a maximiser $\sigma_L^+(h)$ and define

$$\widetilde{\sigma}_x := \begin{cases} \sigma_{L,x}^+(h) \text{ for } x \in \Lambda_L, \\ \quad -1 \text{ on } x \in \partial \Lambda_L. \end{cases}$$

Note that $\widetilde{\sigma}_x \in \mathcal{S}^-$. By definition of the maximum $F_L^-(h)$ and of $\widetilde{\sigma}$, we have

$$\begin{aligned} F_L^+(h) &= \sum_{\substack{x,y \in \Lambda_{L+1} \\ x \sim y}} \sigma_{L,x}^+(h) \sigma_{L,y}^+(h) + \sum_{x \in \Lambda_L} h_x \sigma_{L,x}^+(h) \\ &\leq \sum_{\substack{x,y \in \Lambda_{L+1} \\ x \sim y}} \widetilde{\sigma}_x \widetilde{\sigma}_y + \sum_{x \in \Lambda_L} h_x \widetilde{\sigma}_x + 2\,|\partial \Lambda_L| \\ &\leq F_L^-(h) + 2\,|\partial \Lambda_L|. \end{aligned}$$

A similar argument shows

$$F_L^-(h) \leq F_L^+(h) + 2\,|\partial \Lambda_L|,$$

and a combination of the two previous inequalities completes the proof.

3.2.2. *Question 2.* Let us fix $L \in \mathbb{N}$ and an external field $h : \Lambda_L \to \mathbb{R}$ (to avoid technical difficulties, we assume that h is in the set of full measures in which the ground states $\sigma_{L,x}^-(h)$ and $\sigma_{L,x}^+(h)$ are uniquely defined).

We argue by contradiction and assume that there exists a vertex $z \in \Lambda_L$ such that

$$\sigma_{L,z}^+(h) < \sigma_{L,z}^-(h).$$

We then denote by $\mathcal{C}_z$ the connected component of z in the set $\left\{ x \in \Lambda_L : \sigma_{L,x}^+(h) < \sigma_{L,x}^-(h) \right\}$ and define two configurations $\widetilde{\sigma}^+$ and $\widetilde{\sigma}^-$ as follows

$$\widetilde{\sigma}_x^+ := \begin{cases} \sigma_{L,x}^-(h) \text{ for } x \in \mathcal{C}_z, \\ \sigma_{L,x}^+(h) \text{ for } x \in \Lambda_L \setminus \mathcal{C}_z, \end{cases}$$

and

$$\widetilde{\sigma}^-_x := \begin{cases} \sigma^+_{L,x}(h) \text{ for } x \in \mathcal{C}_z, \\ \sigma^-_{L,x}(h) \text{ for } x \in \Lambda_L \setminus \mathcal{C}_z. \end{cases}$$

Note that $\widetilde{\sigma}^+ \in \mathcal{S}^+$ and $\widetilde{\sigma}^- \in \mathcal{S}^-$. Using that $\sigma^+_{L,x}(h) = 1 \geq -1 = \sigma^-_{L,x}(h)$ for $x \in \partial\Lambda_L$, it can be deduced from the definitions of $\widetilde{\sigma}^+$ and $\widetilde{\sigma}^-$ that

$$\sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \widetilde{\sigma}^+_x\widetilde{\sigma}^+_y + \sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \widetilde{\sigma}^-_x\widetilde{\sigma}^-_y > \sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \sigma^+_{L,x}(h)\sigma^+_{L,y}(h) + \sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \sigma^-_{L,x}(h)\sigma^-_{L,y}(h)$$

and

$$\sum_{x\in\Lambda_L} h_x\widetilde{\sigma}^+_x + \sum_{x\in\Lambda_L} h_x\widetilde{\sigma}^-_x = \sum_{x\in\Lambda_L} h_x\sigma^+_{L,x}(h) + \sum_{x\in\Lambda_L} h_x\sigma^-_{L,x}(h).$$

A combination of the two previous displays shows that

$$\left(\sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \widetilde{\sigma}^+_x\widetilde{\sigma}^+_y + \sum_{x\in\Lambda_L} h_x\widetilde{\sigma}^+_x\right) + \left(\sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \widetilde{\sigma}^-_x\widetilde{\sigma}^-_y + \sum_{x\in\Lambda_L} h_x\widetilde{\sigma}^-_x\right) > F^+_L(h) + F^-_L(h).$$

This is in contradiction with the fact that $\widetilde{\sigma}^+ \in \mathcal{S}^+$ and $\widetilde{\sigma}^- \in \mathcal{S}^-$ and the definitions of the suprema $F^+_L(h)$ and $F^-_L(h)$.

The same argument can be used to show the first **Hint**.

Applying this **Hint** with the particular choice $x = x' = 0$ and $L' = L+1$, we deduce that, for any $y \in \Lambda_L$,

$$\sigma^-_{L+1,y}(\zeta) \geq \sigma^-_{L,y}(\zeta) \text{ and } \sigma^+_{L+1,y}(\zeta) \leq \sigma^+_{L,y}(\zeta).$$

3.2.3. *Question 3.* Using Question 2, we see that the sequences $L \mapsto \sigma^-_{L,x}(\zeta)$ and $L \mapsto \sigma^+_{L,x}(\zeta)$ are respectively increasing and decreasing. Since they are bounded, we deduce that they converge.

For the translation covariance, we treat the case of the ground state with + boundary condition, fix a vertex $x \in \mathbb{Z}^d$. We first note that the following identity holds, for any $L \in \mathbb{N}$,

$$\sigma^+_{x+\Lambda_L,x}(\zeta) = \sigma^+_{L,0}(\tau_x\zeta).$$

We next select an integer $L \in \mathbb{N}$ sufficiently large (depending on x) such that $\Lambda_{L/2} \subseteq (x+\Lambda_L) \subseteq \Lambda_{2L}$. Applying the **Hint**, we deduce that

$$\sigma^+_{2L,x}(\zeta) \leq \sigma^+_{x+\Lambda_L,x}(\zeta) \leq \sigma^+_{L/2,x}(\zeta).$$

Combining the two previous displays and taking the limit $L \to \infty$ implies that

$$\sigma^+_0(\tau_x\zeta) = \lim_{L\to\infty} \sigma^+_{L,0}(\tau_x\zeta) = \lim_{L\to\infty} \sigma^+_{x+\Lambda_L,x}(\zeta) = \sigma^+_x(\zeta).$$

3.2.4. *Question 4.* The result can be obtained as a consequence of Question 1.

3.2.5. *Question 5.* Using the **Hint**, we fix a realization of $\zeta^\perp_L$ introduce the functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$F_1 : t \mapsto \frac{1}{4|\partial\Lambda_L|} F^+_L\left(\frac{t}{\sqrt{|\Lambda_L|}}, \zeta^\perp_L\right) \text{ and } F_2 : t \mapsto \frac{1}{4|\partial\Lambda_L|} F^-_L\left(\frac{t}{\sqrt{|\Lambda_L|}}, \zeta^\perp_L\right).$$

Using the properties established in Question 1, we know that the functions F_1 and F_2 are convex, and that, for any $t \in \mathbb{R}$, $|F_1(t) - F_2(t)| \leq 1$.

By using Question 4, we have the identity

$$F_1'(t) = \frac{1}{4|\partial\Lambda_L|\sqrt{|\Lambda_L|}} \sum_{x\in\Lambda_L} \sigma^+_{L,x}\left(\frac{t}{\sqrt{|\Lambda_L|}}, \zeta^\perp_L\right) \text{ and } F_2'(t) = \frac{1}{4|\partial\Lambda_L|\sqrt{|\Lambda_L|}} \sum_{x\in\Lambda_L} \sigma^-_{L,x}\left(\frac{t}{\sqrt{|\Lambda_L|}}, \zeta^\perp_L\right).$$

We next note that, in dimension $d = 2$, we have $|\partial\Lambda_L| \simeq 8L$, $\sqrt{|\Lambda_L|} \simeq 2L$ and thus (at least for L sufficiently large)

$$F_1'(t) - F_2'(t) \geq \frac{1}{16|\Lambda_L|} \sum_{x\in\Lambda_L} \left(\sigma^+_{L,x}\left(\frac{t}{\sqrt{|\Lambda_L|}}, \zeta^\perp_L\right) - \sigma^-_{L,x}\left(\frac{t}{\sqrt{|\Lambda_L|}}, \zeta^\perp_L\right)\right).$$

Next, using that $\zeta_L^\perp$ and $\hat{\zeta}_L$ are independent and that $\hat{\zeta}_L$ is a Gaussian of variance $\lambda^2/|\Lambda_L|$, we deduce that

$$\mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\left(\sigma^+_{L,x}\left(\hat{\zeta}_L,\zeta_L^\perp\right)-\sigma^-_{L,x}\left(\hat{\zeta}_L,\zeta_L^\perp\right)\right)\geq\delta\;\middle|\;\zeta_L^\perp\right] \tag{3.7}$$
$$=\int_{\mathbb{R}}\mathbb{1}_{\left\{\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\left(\sigma^+_{L,x}\left(\frac{t}{\sqrt{|\Lambda_L|}},\zeta_L^\perp\right)-\sigma^-_{L,x}\left(\frac{t}{\sqrt{|\Lambda_L|}},\zeta_L^\perp\right)\right)\geq\delta\right\}}\frac{e^{-\frac{t^2}{2\lambda^2}}}{\sqrt{2\pi\lambda^2}}\,dt$$
$$\leq\int_{\mathbb{R}}\mathbb{1}_{\left\{F_1'(t)-F_2'(t)\geq\frac{\delta}{16}\right\}}\frac{e^{-\frac{t^2}{2\lambda^2}}}{\sqrt{2\pi\lambda^2}}\,dt$$

By Exercise 1, we have that

$$\mathrm{Leb}\left(\mathrm{Div}(F_1,F_2,\delta/16)\right)\leq\frac{C}{\delta^2},$$

and thus

$$\int_{\mathbb{R}}\mathbb{1}_{\left\{F_1'(t)-F_2'(t)\geq\frac{\delta}{16}\right\}}\frac{e^{-\frac{t^2}{2\lambda^2}}}{\sqrt{2\pi\lambda^2}}\,dt\leq\int_{-C/(2\delta^2)}^{C/(2\delta^2)}\frac{e^{-\frac{t^2}{2\lambda^2}}}{\sqrt{2\pi\lambda^2}}\,dt\leq 1-c_{\lambda,\delta},$$

for some constant $c_{\lambda,\delta}>0$ depending only on λ,δ. A combination of the previous display with (3.7) shows that, for any realization of $\zeta_L^\perp$,

$$\mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\left(\sigma^+_{L,x}\left(\hat{\zeta}_L,\zeta_L^\perp\right)-\sigma^-_{L,x}\left(\hat{\zeta}_L,\zeta_L^\perp\right)\right)\geq\delta\;\middle|\;\zeta_L^\perp\right]\leq 1-c_{\lambda,\delta}.$$

Taking the expectation on both sides of the previous inequality, we obtain

$$\mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\left(\sigma^+_{L,x}\left(\hat{\zeta}_L,\zeta_L^\perp\right)-\sigma^-_{L,x}\left(\hat{\zeta}_L,\zeta_L^\perp\right)\right)\geq\delta\right]\leq 1-c_{\lambda,\delta}.$$

3.2.6. *Question 5.* We first note that, by the translation covariance of the ground states, it is enough to show the result when $x=0$. Additionally, Questions 2 and 3 imply that, for almost every realisation of the disorder ζ,

$$\sigma_0^+(\zeta)\geq\sigma_0^-(\zeta),$$

and thus it is sufficient to show the identity

$$\mathbb{E}\left[\sigma_0^+(\zeta)-\sigma_0^-(\zeta)\right]=0.$$

We argue by contradiction and assume that $\mathbb{E}\left[\sigma_0^+(\zeta)-\sigma_0^+(\zeta)\right]>0$. An application of the ergodic theorem shows that, for almost every realization of the disorder ζ,

$$\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_0^+(\tau_x\zeta)-\sigma_0^-(\tau_x\zeta)\underset{L\to\infty}{\longrightarrow}\mathbb{E}\left[\sigma_0^+(\zeta)-\sigma_0^-(\zeta)\right].$$

Combining the previous display with the translation covariance of the ground states, we obtain the almost-sure convergence

$$\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_x^+(\zeta)-\sigma_x^-(\zeta)\underset{L\to\infty}{\longrightarrow}\mathbb{E}\left[\sigma_0^+(\zeta)-\sigma_0^-(\zeta)\right]>0.$$

We then set $\delta:=\mathbb{E}\left[\sigma_0^+(\zeta)-\sigma_0^-(\zeta)\right]/2$. Since the almost-sure convergence implies the convergence in probability, we deduce that

$$\lim_{L\to\infty}\mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_x^+(\zeta)-\sigma_x^-(\zeta)\leq\delta\right]\longrightarrow 0. \tag{3.8}$$

We then note that Question 2 implies that, for any $L\in\mathbb{N}$,

$$\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_x^+(\zeta)-\sigma_x^-(\zeta)\leq\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_{L,x}^+(\zeta)-\sigma_{L,x}^-(\zeta),$$

and thus

$$\mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_x^+(\zeta)-\sigma_x^-(\zeta)\leq\delta\right]\geq\mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_{L,x}^+(\zeta)-\sigma_{L,x}^-(\zeta)\leq\delta\right].$$

Applying Question 5, we finally deduce that

$$\liminf_{L\to\infty} \mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_x^+(\zeta)-\sigma_x^-(\zeta)\le\delta\right]\ge\liminf_{L\to\infty} \mathbb{P}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_{L,x}^+(\zeta)-\sigma_{L,x}^-(\zeta)\le\delta\right]>0.$$

This is in contradiction with (3.8).

References

[1] Paul Dario, Matan Harel, and Ron Peled. "Quantitative disorder effects in low-dimensional spin systems." Communications in Mathematical Physics 405, no. 9 (2024): 212.