Summer School: Additive Combinatorics

Preserving and Counting Arithmetic Structures

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A CLASS ABOVE

1. Arithmetic Ramsey Theory

- **Arithmetic Ramsey Theory** studies how arithmetic structures in sets of integers are preserved under partitions.
- For $r \in \mathbb{Z}^+$ and a set of integers S, an **r-colouring** of S is a function $\gamma : S \to [r] \coloneqq \{1, 2, \dots, r\}$.
- An r-colouring is **monochromatic** if it is constant on a set S.

2. Schur's Theorem and the Integers Modulo m

- **Theorem** (Schur, 1916). *Every r-colouring of the positive integers contains at least one monochromatic solution to $x + y = z$.*
- The **integers modulo** m is the set of all congruence classes under the congruence relation $x \equiv y \pmod{m}$, that is

$$\mathbb{Z}_m \coloneqq \{[0],\ [1],\ \dots,\ [m-1]\}.$$

- $(\mathbb{Z}_m, +, \times)$ is a commutative ring with multiplicative identity.

3. Key Definition: Modular Schur Numbers

- An **r-partition** of a set S is a set $P = \{S_1, \dots, S_r\}$ where the S_i's are pairwise disjoint and their union equals S.
- A set of integers S is called **l-sum-free modulo** m if $x_1, \dots, x_l \in S$ and $x_1 + \dots + x_l \equiv y \pmod{m}$, then $y \notin S$.
- The set $\{\{1,4\},\{2,3\},\{5,6\},\{7\}\}$ is a 4-partition of [7] into 2-sum-free sets modulo 8.
- The **generalised Schur number modulo** m, $S_m(r,l)$ is defined as the largest positive integer n such that the set $[n] = \{1, 2, \dots, n\}$ admits an r-partition into l-sum-free sets modulo m.

4. Key Definition: r-common

- A triple, (x, y, z) satisfying $L : x + y = z$ is called a **Schur triple**.
- The **density of monochromatic solutions** of L in $\mathbb{Z}_m$, with respect to an r-colouring γ, is denoted by $m_{\mathbb{Z}_m, L}(\gamma)$.
- A random r-colouring of $\mathbb{Z}_m$ has an expected proportion of monochromatic Schur triples of

$$\frac{1}{r^2}.$$

- L is **r-common over** $\mathbb{Z}_m$ if, for all r-colourings γ of $\mathbb{Z}_m$,

$$m_{\mathbb{Z}_m, L}(\gamma) \geq \frac{1}{r^2}.$$

- L is **r-common over** $[n]$ if, for all r-colourings γ of $[n]$,

$$\limsup_{n\to\infty} \left[\min_{\gamma} m_{[n],L}(\gamma)\right] \geq \frac{1}{r^2}.$$

5. Research Objectives

$S_m(r,l)$ is known for all $r, l \in \mathbb{Z}^+$ when $m = 1, 2, 3$. Schur triples are 2-common over $\mathbb{Z}_m$ but not 2-common over $[n]$.

Objectives:

1. Determine $S_m(r,l)$ for some $l, m, r \in \mathbb{Z}^+$.
2. Determine if Schur triples are r-common over $\mathbb{Z}_m$ for some $m, r \in \mathbb{Z}^+$ where $r > 2$.

6. Methods

We use the following well-known theorem to determine when sets are l-sum-free modulo m:

Theorem. *The linear congruence $ax \equiv b \pmod{m}$ has a solution if and only if $d \mid b$, where $d = \gcd(a, m)$. If $d \mid b$, then it has d mutually incongruent solutions modulo m that are given by*

$$x \equiv x_0 + u\left(\frac{m}{d}\right) \pmod{m},\ 0 \leq u \leq d - 1.$$

7. Results: Modular Schur Numbers

To date, we obtained the following results:

Theorem (D'orville et al.[2]). *Let p be a prime and $i, l \in \mathbb{Z}^+$. Suppose $l \geq p^i - 1$ and $l \not\equiv 1 \pmod{p}$. Then*

$$S_{p^i}(r,l) = \begin{cases} r, & \text{if } 1 \leq r \leq p-1 \text{ and } i \geq 1; \\ up - 1, & \text{if } r = p + (u-2),\ i \geq 2 \text{ and } 2 \leq u \leq p^{i-1}; \\ p^i - 1, & \text{if } k \geq p + (p^{i-1} - 2) \text{ and } i \geq 1. \end{cases}$$

[2]D'orville, J., Sim, K. A., Wong, K. B., & Ho, C. K. (2023). *Modular generalisations of Schur numbers.* Submitted.

8. Results: r-common property of Schur triples

Theorem (D'orville et al.[3]). *Let $m, r \in \mathbb{Z}^+$ such that $m \geq 5$. Then Schur triples are not r-common over $\mathbb{Z}_m$ for all r, $\lceil m/2 \rceil \leq r < m$.*

Theorem (D'orville et al.[3]). *Schur triples are not r-common over $[n]$ for all $r \geq 2$.*

[3]D'orville, J., Sim, K. A., Wong, K. B., & Ho, C. K. (2024). *On the c-common property of the Schur equation.* Submitted.

9. Open Problems

Modular Schur numbers

- Determine the values of $S_m(r,l)$ for $l < m$.
- Find a way to apply probablistic arguments to study $S_m(r,l)$.

r-common property of Schur triples

- Are Schur triples not r-common over $\mathbb{Z}_m$ for all $2 < r < m$?
- What's the minimum number of monochromatic Schur triples over all r-colourings of $[n]$ and $\mathbb{Z}_m$?[4]

[4]This is known for 2-colourings of $[n]$ and $\mathbb{Z}_m$. See Robertson and Zeilberger (1998) and Datskovsky (2003), respectively.

10. Acknowledgment and Main References

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