The uses of Dyson-Schwinger equations

Alice Guionnet

June 13, 2025

Abstract These lecture notes were first written for a course I gave at Columbia University in August 2017. They have been updated for a course in Budapest, 2025, stressing new development and ignoring some others. Their goal is to put together the asymptotic analysis of several highly correlated systems such as the eigenvalues of random matrices or random tilings that can be attacked by similar tools, namely the derivation and analysis of the Dyson-Schwinger or loop equations.

Contents

1 Introduction

1.1 Some historical references

These lecture notes concern the study of the asymptotics of large systems of particles in very strong mean field interaction and in particular the study of their fluctuations. Examples are given by the distributions of eigenvalues of Gaussian random matrices, β-ensembles, random tilings and discrete β-ensembles, or several random matrices. These models display a much stronger interaction between the particles than the underlying randomness so that classical tools from probability theory fail. Fortunately, these model have in common that their correlators (basically moments of a large class of test functions) obey an infinite system of equations that we will call the Dyson-Schwinger equations. They are also called loop equations, Master equations or Ward identities. Dyson-Schwinger equations are usually derived from some invariance or some symmetry of the model, for instance by some integration by parts formula. We shall argue in these notes that even though these equations are not closed, they are often asymptotically closed (in the limit where the dimension goes to infinity) so that we can asymptotically solve them and deduce asymptotic expansions for the correlators. This in turn allows to retrieve the global fluctuations of the system, and eventually even more local information such as rigidity.

This strategy has been developed at the formal level in physics [2] for a long time. In particular in the work of Eynard and collaborators [52, 51, 50, 16], it was shown that if one assumes that correlators expand formally in the dimension N, then the coefficients of these expansions obey the so-called topological recursion. For instance, in [30, 31], it was shown that assuming a formal expansion holds, Dyson-Schwinger equations induce recurrence relations on the terms in the expansion which can be solved by algebraic geometry means. These recurrence relations can even be interpreted as topological recursion, so that the coefficients of these expansions can be given combinatorial interpretations. In fact, it was realized in the seminal works of t'Hooft [87] and Brézin-Parisi-Itzykson-Zuber [44] that moments of Gaussian matrices and matrix models can be interpreted as the generating functions for maps. One way to retrieve this result is by using Dyson-Schwinger equations and checking that asymptotically they are similar to the topological recursion formulas obeyed by the enumeration of maps, as found by Tutte [91]. In this case, one first need to analyze the limiting behavior of the system, given by the so-called equilibrium measure or spectral curve, and then the Dyson-Schwinger equations, that is the topological recursion, will provide the large dimension expansion of the observables.

The study of the asymptotics of our large system of particles also starts with the analysis of its limiting behaviour. I usually derive this limiting behaviour as the minimizer of an energy functional appearing as a large deviation rate functional [9], or in concentration of measure estimates [75], but, according to fields, people can prefer to see it as the optimizer of Fekete points [81], or as the solution of a Riemann-Hilbert problem [37]. This study often amounts to the analysis of some equation. The same type of analysis appears in combinatorics when one counts for example triangulations of the sphere. Indeed, it can be seen, thanks to Tutte surgery [91], that the generating function for this enumeration satisfies some equation. Sometimes, one can solve explicitly this equation, for instance thanks to the quadratic method and catalytic variables [25, 28] or [57, Section 2.9]. In our models, we will also be able to derive equations for our equilibrium measure thanks to Dyson-Schwinger equations. But sometimes, these equations may have several solutions, for instance in the setting of a double well potential in β-models. The absence of uniqueness of solutions to these equations prevents the analysis of many interesting models, such as several matrix models at low temperature. In good cases such as the β-models, we may still get uniqueness for instance if we add the information that the equilibrium measure minimizes a strictly convex energy. Dyson-Schwinger equation can then be regarded as the equations satisfied by the critical points of this energy.

The Dyson-Schwinger equations will be our key to get precise informations on the convergence to equilibrium, such as large dimension expansion of the free energy or fluctuations. These types of questions were attacked also in the Riemann Hilbert problems community based on a fine study of the asymptotics of orthogonal polynomials [55, 35, 45, 11, 27, 36]. It seems to me however that such an approach is more rigid as it requires more technical steps and assumptions and can not apply in such a great generality than loop equations. Yet, when it can be used, it provides eventually more detailed information. Moreover, in certain cases, such as the case of potentials with Fisher Hartwig singularities, Riemann Hilbert techniques could be used but not yet loop equations [67, 38].

To study the asymptotic properties of our models we need to get one step further than the formal approach developped in the physics litterature. In other words, we need to show that indeed correlators can expand in the dimension up to some error which is quantified in the large N limit and shown to go to zero. To do so, one needs in general a priori concentration bounds in order to expand the equations around their limits. For β-models, such a priori concentration of measure estimates can be derived thanks to a result of Boutet de Monvel, Pastur and Shcherbina [26] or Maida and Maurel-Segala [75]. It is roughly based on the fact that the logarithm of the density of such models is very close to a distance of the empirical measure to its equilibrium measure, hence implying a priori estimates on this distance. In more general situations where densities are unknown, for instance when one considers the distributions of the traces of polynomials in several matrices, one can rely on abstract concentration of measures estimates for instance in the case where the density is strictly log-concave or the underlying space has a positive Ricci curvature (e.g $SU(N)$) [61]. Dyson-Schwinger equations are then crucial to obtain optimal concentration bounds and asymptotics.

This strategy was introduced by Johansson [68] to derive central limit theorems for β-ensembles with convex potentials. It was further developped by Shcherbina and collaborators [1, 84] and myself, together with Borot [15], to study global fluctuations for β-ensembles when the potential is off-critical in the sense that the equilibrium has a connected support and its density vanishes like a square root at its boundary. These assumptions allow to linearize the Dyson-Schwinger equations around their limit and solve these linearizations by inverting the so-called Master operator. The case where the support of the density has finitely many connected component but the potential is off-critical was adressed in [82, 13]. It displays the additional tunneling effect where eigenvalues may jump from one connected support to the other, inducing discrete fluctuations. However, it can also be solved asymptotically after a detailed analysis of the case where the number of particles in each connected components is fixed, in which case Dyson-Schwinger equations can be asymptotically solved. These articles assumed that the potentials are real analytic in order to use Dyson-Schwinger equations for the Stieltjes functions. We will see that these techniques generalize to sufficiently smooth potentials by using more general Dyson-Schwinger equations. Global fluctuations, together with estimates of the Wasserstein distance, were obtained in [73] for off-critical, one-cut smooth potentials. One can obtain by such considerations much more precise estimates such as the expansion of the partition function up to any order for general off-critical cases with fixed filling fractions, see [13]. Such expansion can also be derived by using Riemann-Hilbert techniques, see [46] in a perturbative setting and [32] in two cut cases and polynomial potential.

But β-models on the real line serve also as toy models for many other models. Borot, Kozlowski and myself considered more general potentials depending on the empirical measure in [19]. We studied also the case of more complicated interactions (in particular sinh interactions) in [20] : the main problems are then due to the non-linearity of the interaction which induces multi-scale phenomenon. The case of critical potentials was tackled recently in [41]. Also Dyson-Schwinger (often called Ward identities) equations are instrumental to study Coulomb gas systems in higher dimension. One however has to deal with the fact that Ward identities are not nice functions of the empirical measure anymore, so that an additional term, the anisotropic term, has to be controlled. This could very nicely be done by Leblé and Serfaty [74] by using local large deviations estimates. Recently we also generalized this approach to study discrete β-ensembles and random tilings [14] by analyzing the so-called Nekrasov's equations in the spirit of Dyson-Schwinger equations.

The same approach can be developed to study multi-matrix questions. Originally, I developed this approach to study fluctuations and large dimension expansion of the free energy with E. Maurel Segala [58, 59] in the context of several random matrices. In this case we restrict ourselves to perturbations of the quadratic potential to insure convergence and stability of our equations. We could extend this study to the case of unitary or orthogonal matrices following the Haar measure (or perturbation of this case) in [33, 60]. This strategy was then applied in a closely related setting by Chatterjee [29], see also [34].

Dyson-Schwinger equations are also central to derive more local results such as rigidity and universality, showing that the eigenvalues are very close to their deterministic locus and that their local fluctuations does not depend much on the model. For instance, in the case of Wigner matrices with non Gaussian entries, a key tool to prove rigidity is to show that the Stieltjes transform approximately satisfies the same quadratic equation than in the Gaussian case up to the optimal scale [48, 47, 5]. Recently, it was also realized that closely connected ideas could lead to universality of local fluctuations, on one hand by using the local relaxation flow [48, 49, 22], by using Lindenberg strategy [85, 86] or by constructing approximate transport maps [84, 6, 54]. Such ideas could be generalized in the discrete Beta ensembles [62] where universality could be derived thanks to optimal rigidity (based on the study of Nekrasov's equations) and comparisons to the continuous setting.

Acknowledgments I would like to thank Columbia University, and in particular Ivan Corwin and Andrei Okounkov, for giving me the opportunity to give these lectures. I thank as well Jonathan Husson and Felix Parraud for carefully reading these lecture notes and giving me many constructive feedbacks.

1.2 A toy model

Let us give some heuristics for the type of analysis we will do in these lectures thanks to a toy model. We will consider the distribution of N real-valued variables $\lambda_1, \dots, \lambda_N$ and denote by

$$\hat{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

their empirical measure : for a test function f, $\hat{\mu}^N(f) = \frac{1}{N} \sum f(\lambda_i)$. Then, the correlators are moments of the type

$$M(f_1, \dots, f_p) = \mathbb{E}[\prod_{i=1}^{p} \hat{\mu}^N(f_i)]$$

where f_i are test functions, which can be chosen to be polynomials, Stieltjes functionals or some more general set of smooth test functions. Dyson-Schwinger equations are usually retrieved from some underlying invariance or symmetries of the model. Let us consider the continuous case where the law of the λ_i's is absolutely continuous with respect to $\prod d\lambda_i$ and given by

$$dP_N^V(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^V} \exp\{-\sum_{i_1=1}^N \sum_{i_2=1}^N V(\lambda_{i_1}, \lambda_{i_2})\} \prod d\lambda_i$$

where V is some symmetric smooth function. Then a way to get equations for the correlators is simply by integration by parts (which is a consequence of the invariance of Lebesgue measure under translation) : Let $f_0, f_1, \ldots, f_\ell$ be continuously differentiable functions. Then

$$\begin{aligned}
\mathbb{E}[\hat{\mu}^N(f_0') \prod_{i=1}^{\ell} \hat{\mu}^N(f_i)] &= \mathbb{E}[\left(\frac{1}{N}\sum_k \partial_{\lambda_k} f_0(\lambda_k))\right) \prod_{i=1}^{\ell} \hat{\mu}^N(f_i)] \\
&= -\frac{1}{N}\mathbb{E}\left[(\frac{dP_N^V}{d\lambda})^{-1} \sum_k f_0(\lambda_k)\partial_{\lambda_k}\left(\prod_{i=1}^{\ell} \hat{\mu}^N(f_i)(\frac{dP_N^V}{d\lambda})\right)\right] \\
&= 2N\mathbb{E}[\left(\int f_0(x_1)\partial_{x_1} V(x_1, x_2) d\hat{\mu}^N(x_1) d\hat{\mu}^N(x_2)\right) \prod_{i=1}^{\ell} \hat{\mu}^N(f_i)] \\
&\quad -\frac{1}{N}\sum_{j=1}^{\ell} \mathbb{E}[(\hat{\mu}^N(f_0 f_j')) \prod_{i\neq j} \hat{\mu}^N(f_i)]
\end{aligned}$$

where we noticed that since V is symmetric $\partial_x V(x,x) = 2\partial_x V(x,y)|_{y=x}$. The case $\ell = 0$ refers to the case $f_1 = \cdots = f_\ell = 1$. We will call the above equations Dyson-Schwinger equations. One would like to analyze the asymptotics of the correlators. The idea is that if we can prove that $\hat{\mu}^N$ converges, then we can linearize the above equations around this limit, and hopefully solve them asymptotically by showing that only few terms are relevant on some scale, solving these simplified equations and then considering the equations at the next order of correction. Typically in the case above, we see that if $\hat{\mu}^N$ converges towards μ^* almost surely (or in L^p) then by the previous equation (with $\ell = 0$) we must have

$$\int f_0(x_1)\partial_{x_1} V(x_1, x_2) d\mu^*(x_1) d\mu^*(x_2) = 0\,. \tag{1}$$

We can then linearize the equations around μ^* and we find that if we set $\Delta_N = \hat{\mu}^N - \mu^*$, we can rewrite the above equation with $\ell = 0$ as

$$\mathbb{E}[\Delta_N(\Xi f_0)] = \frac{1}{N}\mathbb{E}[\hat{\mu}^N(f_0')] - 2\mathbb{E}[\int f_0(x_1)\partial_{x_1} V(x_1, x_2) d\Delta_N(x_1) d\Delta_N(x_2)] \tag{2}$$

where Ξ is the Master operator given by

$$\Xi f_0(x) = 2f_0(x)\int \partial_{x_1} V(x, x_1) d\mu^*(x_1) + 2\int f_0(x_1)\partial_{x_1} V(x_1, x) d\mu^*(x_1)\,.$$

Let us show heuristically how such an equation can be solved asymptotically. Let us assume that we have some a priori estimates that tell us that Δ_N is of order δ_N almost surely (or in all L^k's)[that is that for sufficiently smooth functions g, $\Delta_N(g) = (\hat{\mu}^N - \mu^*)(g)$ is with high probability (i.e with probability greater than $1 - N^{-D}$ for all D and N large enough) at most of order $\delta_N C_g$ for some finite constant C_g]. Then, the right hand side of (2) should be smaller than $\max\{\delta_N^2, N^{-1}\}$ for sufficiently smooth test functions. Hence, if we can invert the master operator Ξ, we see that the expectation of Δ_N is of order at most $\max\{\delta_N^2, N^{-1}\}$. We would like to bootstrap this estimate to show that δ_N is at most of order N^{-1}. This requires to estimate higher moments of Δ_N. Let us do a similar derivation from the Dyson-Schwinger equations when $\ell = 1$ to find that if $\bar{\Delta}_N(f) = \Delta_N(f) - \mathbb{E}[\Delta_N(f)]$,

$$\begin{aligned} \mathbb{E}[\Delta_N(\Xi f_0)\bar{\Delta}_N(f_1)] &= -2\mathbb{E}[\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\Delta_N(x_1)d\Delta_N(x_2)\bar{\Delta}_N(f_{0)}] \\ &\quad + \frac{1}{N}\mathbb{E}[\Delta_N(f_0')\bar{\Delta}_N(f_1)] + \frac{1}{N^2}\mathbb{E}[\hat{\mu}^N(f_0 f_1')]\,. \end{aligned} \tag{3}$$

Again if Ξ is invertible, this allows to bound the covariance $\mathbb{E}[\Delta_N(f_0)\bar{\Delta}_N(f_1)]$ by $\max\{\delta_N^3, \delta_N^2/N, N^{-2}\}$, which is a priori better than δ_N^2 unless δ_N is of order $1/N$. Since $\Delta_N(f) - \bar{\Delta}_N(f)$ is at most of order δ_N^2 by (2), we deduce that also $\mathbb{E}[\Delta_N(f_0)\Delta_N(f_1)]$ is at most of order δ_N^3. We can plug back this estimate into the previous bound and show recursively (by considering higher moments) that δ_N can be taken to be of order $1/N$ up to small corrections. We then deduce that

$$C(f_0, f_1) = \lim_{N\to\infty} N^2 \mathbb{E}[(\Delta_N - \mathbb{E}[\Delta_N])(f_0)(\Delta_N - \mathbb{E}[\Delta_N])(f_1)] = \mu^*(\Xi^{-1} f_0 f_1')$$

and

$$m(f_0) = \lim_{N\to\infty} N\mathbb{E}[\Delta_N(f_0)] = \mu^*((\Xi^{-1}f_0)')\,.$$

We can consider higher order equations (with $\ell \geq 1$) to deduce higher orders of corrections, and the convergence of higher moments.

1.3 Rough plan of the lecture notes

We will apply these ideas in several cases where V has a logarithmic singularity in which case the self-interaction term in the potential has to be treated with more care. More precisely we will examine the following models.

1. *The law of the GUE.* We consider the case where the λ_i are the eigenvalues of the GUE and we take polynomial test functions. In this case the operator Ξ is triangular and easy to invert. Convergence towards μ^* and a priori estimates on Δ_N can also be proven from the Dyson-Schwinger equations.

2. *The Beta ensembles.* We take smooth test functions. Convergence of $\hat{\mu}^N$ is proven by large deviation principle and quantitative estimates on δ_N

are obtained by concentration of measure. The operator Ξ is invertible if μ^* has a single cut, with a smooth density which vanishes like a square root at the boundary of the support. We then obtain full expansion of the correlators. In the case where the equilibrium measure has p connected components in its support, we can still follow the previous strategy if we fix the number of eigenvalues in a small neighborhood of each connected pieces (the so-called filing fractions). Summing over all possible choices of filing fractions allows to estimate the partition functions as well as prove a form of central limit theorem depending on the fluctuations of the filling fractions.

3. *Discrete Beta ensembles.* These distributions include the law of random tilings and the λ_i's are now discrete. Integration by parts does not give nice equations a priori but Nekrasov found a way to write new equations by showing that some observables are analytic. These equations can in turn be analyzed in a spirit very similar to continuous Beta-ensembles.

We will discuss also one idea related with our approach based on Dyson-Schwinger to study more local questions, in particular universality of local fluctuations. The first is based on the construction of approximate transport maps as introduced in [6]. The point is that the construction of this transport maps goes through solving a Poisson equation $Lf = g$ where L is the generator of the Langevin dynamics associated with our invariant measure. It is symmetric with respect to this invariant measure and therefore closely related with integration by parts. In fact, solving this Poisson equation is at the large N limit closely related with inverting the master operator Ξ above, and in general follows the strategy we developed to analyze Dyson-Schwinger equations. Another strategy to show universality of local fluctuations is by analyzing the Dyson-Schwinger equations but for less smooth test functions, that is prove local laws. We will not developp this approach here. These ideas were developed in [62] for discrete beta-ensembles, based on a strategy initiated in [23]. The argument is to show that optimal bounds on Stieltjes functionals can be derived from Dyson-Schwinger equation away from the support of the equilibrium measure, but at some distance. It is easy to get it at distance of order $1/\sqrt{N}$, by straightforward concentration inequalities. To get estimates up to distance of order $1/N$, the idea is to localize the measure. Rigidity follows from this approach, as well as universality eventually.

2 The example of the GUE

In this section, we show how to derive topological expansions from Dyson-Schwinger equations for the simplest model : the GUE. The Gaussian Unitary Ensemble is the sequence of $N \times N$ hermitian matrices $X_N, N \geq 0$ such that $(X_N(ij))_{i\leq j}$ are independent centered Gaussian variables with variance $1/N$ that are complex outside of the diagonal (with independent real and imaginary parts). Then, we shall discuss the following expansion, true for all integer k

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}(X_N^k)] = \sum_{g\geq 0} \frac{1}{N^{2g}} M_g(k)\,.$$

This expansion is called a topological expansion because $M_g(k)$ is the number of maps of genus g which can be build by matching the edges of a vertex with k labelled half-edges. We remind here that a map is a connected graph properly embedded into a surface (i.e so that edges do not cross). Its genus is the smallest genus of a surface so that this can be done. This identity is well known [93] and was the basis of several breakthroughs in enumerative geometry [65, 69]. It can be proven by expanding the trace into products of Gaussian entries and using Wick calculus to compute these moments. In this section, we show how to derive it by using Dyson-Schwinger equations.

2.1 Combinatorics versus analysis

In order to calculate the electromagnetic momentum of an electron, Feynman used diagrams and Schwinger used Green's functions. Dyson unified these two approaches thanks to Dyson-Schwinger equations. On one hand they can be thought as equations for the generating functions of the graphs that are enumerated, on the other they can be seen as equations for the invariance of the underlying measure. A baby version of this idea is the combinatorial versus the analytical characterization of the Gaussian law $\mathcal{N}(0,1)$. Let X be a random variable with law $\mathcal{N}(0,1)$. On one hand it is the unique law with moments given by the number of matchings :

$$\mathbb{E}\left[X^n\right] = \#\left\{\text{pair partitions of } n \text{ points}\right\} =: P_n\,. \tag{4}$$

On the other hand, it is also defined uniquely by the integration by parts formula

$$\mathbb{E}\left[Xf(X)\right] \quad = \quad \mathbb{E}\left[f'(X)\right] \tag{5}$$

$$\tag{6}$$

for all smooth functions f going to infinity at most polynomially. If one applies the latter to $f(x) = x^n$ one gets

$$m_{n+1} := \mathbb{E}\left[X^{n+1}\right] \quad = \quad \mathbb{E}\left[nX^{n-1}\right] = nm_{n-1}\,.$$

This last equality is the induction relation for the number P_{n+1} of pair partitions of $n+1$ points by thinking of the n ways to pair the first point. Since $P_0 = m_0 = 1$ and $P_1 = m_1 = 0$, we conclude that $P_n = m_n$ for all n. Hence, the integration by parts formula and the combinatorial interpretation of moments are equivalent.

2.2 GUE : combinatorics versus analysis

When instead of considering a Gaussian variable we consider a matrix with Gaussian entries, namely the GUE, it turns out that moments are as well described both by integration by parts equations and combinatorics. In fact moments of GUE matrices can be seen as generating functions for the enumeration of interesting graphs, namely maps, which are sorted by their genus. We shall describe the full expansion, the so-called topological expansion, at the end of this section and consider more general colored cases in section **??**. In this section, we discuss the large dimension expansion of moments of the GUE up to order $1/N^2$ as well as central limit theorems for these moments, and characterize these asymptotics both in terms of equations similar to the previous integration by parts, and by the enumeration of combinatorial objects.

Let us be more precise. A matrix $X = (X_{ij})_{1\le i,j\le N}$ from the GUE is the random $N \times N$ Hermitian matrix so that for $k < j$, $X_{kj} = X^{\mathbb{R}}_{kj} + iX^{i\mathbb{R}}_{kj}$, with two independent real centered Gaussian variables with covariance $1/2N$ (denoted later $\mathcal{N}\left(0, \frac{1}{2N}\right)$) variables $X^{\mathbb{R}}_{kj}, X^{i\mathbb{R}}_{kj}$) and for $k \in \{1, \dots, N\}$, $X_{kk} \sim \mathcal{N}\left(0, \frac{1}{N}\right)$. then, we shall prove that

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}(X^k)] = M_0(k) + \frac{1}{N^2}M_1(k) + o(\frac{1}{N^2}) \tag{7}$$

where

- $M_0(k) = C_{k/2}$ denotes the Catalan number : it vanishes if k is odd and is the number of non-crossing pair partitions of $2k$ (ordered) points, that is pair partitions so that any two blocks (a, b) and (c, d) is such that $a < b < c < d$ or $a < c < d < b$. C_k can also be seen to be the number of rooted trees embedded into the plane and k edges, that is trees with a distinguished edge and equipped with an exploration path of the vertices $v_1 \to v_2 \to \cdots \to v_{2k}$ of length $2k$ so that (v_1, v_2) is the root and each edge is visited twice (once in each direction). C_k can also be seen as the number of planar maps build over one vertex with valence k : namely take a vertex with valence k, draw it on the plane as a point with k half-edges. Choose a root, that is one of these half-edges. Then the set of half-edges is in bijection with k ordered points (as we drew them on the plane which is oriented). A matching of the half-edges is equivalent to a pairing of these points. Hence, we have a bijection between the graphs build over one vertex of valence k by matching the end-points of the half-edges and the pair partitions of k ordered points. The pairing is non-crossing iff the matching gives a planar graph, that is a graph that is properly embedded into the plane (recall that an embedding of a graph in a surface is proper iff the edges of the graph do not cross on the surface). Hence, $M_0(k)$ can also be interpreted as the number of planar graphs build over a rooted vertex with valence k. Recall that the genus g of a graph (that is the minimal genus of a surface in which it can be properly embedded) is given by Euler formula :

$$2 - 2g = \#Vertices + \#Faces - \#Edges\,,$$

where the faces are defined as the pieces of the surface in which the graph is embedded which are separated by the edges of the graph. If the surface as minimal genus, these faces are homeomorphic to discs.

- $M_1(k)$ is the number of graphs of genus one build over a rooted vertex with valence k. Equivalently, it is the number of rooted trees with $k/2$ edges and exactly one cycle.

Moreover, we shall prove that for any $k_1, \dots, k_p$ $(\mathrm{Tr}(X^{k_j}) - \mathbb{E}[\mathrm{Tr}(X^{k_j})])_{1 \le j \le p}$ converges in moments towards a centered Gaussian vector with covariance

$$M_0(k, \ell) = \lim_{N \to \infty} \mathbb{E}\left[(\mathrm{Tr}(X^k) - \mathbb{E}[\mathrm{Tr}(X^k)])(\mathrm{Tr}(X^\ell) - \mathbb{E}[\mathrm{Tr}(X^\ell)])\right] .$$

$M_0(k, \ell)$ is the number of connected planar rooted graphs build over a vertex with valence k and one with valence ℓ. Here, both vertices have labelled half-edges and two graphs are counted as equal only if they correspond to matching half-edges with the same labels (and this despite of symmetries). Equivalently $M_0(k, \ell)$ is the number of rooted trees with $(k+\ell)/2$ edges and an exploration path with $k+\ell$ steps such that k consecutive steps are colored and at least an edge is explored both by a colored and a non-colored step of the exploration path.

Recall here that convergence in moments means that all mixed moments converge to the same mixed moments of the Gaussian vector with covariance M. We shall use that the moments of a centered Gaussian vector are given by Wick formula :

$$m(k_1, \dots, k_p) = \mathbb{E}[\prod_{i=1}^p X_{k_i}] = \sum_{\pi} \prod_{\text{blocks } (a,b) \text{ of } \pi} M(k_a, k_b)$$

which is in fact equivalent to the induction formula we will rely on :

$$m(k_1, \dots, k_p) = \sum_{i=2}^p M(k_1, k_i) m(k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_p) .$$

Convergence in moments towards a Gaussian vector implies of course the standard weak convergence as convergence in moments implies that the second moments of $Z_N := (\mathrm{Tr}(X^{k_j}) - \mathbb{E}[\mathrm{Tr}(X^{k_j})])_{1 \le j \le p}$ are uniformly bounded, hence the law of Z_N is tight. Moreover, any limit point has the same moments than the Gaussian vector. Since these moments do not blow too fast, there is a unique such limit point, and hence the law of Z_N converges towards the law of the Gaussian vector with covariance M. We will discuss at the end of this section how to generalize the central limit theorem to differentiable test functions, that is show that $Z_N(f) = \mathrm{Tr} f(X) - \mathbb{E}[\mathrm{Tr} f(X)]$ converges towards a centered Gaussian variable for any bounded differentiable function. This requires more subtle uniform estimates on the covariance of $Z_N(f)$ for which we will use Poincaré's inequality.

The asymptotic expansion (7) as well as the central limit theorem can be derived using combinatorial arguments and Wick calculus to compute Gaussian moments. This can also be obtained from the Dyson-Schwinger (DS) equation, which we do below.

2.2.1 Dyson-Schwinger Equations

Let :

$$Y_k := \mathrm{Tr} X^k - \mathbb{E}\mathrm{Tr} X^k$$

We wish to compute for all integer numbers $k_1, \ldots, k_p$ the correlators :

$$\mathbb{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^{p} Y_{k_i}\right].$$

By integration by parts, one gets the following Dyson-Schwinger equations

Lemma 2.1. *For any integer numbers $k_1, \ldots, k_p$, we have*

$$\mathbb{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^{p} Y_{k_i}\right] = \mathbb{E}\left[\frac{1}{N}\sum_{\ell=0}^{k_1-2} \mathrm{Tr} X^{\ell} \mathrm{Tr} X^{k_1-2-p} \prod_{i=2}^{p} Y_{k_i}\right]$$
$$+\mathbb{E}\left[\sum_{i=2}^{p} \frac{k_i}{N} \mathrm{Tr} X^{k_1+k_i-2} \prod_{j=2, j\neq i}^{p} Y_{k_j}\right] \qquad (8)$$

Proof. Indeed, we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^{p} Y_{k_i}\right] &= \sum_{i,j=1}^{N} \mathbb{E}\left[X_{ij} (X^{k_1-1})_{ji} \prod_{i=2}^{p} Y_{k_i}\right] \\ &= \frac{1}{N}\sum_{i,j=1}^{N} \mathbb{E}\left[\partial_{X_{ji}}\left((X^{k_1-1})_{ji} \prod_{i=2}^{p} Y_{k_i}\right)\right]\end{aligned}$$

where we noticed that since the entries are Gaussian independent complex variables, for any smooth test function f,

$$\mathbb{E}[X_{ij} f(X_{k\ell}, k \leq \ell)] = \frac{1}{N}\mathbb{E}[\partial_{X_{ji}} f(X_{k\ell}, k \leq \ell)]. \qquad (9)$$

But, for any $i, j, k, \ell \in \{1, \ldots, N\}$ and $r \in \mathbb{N}$

$$\partial_{X_{ji}} (X^r)_{k\ell} = \sum_{s=0}^{r-1} (X^s)_{kj} (X^{r-s-1})_{i\ell}$$

where $(X^0)_{ij} = 1_{i=j}$. As a consequence

$$\partial_{X_{ji}}(Y_r) = r X_{ij}^{r-1}.$$

The Dyson-Schwinger equations follow readily. ◇

Exercise 2.2. *Show that*

1. *If X is a GUE matrix,* (9) *holds. Deduce* (2.1).

2. *take X to be a GOE matrix, that is a symmetric matrix with real independent Gaussian entries $N_{\mathbb{R}}(0, \frac{1}{N})$ above the diagonal, and $N_{\mathbb{R}}(0, \frac{2}{N})$ on the diagonal. Show that*

$$\mathbb{E}[X_{ij} f(X_{k\ell}, k \leq \ell)] = \frac{1}{N}\mathbb{E}[\partial_{X_{ji}} f(X_{k\ell}, k \leq \ell)] + \frac{1}{N}\mathbb{E}[\partial_{X_{ij}} f(X_{k\ell}, k \leq \ell)] \,.$$

   Deduce that a formula analogous to (2.1) *holds provided we have an additional term $N^{-1}\mathbb{E}\left[k_1 \mathrm{Tr} X^{k_1} \prod_{i=2}^{p} Y_{k_i}\right]$.*

2.2.2 Dyson-Schwinger equation implies genus expansion

We will show that the DS equation (2.1) can be used to show that :

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr} X^k\right] = M_0(k) + \frac{1}{N^2} M_1(k) + o(\frac{1}{N^2})$$

Next orders can be derived similarly. Let :

$$m_k^N := \mathbb{E}\left[\frac{1}{N}\mathrm{Tr} X^k\right]$$

By the DS equation (with no Y terms), we have that :

$$m_k^N = \mathbb{E}\left[\sum_{\ell=0}^{k-2} \frac{1}{N}\mathrm{Tr} X^\ell \frac{1}{N}\mathrm{Tr} X^{k-\ell-2}\right] \,. \tag{10}$$

We now assume that we have the self-averaging property that for all $\ell \in \mathbb{N}$:

$$\mathbb{E}\left[\left(\frac{1}{N}\mathrm{Tr} X^\ell - \mathbb{E}\left[\frac{1}{N}\mathrm{Tr} X^\ell\right]\right)^2\right] = o(1)$$

as $N \to \infty$ as well as the boundedness property

$$\sup_N m_\ell^N < \infty \,.$$

We will show both properties are true in Lemma 2.3. If this is true, then the above expansion (10) gives us :

$$m_k^N = \sum_{\ell=0}^{k-2} m_\ell^N m_{k-\ell-2}^N + o(1)$$

As $\{m_\ell^N, \ell \leq k\}$ are uniformly bounded, they are tight and so any limit point $\{m_\ell, \ell \leq k\}$ satisfies

$$m_k = \sum_{\ell=0}^{k-2} m_\ell m_{k-\ell-2}, m_0 = 1, m_1 = 0 \,.$$

This equation has clearly a unique solution.

On the other hand, let $M_0(k)$ be the number of maps of genus 0 with one vertex with valence k. These satisfy the Catalan recurrence :

$$M_0(k) = \sum_{\ell=0}^{k-2} M_0(\ell) M_0(k-\ell-2)$$

This recurrence is shown by a Catalan-like recursion argument, which goes by considering the matching of the first half edge with the ℓth half-edge, dividing each map of genus 0 into two sub-maps (both still of genus 0) of size ℓ and $k-\ell-2$, for $\ell \in \{0, \ldots, k-2\}$.

Since m and M_0 both satisfy the same recurrence (and $M_0(0) = m_0^N = 1, M_0(1) = m_1^N = 0$), we deduce that $m = M_0$ and therefore we proved by induction (assuming the self-averaging works) that :

$$m_k^N = M_0(k) + o(1) \text{ as } N \to \infty$$

It remains to prove the self-averaging and boundedness properties.

Lemma 2.3. *There exists finite constants D_k and E_k, $k \in \mathbb{N}$, independent of N, so that for integer number ℓ, every integer numbers $k_1, \ldots, k_\ell$ then :*

$$\text{a)} \qquad c^N(k_1, \ldots, k_p) := \mathbb{E}\left[\prod_{i=1}^{\ell} Y_{k_i}\right] \textit{ satisfies } |c^N(k_1, \ldots, k_p)| \leq D_{\sum k_i}$$

and

$$\text{b)} \qquad m_{k_1}^N := \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^{k_1}\right] \textit{ satisfies } |m_{k_1}^N| \leq E_{k_1}\,.$$

Proof. The proof is by induction on $k = \sum k_i$. It is clearly true for $k = 0, 1$ where $E_0 = 1, E_1 = 0$ and $D_k = 0$. Suppose the induction hypothesis holds for $k-1$. To see that $b)$ holds, by the DS equation, we first observe that :

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right] &= \mathbb{E}\left[\sum_{\ell=0}^{k-2} \frac{1}{N}\mathrm{Tr}X^\ell \frac{1}{N}\mathrm{Tr}X^{k-\ell-2}\right] \\
&= \sum_{\ell=0}^{k-2} (m_\ell^N m_{k-\ell-2}^N + \frac{1}{N^2} c^N(\ell, k-\ell-2))
\end{aligned}$$

Hence, by the induction hypothesis we deduce that

$$\left|\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right]\right| \leq \sum_{\ell=0}^{k-2} (E_\ell E_{k-2-\ell} + D_{k-2}) := E_k\,.$$

To see that $a)$ holds, we use the DS equation as follows

$$\begin{aligned}
\mathbb{E}\left[Y_{k_1}\prod_{j=2}^{p}Y_{k_j}\right] &= \mathbb{E}\left[\mathrm{Tr}X_{k_1}\prod_{j=2}^{p}Y_{k_j}\right]-\mathbb{E}\left[\mathrm{Tr}X_{k_1}\right]\mathbb{E}\left[\prod_{j=2}^{p}Y_{k_j}\right] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{\ell=0}^{k-2}\mathrm{Tr}X^{\ell}\mathrm{Tr}X^{k_1-\ell-2}\prod_{j=2}^{p}Y_{k_j}\right] \\
&\quad +\mathbb{E}\left[\sum_{i=2}^{p}\frac{k_i}{N}\mathrm{Tr}X^{k_1+k_i-2}\prod_{j=2,j\neq i}^{p}Y_{k_j}\right] \\
&\quad -\mathbb{E}\left[\frac{1}{N}\sum_{\ell=0}^{k-2}\mathrm{Tr}X^{\ell}\mathrm{Tr}X^{k_1-\ell-2}\right]\mathbb{E}\left[\prod_{j=2}^{p}Y_{k_j}\right].
\end{aligned}$$

We next substract the last term to the first and observe that

$$\begin{aligned}
&\mathrm{Tr}X^{\ell}\mathrm{Tr}X^{k_1-\ell-2}-\mathbb{E}[\mathrm{Tr}X^{\ell}\mathrm{Tr}X^{k_1-\ell-2}] \\
&\qquad = NY_{\ell}m^N_{k_1-2-\ell}+NY_{k_1-2-\ell}m^N_{\ell}+Y_{\ell}Y_{k_1-2-\ell}-c^N(\ell,k_1-2-\ell)
\end{aligned}$$

to deduce

$$\begin{aligned}
\mathbb{E}\left[Y_{k_1}\prod_{j=2}^{p}Y_{k_j}\right] &= 2\sum_{\ell=0}^{k_1-2}m^N_{\ell}c^N(k_1-2-\ell,k_2,\dots,k_p) \\
&+\sum_{i=2}^{p}k_i m^N_{k_1+k_i-2}c^N(k_2,..,k_{i-1},k_{i+1},.,k_p) \\
&-\frac{1}{N}\sum_{\ell=0}^{k_1-2}[c^N(\ell,k_1-2-\ell)c^N(k_2,\dots,k_p)-c^N(\ell,k_1-2-\ell,k_2,\dots,k_p)] \\
&+\frac{1}{N}\sum_{i=2}^{p}k_i c^N(k_1+k_i-2,k_2,..,k_{i-1},k_{i+1},.,k_p) \qquad (11)
\end{aligned}$$

which is bounded uniformly by our induction hypothesis. ◇

As a consequence, we deduce

Corollary 2.4. *For all $k\in\mathbb{N}$, $\frac{1}{N}\mathrm{Tr}(X^k)$ converges almost surely towards $M_0(k)$.*

Proof. Indeed by Borel Cantelli Lemma it is enough to notice that it follows from the summability of

$$P\left(|\mathrm{Tr}(X^k-\mathbb{E}\left(\mathrm{Tr}(X^k)\right)|\geq N\varepsilon\right)\leq\frac{c^N(k,k)}{\varepsilon^2N^2}\leq\frac{D_{2k}}{\varepsilon^2N^2}.$$

◇

2.3 Central limit theorem

The above self averaging properties prove that $m_k^N = M_0(k) + o(1)$. To get the next order correction we analyze the **limiting covariance** $c^N(k,\ell)$. We will show that

Lemma 2.5. *For all $k, \ell \in \mathbb{N}$, $c^N(k,\ell)$ converges as N goes to infinity towards the unique solution $M_0(k,\ell)$ of the equation*

$$M_0(k,\ell) \quad = \quad 2\sum_{p=0}^{\ell-2} M_0(p)M_0(k-2-p,\ell) + \ell M_0(k+\ell-2)$$

so that $M_0(k,\ell) = 0$ if $k+\ell \leq 1$.

As a consequence we will show that

Corollary 2.6. $N^2(m_k^N - M_0(k)) = m_k^1 + o(1)$ *where the numbers $(m_k^1)_{k\geq 0}$ are defined recursively by :*

$$m_k^1 = 2\sum_{\ell=0}^{k-2} m_\ell^1 M_0\left(k-\ell-2\right) + \sum_{\ell=0}^{k-2} M_0(\ell, k-\ell-2)$$

Proof. (Of Lemma 2.5) Observe that $c^N(k,\ell)$ converges for $K = k+\ell \leq 1$ (as it vanishes uniformly). Assume you have proven convergence towards $M_0(k,\ell)$ up to K. Take $k_1 + k_2 = K+1$ and use (11) with $p=1$ to deduce that $c^N(k_1,k_2)$ satisfies

$$c^N(k_1,k_2) = 2\sum_{\ell=0}^{k_1-2} m_\ell^N c^N(k_1-\ell-2,k_2) + k_2 m_{k_1+k_2-2}^N + \frac{1}{N}\sum c^N(\ell, k_1-\ell-2, k_2)\,.$$

Lemma 2.3 implies that the last term is at most of order $1/N$ and hence we deduce by our induction hypothesis that $c(k_1,k_2)$ converges towards $M_0(k_1,k_2)$ which is given by the induction relation

$$M_0(k_1,k_2) \quad = \quad 2\sum_{\ell=0}^{k_1} M_0(\ell)M_0(k_1-2-\ell,k_2) + k_2 M_0(k_1+k_2-2)\,.$$

Moreover clearly $M_0(k_1,k_2) = 0$ if $k_1+k_2 \leq 1$. There is a unique solution to this equation. $\diamond$

Exercise 2.7. *Show by induction that*

$M_0(k,\ell) = \#\{$*planar maps with 1 vertex of degree ℓ and one vertex of degree k*$\}$

Proof. (of Corollary 2.6) Again we prove the result by induction over k. It is fine for $k = 0, 1$ where $c^1_k = 0$. By (11) with $p = 0$ we have :

$$\begin{aligned} N^2(m^N_k - M_0(k)) &= 2\sum M_0(\ell)N^2\left(m^N_{k-\ell-2} - M_0(k-2-\ell)\right) \\ &\quad + \sum N^2\left(m^N_\ell - M_0(\ell)\right)(m_{k-\ell-2} - M_0(k-2-\ell)) \\ &\quad + \sum c^N(\ell, k-\ell-2) \end{aligned}$$

from which the result follows by taking the large N limit on the right hand side. $\diamond$

Exercise 2.8. *Show that $c^1_k = m_1(k)$ is the number of planar maps with genus 1 build on a vertex of valence k.(The proof goes again by showing that $m_1(k)$ satisfies the same type of recurrence relations as c^1_k by considering the matching of the root : either it cuts the map of genus 1 into a map of genus 1 and a map of genus 0, or there remains a (connected) planar maps.)*

Theorem 2.9. *For any polynomial function $P = \sum \lambda_k x^k$, $Z_N(P) = \mathrm{Tr}P - \mathbb{E}[\mathrm{Tr}P]$ converges in moments towards a centered Gaussian variable $Z(P)$ with covariance given by*

$$\mathbb{E}[Z(P)\bar{Z}(P)] = \sum \lambda_k \bar{\lambda}_{k'} M_0(k, k') .$$

Proof. It is enough to prove the convergence of the moments of the Y_k's. Let

$$c^N(k_1, \ldots, k_p) = \mathbb{E}\left[Y_{k_1} \cdots Y_{k_p}\right] .$$

Then we claim that, as $N \to \infty$, $c^N(k_1, \ldots, k_p)$ converges to $G(k_1, \ldots k_p)$ given by :

$$G(k_1, \ldots, k_p) = \sum_{i=2}^{k} M_0(k_1, k_i)G(k_2, \ldots, \hat{k}_i, \ldots, k_p) \tag{12}$$

where $\hat{}$ is the absentee hat.

This type of moment convergence is equivalent to a Wick formula and is enough to prove (by the moment method) that $Y_{k_1}, \ldots, Y_{k_p}$ are jointly Gaussian. Again, we will prove this by induction by using the DS equations. Now assume that (12) holds for any $k_1, \ldots, k_p$ such that $\sum_{i=1}^p k_i \le k$. (induction hypothesis) We use (11). Notice by the a priori bound on correlators of Lemma 2.3(a) that the terms with a $1/N$ are negligible in the right hand side and m^N_k is close to $M_0(k)$, yielding

$$\mathbb{E}\left[Y_{k_1} \prod_{j=2}^p Y_{k_j}\right] = 2\sum_{\ell=0}^{k_1-2} M_0(\ell)c^N(k_1 - 2 - \ell, k_2, \ldots, k_p)$$

$$+ \sum_{i=2}^p k_i M_0(k_1 + k_i - 2)c^N(k_2, .., k_{i-1}, k_{i+1}, ., k_p) + O(\frac{1}{N})$$

By using the induction hypothesis, this gives rise to :

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{p} Y_{k_i}\right] &= 2\sum M_0(\ell)G(k_1-\ell-2,k_2,\ldots,k_p)\\ &\quad+\sum k_i M_0(k_i+k_j-2)G(k_2,\ldots,\hat{k}_i,\ldots k_p)+o(1)\end{aligned}$$

It follows that

$$G(k_1,\ldots,k_p)=2\sum M_0(\ell)G(k_1-\ell-2,k_2,\ldots,k_p)+\sum k_i M_0(k_i+k_j-2)G(k_2,\ldots,\hat{k}_i,\ldots k_p)\,.$$

But using the induction hypothesis, we get

$$G(k_1,\ldots,k_p)=\sum_{i=2}^{p}(2\sum M_0(\ell)M(k_1-\ell-2,k_i)+k_i M_0(k_i+k_j-2))G(k_2,\ldots,\hat{k}_i,\ldots k_p)$$

which yields the claim since

$$M_0(k_1,k_i)=2\sum M_0(\ell)M(k_1-\ell-2,k_i)+k_i M_0(k_1+k_i-2)\,.$$

$\diamond$

2.4 Generalization

One can generalize the previous results to smooth test functions rather than polynomials. We have

Lemma 2.10. *Let σ be the semi-circle law given by*

$$d\sigma(x)=\frac{1}{2\pi}\sqrt{4-x^2}dx\,.$$

1. *For any bounded continuous function f with polynomial growth at infinity*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)=\int f(x)d\sigma(x)\qquad a.s.$$

2. *For any C^2 function f with polynomial growth at infinity $Z(f)=\sum f(\lambda_i)-\mathbb{E}(\sum f(\lambda_i))$ converges in law towards a centered Gaussian variable.*

Our proof will only show convergence : the covariance is well known and can be found for instance in [79, (3.2.2)].

Exercise 2.11. *Show that for all $n\in\mathbb{N}$, $\int x^n d\sigma(x)=M_0(n)$.*

Proof. The convergence of $\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)$ follows since polynomials are dense in the set of continuous functions on compact sets by Weierstrass theorem. Indeed, our bounds on moments imply that we can restrict ourselves to a neighborhood of $[-2,2]$:

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i^{2p}1_{|\lambda_i|\geq M}\leq \frac{1}{M^{2k}}\frac{1}{N}\sum\lambda_i^{2k+2p}$$

has moments asymptotically bounded by $\sigma(x^{2k+2p})/M^{2k}\leq 2^{2p}(2/M)^{2k}$. This allows to approximate moments by truncated moments and then use Weierstrass theorem.

To derive the central limit theorem, one can use concentration of measure inequalities such as Poincaré inequality. Indeed, Poincaré inequalities for Gaussian variables read : for any C^1 real valued function F on $\mathbb{C}^{N(N-1)/2}\times\mathbb{R}^N$

$$\mathbb{E}\left[(F(X_{k\ell},k,l)-\mathbb{E}[F(X_{k\ell},k,l)])^2\right]\leq\frac{2}{N}\mathbb{E}\left[\sum_{i,j}\left|\partial_{X_{ij}}F(X_{k\ell},k,l)\right|^2\right].$$

Taking $F=\mathrm{Tr}f(X)$ we find that $\partial_{X_{ij}}F(X_{k\ell},k,l)=f'(X)_{ji}$. Indeed, we proved this point for polynomial functions f so that we deduce

$$\mathbb{E}\left[(\mathrm{Tr}(f(X))-\mathbb{E}[\mathrm{Tr}(f(X))])^2\right]\leq\frac{2}{N}\mathbb{E}\left[\mathrm{Tr}(f'(X)^2)\right].$$

Hence, if we take a C^1 function f, whose derivative is approximated by a polynomial P_ε on $[-M,M]$ (with $M>2$) up to an error $\varepsilon>0$, and whose derivative grows at most like x^{2K} for $|x|\geq M$, we find

$$\mathbb{E}\left[(\mathrm{Tr}f(X)-\mathbb{E}[\mathrm{Tr}f(X)]-(\mathrm{Tr}P_\varepsilon(X)-\mathbb{E}[\mathrm{Tr}P_\varepsilon(X)]))^2\right]$$
$$\leq 4\mathbb{E}\left[(\varepsilon^2+\frac{1}{N}\sum(P_\varepsilon^2(\lambda_i)+\lambda_i^{2K})1_{|\lambda_i|\geq M})\right]$$

where the right hand side goes to zero as N goes to infinity and then ε goes to zero. This shows the convergence of the covariance of $Z(f)$. We then proceed similarly to show that the approximation is good in any L^p, hence deriving the convergence in moments.

$\diamond$

2.5 GUE topological expansion

The "topological expansion" reads

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left[X^k\right]\right]=\sum_{g\geq 0}\frac{1}{N^{2g}}M_g(k)$$

where $M_g(k)$ is the number of rooted maps of genus g build over a vertex of degree k. Here, a "map" is a connected graph properly embedded in a surface and a "root" is a distinguished oriented edge. A map is assigned a genus, given by the smallest genus of a surface in which it can be properly embedded. This complete expansion (not that the above series is in fact finite) can be derived as well either by Wick calculus or by Dyson-Schwinger equations : we leave it as an exercise to the reader. We will see later that cumulants of traces of moments of the GUE are related with the enumeration of maps with several vertices.

3 Beta-ensembles

Their distribution is the probability measure on $\mathbb{R}^N$ given by

$$\mathrm{d}P_N^{\beta,V}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{\beta,V}}\Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)}\prod_{i=1}^N \mathrm{d}\lambda_i$$

where $\Delta(\lambda)=\prod_{i<j}|\lambda_i-\lambda_j|$.

Remark 3.1. *In the case $V(X)=\frac{1}{2}x^2$ and $\beta=2$, $P_N^{2,x^2/4}$ is exactly the law of the eigenvalues for a matrix taken in the GUE as we were considering in the previous chapter (the case $\beta=1$ corresponds to GOE and $\beta=4$ to GSE). This is left as a (complicated) exercise, see e.g. [3].*

β ensembles also represent strongly interacting particle systems. It turns out that both global and local statistics could be analyzed in some details. In these lectures, we will discuss global asymptotics in the spirit of the previous chapter. This section is strongly inspired from [15]. However, in that paper, only Stieltjes functions were considered, so that closed equations for correlators were only retrieved under the assumption that V is analytic. In this section, we consider more general correlators, allowing sufficiently smooth (but not analytic) potentials. We did not try to optimize the smoothness assumption.

3.1 Law of large numbers and large deviation principles

Notice that we can rewrite the density of β-ensembles as :

$$\begin{aligned}\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} &= \frac{1}{Z_N^{\beta,V}}\exp\left\{\frac{1}{2}\beta\sum_{i\neq j}\ln|\lambda_i-\lambda_j| - \beta N\sum V(\lambda_i)\right\}\\ \text{"}=\text{"}\ & \frac{1}{Z_N^{\beta,V}}\exp\left\{-\beta N^2\mathcal{E}\left(\hat{\mu}_N\right)\right\}\end{aligned}$$

where $\hat{\mu}_N$ is the empirical measure (total mass 1), and for any probability measure μ on the real line, we denote by $\mathcal{E}$ the energy

$$\mathcal{E}(\mu)=\int\int[\frac{1}{2}V(x)+\frac{1}{2}V(y)-\frac{1}{2}\ln|x-y|]\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

(the "=" is in quotes because we have thrown out the fact that $\ln|x-y|$ is not well defined for a Dirac mass on the "self-interaction" diagonal terms)

Assumption 3.2. *Assume that* $\liminf_{|x|\to\infty} \frac{V(x)}{\ln(|x|)} > 1$ *(i.e.* $V(x)$ *goes to infinity fast enough to dominate the log term at infinity) and* V *is continuous.*

Theorem 3.3. *If Assumption 3.2 holds, the empirical measure converges almost surely for the weak topology*

$$\hat{\mu}_N \Rightarrow \mu_V^{\mathrm{eq}}\,, a.s$$

where μ_V^{eq} *is the equilibrium measure for* V*, namely the minimizer of* $\mathcal{E}(\mu)$*.*

One can derive this convergence from a related large deviation principle [10] that we now state.

Theorem 3.4. *If Assumption 3.2 holds, the law of* $\hat{\mu}_N$ *under* $P_N^{\beta,V}$ *satisfies a large deviation principle with speed* N^2 *and good rate function*

$$I(\mu) = \beta\mathcal{E}(\mu) - \beta \inf_{\nu\in\mathcal{P}(\mathbb{R})} \mathcal{E}(\nu)\,.$$

In other words, I *has compact level sets and for any closed set* F *of* $\mathcal{P}(\mathbb{R})$*,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln P_N^{\beta,V}(\hat{\mu}_N \in F) \le -\inf_F I$$

whereas for any open set O *of* $\mathcal{P}(\mathbb{R})$*,*

$$\liminf_{N\to\infty} \frac{1}{N^2} \ln P_N^{\beta,V}(\hat{\mu}_N \in O) \ge -\inf_O I$$

To deduce the convergence of the empirical measure, we first prove the existence and uniqueness of the minimizers of $\mathcal{E}$.

Lemma 3.5. *Suppose Assumption 3.2 holds, then :*

- *There exists a unique minimizer* μ_V^{eq} *to* $\mathcal{E}$*. It is characterized by the fact that there exists a finite constant* C_V *such that the effective potential*

$$V_{\mathrm{eff}}(x) := V(x) - \int \ln|x-y|\, d\mu_V^{\mathrm{eq}}(y) - C_V$$

  vanishes on the support of μ_V^{eq} *and is non negative everywhere.*

- *For any probability measure* μ*, we have the decomposition*

$$\mathcal{E}(\mu) = \mathcal{E}(\mu_V^{\mathrm{eq}}) + \int_0^\infty \frac{ds}{s} \left| \int e^{isx} d(\mu - \mu_V^{\mathrm{eq}})(x) \right|^2 + \int V_{\mathrm{eff}}(x) d\mu(x)\,. \quad (13)$$

Proof. We notice that with $f(x,y) = \frac{1}{2}V(x) + \frac{1}{2}V(y) - \frac{1}{2}\ln|x-y|$,

$$\mathcal{E}(\mu) = \int f(x,y) d\mu(x) d\mu(y) = \sup_{M\ge 0} \int f(x,y) \wedge M d\mu(x) d\mu(y)$$

by monotone convergence theorem. Observe also that the growth assumption we made on V insures that there exists $\gamma > 0$ and $C > -\infty$ such that

$$f(x,y) \geq \gamma(\ln(|x|+1) + \ln(|y|+1)) + C\,, \tag{14}$$

so that $f \wedge M$ is a bounded continuous function. Hence, $\mathcal{E}$ is the supremum of the bounded continuous functions $\mathcal{E}_M(\mu) := \int\int f(x,y) \wedge M d\mu(x)d\mu(y)$, defined on the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$, equipped with the weak topology. Hence $\mathcal{E}$ is lower semi-continuous. Moreover, the lower bound (14) on f yields

$$L_M := \{\mu \in \mathcal{P}(\mathbb{R}) : \mathcal{E}(\mu) \leq M\} \subset \left\{ \int \ln(|x|+1) d\mu(x) \leq \frac{M-C}{2\gamma} \right\} =: K_M \tag{15}$$

where K_M is compact. Hence, since L_M is closed by lower semi-continuity of $\mathcal{E}$ we conclude that L_M is compact for any real number M. This implies that $\mathcal{E}$ achieves its minimal value. Let μ_V^{eq} be a minimizer. Writing that $\mathcal{E}(\mu_V^{\mathrm{eq}} + \epsilon\nu) \geq \mathcal{E}(\mu_V^{\mathrm{eq}})$ for any measure ν with zero mass so that $\mu_V^{\mathrm{eq}} + \epsilon\nu$ is positive for ϵ small enough gives the announced characterization in terms of the effective potential V_{eff}.

For the second point, take μ with $\mathcal{E}(\mu) < \infty$ and write

$$V = V_{\mathrm{eff}} + \int \ln|.-y| \,\mathrm{d}\mu_V^{\mathrm{eq}}(y) + C_V$$

so that

$$\mathcal{E}(\mu) = \mathcal{E}(\mu_V^{\mathrm{eq}}) - \frac{1}{2}\int\int \ln|x-y|\, d(\mu - \mu_V^{\mathrm{eq}})(x) d(\mu - \mu_V^{\mathrm{eq}})(y) + \int V_{\mathrm{eff}}(x) d\mu(x)\,.$$

On the other hand, we have the following equality for all $x, y \in \mathbb{R}$

$$\ln|x-y| = \int_0^\infty \frac{1}{2t}\left(e^{-\frac{1}{2t}} - e^{-\frac{|x-y|^2}{2t}} \right) dt\,.$$

One can then argue [9] that for all probability measure μ with $\mathcal{E}(\mu) < \infty$(in particular with no atoms), we can apply Fubini's theorem and the fact that $\mu - \mu_V^{\mathrm{eq}}$ is massless, to show that

$$\begin{aligned}
\Sigma(\mu) &:= \int\int \ln|x-y|\, d(\mu - \mu_V^{\mathrm{eq}})(x) d(\mu - \mu_V^{\mathrm{eq}})(y) \\
&= -\int_0^\infty \frac{1}{2t} \int\int e^{-\frac{|x-y|^2}{2t}} d(\mu - \mu_V^{\mathrm{eq}})(x) d(\mu - \mu_V^{\mathrm{eq}})(y) dt \\
&= -\int_0^\infty \frac{1}{2\sqrt{2\pi t}} \int e^{-\frac{1}{2}t\lambda^2} \left| \int e^{i\lambda x} d(\mu - \mu_V^{\mathrm{eq}})(x) \right|^2 d\lambda dt \\
&= -\int_0^\infty \left| \int e^{iyx} d(\mu - \mu_V^{\mathrm{eq}})(x) \right|^2 \frac{dy}{y}
\end{aligned}$$

This term is concave non-positive in the measure μ as it is quadratic in μ, and in fact non degenerate as it vanishes only when all Fourier transforms of μ equal those of $\mu_V^{\rm eq}$, implying that $\mu = \mu_V^{\rm eq}$. Therefore $\mathcal{E}$ is as well strictly convex as it differs from this function only by a linear term. Its minimizer is thus unique. $\diamond$

Remark 3.6. *Note that the characterization of $\mu_V^{\rm eq}$ implies that it is compactly supported as $V_{\rm eff}$ goes to infinity at infinity.*

Remark 3.7. *It can be shown that the equilibrium measure has a bounded density with respect to Lebesgue measure if V is C^2. Indeed, if f is C^1 from $\mathbb{R} \to \mathbb{R}$ and ε small enough so that $\varphi_\varepsilon(x) = x + \varepsilon f(x)$ is a bijection, we know that*

$$I(\varphi_\varepsilon \# \mu_V^{\rm eq}) \geq I(\mu_V^{\rm eq}) \,,$$

where we denoted by $\varphi \# \mu$ the pushforward of μ by φ given, for any test function g, by :

$$\int g(y) d\varphi \# \mu(y) = \int g(\varphi(x)) d\mu(x) \,.$$

As a consequence, we deduce by arguing that the term linear in ε must vanish that

$$\frac{1}{2} \int \int \frac{f(x) - f(y)}{x - y} d\mu_V^{\rm eq}(x) d\mu_V^{\rm eq}(y) = \int V'(x) f(x) d\mu_V^{\rm eq}(x) \,.$$

By linearity, we may now take f to be complex valued and given by $f(x) = (z - x)^{-1}$. We deduce that the Stieltjes transform $S_{\rm eq}(z) = \int (z-x)^{-1} d\mu_V^{\rm eq}(x)$ satisfies

$$\frac{1}{2} S_{\rm eq}(z)^2 = \int \frac{V'(x)}{z - x} d\mu_V^{\rm eq}(x) = S_{\rm eq}(z) V'(\Re(z)) + f(z)$$

with

$$f(z) = \int \frac{V'(x) - V'(\Re(z))}{z - x} d\mu_V^{\rm eq}(x) \,.$$

f is bounded on compacts if V is C^2. Moreover, we deduce that

$$S(z) = V'(\Re(z)) - \sqrt{V'(\Re(z))^2 + 2f(z)} \,.$$

But we can now let z going to the real axis and we deduce from Theorem **??** *that $\mu_V^{\rm eq}$ has bounded density $\sqrt{V'(x)^2 - 4f(x)}$.*

Note also that it follows, since $V'(x)^2 - 4f(x)$ is smooth that when the density of $\mu_V^{\rm eq}$ vanishes at a it vanishes like $|x - a|^{q/2}$ for some integer number $q \geq 1$.

Because the proof of the large deviation principle will be roughly the same in the discrete case, we detail it here.

Proof of Theorem 3.4 We first consider the non-normalized measure

$$\frac{\mathrm{d}Q_N^{\beta,V}}{\mathrm{d}\lambda} \quad = \quad \exp\left\{ \frac{1}{2}\beta \sum_{i \neq j} \ln|\lambda_i - \lambda_j| - \beta N \sum V(\lambda_i) \right\}$$

and prove that it satisfies a weak large deviation principle, that is that for any probability measure μ,

$$\begin{aligned} -\beta\mathcal{E}(\mu) &= \limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln Q_N^{\beta,V}(d(\hat\mu_N,\mu)<\delta) \\ &= \liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\ln Q_N^{\beta,V}(d(\hat\mu_N,\mu)<\delta) \end{aligned}$$

where d is a distance compatible with the weak topology, such as the Vasershtein distance.

To prove the upper bound observe that for any $M>0$

$$\begin{aligned} Q_N^{\beta,V}(d(\hat\mu_N,\mu)<\delta) &\le \int_{d(\hat\mu^N,\mu)<\delta} e^{-\beta N^2\int_{x\neq y} f(x,y)\wedge M d\hat\mu^N(x)d\hat\mu^N(y)}\prod e^{-\beta V(\lambda_i)}d\lambda_i \\ &= e^{\beta NM}\int_{d(\hat\mu^N,\mu)<\delta} e^{-\beta N^2\int f(x,y)\wedge M d\hat\mu^N(x)d\hat\mu^N(y)}\prod e^{-\beta V(\lambda_i)}d\lambda_i \end{aligned}$$

where in the first line we used that the λ_i are almost surely distinct. Now, using that for any finite M, $\mathcal{E}_M$ is continuous, we get

$$Q_N^{\beta,V}(d(\hat\mu_N,\mu)<\delta) \le e^{\beta NM}e^{-\beta N^2\mathcal{E}_M(\mu)+N^2o(\delta)}(\int e^{-\beta V(\lambda)}d\lambda)^N$$

Taking first the limit N going to infinity, then δ going to zero and finally M going to infinity yields

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln Q_N^{\beta,V}(d(\hat\mu_N,\mu)<\delta)\le -\beta\mathcal{E}(\mu)\,.$$

To get the lower bound, we may choose μ with no atoms as otherwise $\mathcal{E}(\mu)=+\infty$. We can also assume μ compactly supported, as we can approximate it by $\mu_M(dx)=1_{|x|\le M}d\mu/\mu([-M,M])$ and it is not hard to see that $\mathcal{E}(\mu_M)$ goes to $\mathcal{E}(\mu)$ as M goes to infinity. Let x_i be the i^{th} classical location of the particles given by $\mu((-\infty,x_i])=i/N$. $x_i<x_{i+1}$ and we have for N large enough and $p>0$, if $u_i=\lambda_i-x_i$,

$$\Omega=\cap_i\{|u_i|\le N^{-p}, u_i\le u_{i+1}\}\subset\{d(\hat\mu_N,\mu)<\delta\}$$

so that we get the lower bound

$$Q_N^{\beta,V}(d(\hat\mu_N,\mu)<\delta) \ge \int_\Omega \prod_{i>j}|x_i-x_j+u_i-u_j|^\beta\prod_{i=1}^N\exp\left(-N\beta V\left(x_i+u_i\right)\right)du_i$$

Observe that by our ordering of x and u, we have $|x_i-x_j+u_i-u_j|\ge\max\{|x_i-x_j|,|u_i-u_j|\}$ and therefore

$$\prod_{i>j}|x_i-x_j+u_i-u_j|^\beta\ge\prod_{i>j+1}|x_i-x_j|^\beta\prod_i|x_{i+1}-x_i|^{\beta/2}\prod_i|u_{i+1}-u_i|^{\beta/2}$$

where for $i > j+1$

$$\ln|x_i - x_j| \geq \int_{x_{i-1}}^{x_i} \int_{x_j}^{x_{j+1}} \ln|x-y| d\mu(x) d\mu(y)$$

whereas

$$\ln|x_i - x_{i-1}| \geq 2 \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} 1_{x>y} \ln|x-y| d\mu_V^{\mathrm{eq}}(x) d\mu_V^{\mathrm{eq}}(y) .$$

We deduce that

$$\sum_{i>j+1} \ln|x_i - x_j| + \frac{1}{2} \sum_i \ln|x_{i+1} - x_i| \geq \frac{N^2}{2} \int \int \ln|x-y| d\mu_V^{\mathrm{eq}}(x) d\mu_V^{\mathrm{eq}}(y) .$$

Moreover, V is continuous and μ compactly supported, so that

$$\frac{1}{N} \sum_{i=1}^N V(x_i + u_i) = \frac{1}{N} \sum_{i=1}^N V(x_i) + o(1) . \tag{16}$$

Hence, we conclude that

$$\begin{aligned} Q_N^{\beta,V}(d(\hat{\mu}_N, \mu) < \delta) \quad &\geq \quad \exp\{-\beta N^2 \mathcal{E}(\mu)\} \int_\Omega \prod_i |u_{i+1} - u_i|^{\beta/2} \prod du_i \\ &\geq \quad \exp\{-\beta N^2 \mathcal{E}(\mu) + o(N^2)\} \end{aligned} \tag{17}$$

which gives the lower bound. To conclude, it is enough to prove exponential tightness. But with K_M as in (15) we have by (14)

$$Q_N^{\beta,V}(K_M^c) \leq \int_{K_M^c} e^{-2\gamma N(N-1) \int \ln(|x|+1) d\hat{\mu}^N(x) - CN^2} \prod d\lambda_i \leq e^{N^2(C' - 2\gamma M)}$$

with some finite constant C' independent of M. Hence, exponential tightness follows :

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \ln Q_N^{\beta,V}(K_M^c) = -\infty$$

from which we deduce a full large deviation principle for $Q_N^{\beta,V}$ and taking $F = O$ be the whole set of probability measures, we get in particular that

$$\lim_{N\to\infty} \frac{1}{N^2} \ln Z_N^{\beta,V} = -\beta \inf \mathcal{E} .$$

$\diamond$

We also have large deviations from the support : the probability that some eigenvalue is away from the support of the equilibrium measure decays exponentially fast if V_{eff} is positive there. This was proven for the quadratic potential in [8], then in [3] but with the implicit assumption that there is convergence of the support of the eigenvalues towards the support of the limiting equilibrium measure. In [12, 19], it was proved that large deviations estimate for the support hold in great generality. Hence, if the effective potential is positive outside of the support S of the equilibrium measure, there is no eigenvalue at positive distance of the support with exponentially large probability. It was shown in [53] that if the effective potential is not strictly positive outside of the support of the limiting measure, eigenvalues may deviate towards the points where it vanishes. For completeness, we summarize the proof of this large deviation principle below.

Theorem 3.8. *Let S be the support of μ_V^{eq}. Assume Assumption 3.2 and that V is C^2. Then, for any closed set F in S^c*

$$\limsup_{N\to\infty} \frac{1}{N} \ln P_N^{\beta,V}(\exists i \in \{1,N\} : \lambda_i \in F) \le -\beta \inf_F V_{\mathrm{eff}} ,$$

whereas for any open set $O \subset S^c$

$$\liminf_{N\to\infty} \frac{1}{N} \ln P_N^{\beta,V}(\exists i \in \{1,N\} : \lambda_i \in O) \ge -\beta \inf_O V_{\mathrm{eff}} .$$

Proof. Observe first that V_{eff} is continuous as V is and $x \to \int \ln|x-y| d\mu_V^{\mathrm{eq}}(y)$ is continuous by Remark 3.7. Hence, as V_{eff} goes to infinity at infinity, it is a good rate function.

We shall use the representation

$$\frac{\Upsilon_N^{\beta,V}(\mathsf{F})}{\Upsilon_N^{\beta,V}(\mathbb{R})} \le P_N^{\beta,V}\left[\exists i \quad \lambda_i \in \mathsf{F}\right] \le N\, \frac{\Upsilon_N^{\beta,V}(\mathsf{F})}{\Upsilon_N^{\beta,V}(\mathbb{R})} \tag{18}$$

where, for any measurable set X :

$$\Upsilon_N^{\beta,V}(\mathsf{X}) = P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{\mathsf{X}} d\xi\, e^{\left\{-N\beta\, V(\xi)+(N-1)\beta \int d\hat{\mu}_{N-1}(\lambda)\ln|\xi-\lambda|\right\}}\right] . \tag{19}$$

We shall hereafter estimate $\frac{1}{N}\ln \Upsilon_N^{\beta,V}(\mathsf{X})$. We first prove a lower bound for $\Upsilon_N^{\beta,V}(\mathsf{X})$ with X open. For any $x \in \mathsf{X}$ we can find $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset X$. Let $\delta_\varepsilon(V) = \sup\{|V(x)-V(y)|, |x-y| \le \varepsilon\}$. Using twice Jensen inequality, we lower bound $\Upsilon_N^{\beta,V}(\mathsf{X})$ by

$$\begin{aligned}
&\ge\ P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{x-\varepsilon}^{x+\varepsilon} d\xi e^{\left\{-N\beta\, V(\xi)+(N-1)\beta \int d\hat{\mu}_{N-1}(\eta)\ln|\xi-\eta|\right\}}\right] \\
&\ge\ e^{-N\beta\left(V(x)+\delta_\varepsilon(V)\right)}\, P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{x-\varepsilon}^{x+\varepsilon} d\xi e^{\left\{(N-1)\beta\ \int d\hat{\mu}_{N-1}(\lambda)\ln|\xi-\lambda|\right\}}\right] \\
&\ge\ 2\varepsilon\ e^{-N\beta\left(V(x)+\delta_\varepsilon(V)\right)}\ e^{\left\{(N-1)\beta\, P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int d\hat{\mu}_{N-1}(\lambda)\, H_{x,\varepsilon}(\lambda)\right]\right\}} \\
&\ge\ 2\varepsilon\, e^{-N\beta\left(V(x)+\delta_\varepsilon(V)\right)}\ e^{\left\{(N-1)\beta\, P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int d\hat{\mu}_{N-1}(\lambda)\, \phi_{x,K}(\lambda) H_{x,\varepsilon}(\lambda)\right]\right\}}
\end{aligned} \tag{20}$$

where we have set :

$$H_{x,\varepsilon}(\lambda) = \int_{x-\varepsilon}^{x+\varepsilon} \frac{d\xi}{2\varepsilon}\, \ln|\xi - \lambda| \tag{21}$$

and $\phi_{x,K}$ is a continuous function which vanishes outside of a large compact K including the support of μ_V^{eq}, is equal to one on a ball around x with radius $1+\varepsilon$ (note that H is non-negative outside $[x-(1+\varepsilon), x+1+\varepsilon]$ resulting with the lower bound (20)) and on the support of μ_V^{eq}, and takes values in $[0,1]$. For any fixed $\varepsilon > 0$, $\phi_{x,K} H_{x,\varepsilon}$ is bounded continuous, so we have by Theorem 3.4 (note that it applies as well when the potential depends on N as soon as it converges uniformly on compacts) that :

$$\Upsilon_N^{\beta,V}(\mathsf{X}) \geq 2\varepsilon\, e^{-\frac{N\beta}{2}\left(V(x)+\delta_\varepsilon(V)\right)} e^{\left\{(N-1)\beta \int d\mu_V^{\mathrm{eq}}(\lambda)\, \phi_{x,K}(\lambda)\, H_{x,\varepsilon}(\lambda) + NR(\varepsilon,N)\right\}} \tag{22}$$

with $\lim_{N\to\infty} R(\varepsilon, N) = 0$ for all $\varepsilon > 0$. Letting $N \to \infty$, we deduce since $\int d\mu_V^{\mathrm{eq}}(\lambda)\, \phi_{x,K}(\lambda)\, H_{x,\varepsilon}(\lambda) = \int d\mu_V^{\mathrm{eq}}(\lambda)\, H_{x,\varepsilon}(\lambda)$ that :

$$\liminf_{N\to\infty} \frac{1}{N}\, \ln \Upsilon_N^{\beta,V}(\mathsf{X}) \geq -\beta\, \delta_\varepsilon(V) - \beta\Big(V(x) - \int d\mu_V^{\mathrm{eq}}(\lambda)\, H_{x,\varepsilon}(\lambda)\Big) \tag{23}$$

Exchanging the integration over ξ and x, observing that $\xi \to \int d\mu_V^{\mathrm{eq}}(\lambda)\, \ln|\xi - \lambda|$ is continuous by Remark 3.7 and then letting $\varepsilon \to 0$, we conclude that for all $x \in \mathsf{X}$,

$$\liminf_{N\to\infty} \frac{1}{N}\, \ln \Upsilon_N^{\beta,V}(\mathsf{X}) \geq -\beta\, V_{\mathrm{eff}}(x)\,. \tag{24}$$

We finally optimize over $x \in \mathsf{X}$ to get the desired lower bound. To prove the upper bound, we note that for any $M > 0$,

$$\Upsilon_N^{\beta,V}(\mathsf{X}) \leq P_{N-1}^{\beta,\frac{NV}{N-1}} \left[\int_{\mathsf{X}} d\xi\, e^{\left\{-N\beta\, V(\xi) + (N-1)\beta \int d\hat{\mu}_{N-1}(\lambda) \ln \max\left(|\xi-\lambda|, M^{-1}\right)\right\}}\right].$$

Observe that there exists C_0 and $c > 0$ and d finite such that for $|\xi|$ larger than C_0 :

$$W_\mu(\xi) = V(\xi) - \int d\mu(\lambda) \ln \max\big(|\xi - \lambda|, M^{-1}\big) \geq c \ln|\xi| + d$$

by the confinement Hypothesis (3.2), and this for all probability measures μ on $\mathbb{R}$. As a consequence, if $\mathsf{X} \subset [-C, C]^c$ for some C large enough, we deduce that :

$$\Upsilon_N^{\beta,V}(\mathsf{X}) \leq \int_{\mathsf{X}} d\xi e^{-(N-1)\frac{\beta}{2}(c \ln|\xi| + d)} \leq e^{-N\frac{\beta}{4} c \ln C} \tag{25}$$

where the last bound holds for N large enough. Combining (24), (25) and (18) shows that

$$\limsup_{C\to\infty} \limsup_{N\to\infty} \frac{1}{N} \ln P_N^{\beta,V}\big[\exists i \quad |\lambda_i| \geq C\big] = -\infty\,.$$

Hence, we may restrict ourselves to X bounded. Moreover, the same bound extends to $P_{N-1}^{\beta,\frac{NV}{N-1}}$ so that we can restrict the expectation over $\hat{\mu}_{N-1}$ to probability measures supported on $[-C,C]$ up to an arbitrary small error $e^{-Ne(C)}$, provided C is large enough and where $e(C)$ goes to infinity with C. Recall also that $V(\xi)-2\int d\hat{\mu}_{N-1}(\lambda)\ln\max\bigl(|\xi-\lambda|,M^{-1}\bigr)$ is uniformly bounded from below by a constant D. As $\lambda\to\ln\max\bigl(|\xi-\lambda|,M^{-1}\bigr)$ is bounded continuous on compacts, we can use the large deviation principles of Theorem 3.4 to deduce that for any $\varepsilon>0$, any $C\geq C_0$,

$$\begin{aligned}\Upsilon_N^{\beta,V}(\mathsf{X}) \quad \leq \quad & e^{N^2\tilde{R}(\varepsilon,N,C)}+e^{-N(e(C)-\frac{\beta}{2}D)} \\ & +\int_{\mathsf{X}} d\xi\, e^{\left\{-N\beta V(\xi)+(N-1)\beta\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\left(|\xi-\lambda|,M^{-1}\right)+NM\varepsilon\right\}}\end{aligned} \tag{26}$$

with $\limsup_{N\to\infty}\tilde{R}(\varepsilon,N,C)$ equals to

$$\limsup_{N\to\infty}\frac{1}{N^2}\ln P_{N-1}^{\beta,\frac{NV}{N-1}}\left(\{\hat{\mu}_{N-1}([-C,C])=1\}\cap\{d(\hat{\mu}_{N-1},\mu_V^{\mathrm{eq}})>\varepsilon\}\right) \quad < \quad 0.$$

Moreover, $\xi\to V(\xi)-\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\bigl(|\xi-\lambda|,M^{-1}\bigr)$ is bounded continuous so that a standard Laplace method yields,

$$\limsup_{N\to\infty}\frac{1}{N}\ \ln\Upsilon_N^{\beta,V}(\mathsf{X})$$

$$\leq \max\left\{-\inf_{\xi\in\mathsf{X}}\left[\beta\Bigl(V(\xi)-\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\left(|\xi-\lambda|,M^{-1}\right)\Bigr)\right],-(e(C)-\frac{\beta}{2}D)\right\}.$$

We finally choose C large enough so that the first term is larger than the second, and conclude by monotone convergence theorem that $\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\bigl(|\xi-\lambda|,M^{-1}\bigr)$ decreases as M goes to infinity towards $\int d\mu_V^{\mathrm{eq}}(\lambda)\ln|\xi-\lambda|$. This completes the proof of the large deviation.

$\diamond$

Hereafter we shall assume that

Assumption 3.9. *V_{eff} is positive outside S.*

Remark 3.10. *As a consequence of Theorem 3.8, we see that up to exponentially small probabilities, we can modify the potential at a distance ϵ of the support. Later on, we will assume we did so in order that V'_{eff} does not vanish outside S.*

In these notes we will also use that particles stay smaller than M for some M large enough with exponentially large probability.

Theorem 3.11. *Assume Assumption 3.2 holds. Then, there exists M finite so that*

$$\limsup_{N\to\infty}\frac{1}{N}\ln P_N^{\beta,V}(\exists i\in\{1,\dots,N\}:|\lambda_i|\geq M)<0\,.$$

Here, we do not need to assume that the effective potential is positive everywhere, we only use it is large at infinity. The above shows that latter on, we can always change test functions outside of a large compact $[-M, M]$ and hence that L^2 norms are comparable to L^∞ norms.

3.2 Concentration of measure

We next define a distance on the set of probability measures on $\mathbb{R}$ which is well suited for our problem.

Definition 3.12. *For μ, μ' probability measures on $\mathbb{R}$, we set*

$$D(\mu, \mu') = \left(\int_0^\infty \left| \int e^{iyx} d(\mu - \mu')(x) \right|^2 \frac{dy}{y} \right)^{\frac{1}{2}} .$$

It is easy to check that D defines a distance on $\mathcal{P}(\mathbb{R})$ (taking eventually the value $+\infty$, for instance on measure with Dirac masses). Moreover, we have the following property

Property 3.13. *Let $f \in L^1(dx)$ such that $\hat{f}$ belongs to $L^1(dt)$, and set $\|f\|_{1/2} = \left(\int t|\hat{f}_t|^2 dt\right)^{1/2}$.*

- *Assume also f continuous. Then for any probability measures μ, μ'*

$$\left| \int f(x) d(\mu - \mu')(x) \right| \leq 2\|f\|_{1/2} D(\mu, \mu') .$$

- *Assume moreover $f, f' \in L^2$. Then*

$$\|f\|_{1/2} \leq 2(\|f\|_{L^2} + \|f'\|_{L^2}) . \tag{27}$$

Proof. For the first point we just use inverse Fourier transform and Fubini to write that

$$\begin{aligned} |\int f(x) d(\mu - \mu')(x)| &= |\int \hat{f}_t \widehat{\mu - \mu'}_t dt| \\ &\leq 2 \int_0^\infty t^{1/2} |\hat{f}_t| t^{-1/2} |\widehat{\mu - \mu'}_t| dt \leq 2D(\mu, \mu') \|f\|_{1/2} \end{aligned}$$

where we finally used Cauchy-Schwarz inequality. For the second point, we observe that

$$\|f\|_{1/2}^2 = \int_0^\infty t|\hat{f}_t|^2 dt \leq \frac{1}{2}(\int |\hat{f}_t|^2 dt + \int |t\hat{f}_t|^2 dt) = \frac{\pi}{2}(\|f\|_{L^2}^2 + \|f'\|_{L^2})$$

from which the result follows. ◇

We are going to show that $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ satisfies concentration inequalities for the D-distance. However, the distance between $\hat{\mu}_N$ and μ_V^{eq} is infinite as $\hat{\mu}_N$ has atoms. Hence, we are going to regularize $\hat{\mu}_N$ so that it has finite energy, following an idea of Maurel-Segala and Maida [75]. First define $\tilde{\lambda}$ by $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_i = \tilde{\lambda}_{i-1} + \max\{\sigma_N, \lambda_i - \lambda_{i-1}\}$ where σ_N will be chosen to be like N^{-p}. Remark that $\tilde{\lambda}_i - \tilde{\lambda}_{i-1} \geq \sigma_N$ whereas $|\lambda_i - \tilde{\lambda}_i| \leq N\sigma_N$. Define $\tilde{\mu}_N = \mathbb{E}_U\left[\frac{1}{N}\sum \delta_{\tilde{\lambda}_i + U_i}\right]$ where U_i are independent and equi-distributed random variables uniformly distributed on $[0, N^{-q}]$ (i.e. we smooth the measure by putting little rectangles instead of Dirac masses and make sure that the eigenvalues are at least distance N^{-p} apart). For further use, observe that we have uniformly $|\tilde{\lambda}_i + U_i - \lambda_i| \leq N^{1-p} + N^{-q}$. In the sequel we will take $q = p+1$ so that the first error term dominates. Then we claim that

Lemma 3.14. *Assume V is C^1. For $3 < p+1 \leq q$ there exists $C_{p,q}$ finite and $c > 0$ such that*

$$P_N^{\beta,V}\left(D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}}) \geq t\right) \leq e^{C_{p,q} N \ln N - \beta N^2 t^2} + e^{-cN}$$

Remark 3.15. *Using that the logarithm is a Coulomb interaction, Serfaty et al could improve the above bounds to get the exact exponent in the term in $N \ln N$, as well as the term in N. This allows to prove central limit theorems under weaker conditions. Our approach seems however more robust and extends to more general interactions [19].*

Corollary 3.16. *Assume V is C^1. For all $q > 2$ there exists C finite and $c, c_0 > 0$ such that*

$$\mathbf{P}\left(\sup_{\varphi} \frac{1}{N^{-q+2}\|\varphi\|_L + c_0 N^{-1/2}\sqrt{\ln N}\|\varphi\|_{\frac{1}{2}}} \left|\int \varphi d\left(\hat{\mu}_N - \mu_V^{\mathrm{eq}}\right)\right| \geq 1\right) \leq e^{-cN}$$

Moreover

$$\left|\int \frac{\varphi(x) - \varphi(y)}{x - y} d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(x) d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(y)\right| \leq C\frac{1}{N}\ln N \|\varphi\|_{C^2} \qquad (28)$$

with probability greater than $1 - e^{-cN}$. Here $\|\varphi\|_L = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|}$ and $\|\varphi\|_{C^k} = \sum_{\ell \leq k} \|\varphi^{(\ell)}\|_\infty$. Note that we can modify φ outside a large set $[-M.M]$ up to modify the constant c.

Proof. We take $q = p+1$. The triangle inequality yields :

$$\begin{aligned}
|\int \varphi \mathrm{d}\left(\hat{\mu}_N - \mu_V^{\mathrm{eq}}\right)| &= \left|\int \varphi \mathrm{d}\left(\hat{\mu}_N - \tilde{\mu}_N\right) + \int \varphi \mathrm{d}\left(\tilde{\mu}_N - \mu_V^{\mathrm{eq}}\right)\right| \\
&\leq \left|\frac{1}{N}\sum_{i=1}^N \mathbb{E}_U[\varphi(\lambda_i) - \varphi(\tilde{\lambda}_i + U)]\right| + |\int \hat{\varphi}(\lambda)(\widehat{\tilde{\mu}_N - \mu_V^{\mathrm{eq}}})(\lambda) d\lambda| \\
&\leq \|\varphi\|_L N^{-q+2} + 2\|\varphi\|_{\frac{1}{2}} D\left(\tilde{\mu}_N, \mu_V^{\mathrm{eq}}\right)
\end{aligned}$$

where we noticed that $|\lambda_i - \tilde{\lambda}_i|$ is bounded by N^{-p+1} and U by N^{-q} and used Cauchy-Schwartz inequality. We finally use (27) to see that on $\{|\lambda_i| \le M\}$ we have by the previous lemma that for all φ

$$\left| \int \varphi \mathrm{d} \left(\hat{\mu}_N - \mu_V^{\mathrm{eq}} \right) \right| \le N^{-p+1} \|\varphi\|_L + t\|\varphi\|_{\frac{1}{2}}$$

with probability greater than $1 - e^{C_{p,q} N \ln N - \frac{\beta}{2} N^2 t^2}$. We next choose $t = c_0 \sqrt{\ln N / N}$ with $c_0^2 = 4|C_{p,q}|/\beta$ so that this probability is greater than $1 - e^{-c_0^2/2N \ln N}$. Theorem 3.11 completes the proof of the first point since it shows that the probability that one eigenvalue is greater than M decays exponentially fast.

We next consider

$$L_N(\phi) := \int \frac{\phi(x) - \phi(y)}{x - y} d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(x) d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(y)$$

on $\{\max |\lambda_i| \le M\}$. Hence we can replace ϕ by $\phi \chi_M$ where χ_M is a smooth function, equal to one on $[-M, M]$ and vanishing outside $[-M-1, M+1]$. Hence assume that ϕ is compactly supported. If we denote by $\tilde{L}_N(\phi)$ the quantity defined as $L_N(\phi)$ but with $\tilde{\mu}_N$ instead of $\hat{\mu}^N$ we have that

$$\left| \tilde{L}_N(\phi) - L_N(\phi) \right| \le 2\|\phi^{(2)}\|_\infty N^{-q+2} .$$

We can now replace ϕ by its Fourier representation to find that

$$\tilde{L}_N(\phi) = \int dt i t \hat{\phi}(t) \int_0^1 d\alpha \int e^{i\alpha t x} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x) \int e^{i(1-\alpha)tx} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x) .$$

We can then use Cauchy-Schwartz inequality to deduce that

$$\begin{aligned} |\tilde{L}_N(\phi)| &\le \int dt |t\hat{\phi}(t)| \int_0^1 d\alpha | \int e^{i\alpha t x} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)|^2 \\ &= \int dt |t\hat{\phi}(t)| \int_0^1 \frac{t d\alpha}{t\alpha} | \int e^{i\alpha t x} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)|^2 \\ &\le \int dt |t\hat{\phi}(t)| D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}})^2 \\ &\le C D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}})^2 \|\phi\|_{C^2} \end{aligned} \tag{29}$$

where we noticed that

$$\begin{aligned} \int dt |t\hat{\phi}(t)| &\le \left(\int dt |t\hat{\phi}(t)|^2 (1 + t^2) \right)^{1/2} \left(\int dt (1 + t^2)^{-1} \right)^{1/2} \\ &\le C(\|\phi^{(2)}\|_{L^2} + \|\phi'\|_{L^2}) \le C\|\phi\|_{C^2} \end{aligned}$$

as we compactified ϕ. The conclusion follows from Theorem 3.11. ⋄

We next prove Lemma 3.14. We first show that :

$$Z_N^{\beta,V} \geq \exp\left(-N^2\beta\mathcal{E}(\mu_V^{\mathrm{eq}}) + CN\ln N\right)$$

The proof is exactly as in the proof of the large deviation lower bound of Theorem 3.4 except we take $\mu = \mu_V^{\mathrm{eq}}$ and V is C^1, so that

$$\frac{1}{N}\sum_{i=1}^N V(x_i+u_i) = \frac{1}{N}\sum_{i=1}^N V(x_i) + O(\frac{1}{N})\,.$$

This allows to improve the lower bound (17) into

$$\begin{aligned} Z_N^{\beta,V} &\geq Q_N^{\beta,V}(d(\hat\mu_N,\mu_V^{\mathrm{eq}})<\delta) \\ &\geq \exp\{-\beta N^2\mathcal{E}(\mu_V^{\mathrm{eq}}) + CN\ln N\} \end{aligned} \tag{30}$$

Now consider the unnormalized density of $Q_N^{\beta,V} = Z_N^{\beta,V}P_N^{\beta,V}$ on the set where $|\lambda_i| \leq M$ for all i

$$\begin{aligned} \frac{dQ_N^{\beta,V}(\lambda)}{d\lambda} &= \prod_{i<j}|\lambda_i-\lambda_j|^\beta \exp\left(-N\beta\sum V(\lambda_i)\right) \\ &\leq \prod_{i<j}\left|\tilde\lambda_i-\tilde\lambda_j\right|^\beta \exp\left(-N\beta\sum V(\tilde\lambda_i)\right) \end{aligned}$$

because the $\tilde\lambda$ only increased the differences. Observe that for $|\lambda_i| \leq M$,

$$|V(\lambda_i) - V(\tilde\lambda_i+U_i)| \leq \sup_{|x|\leq M+1}|V'(x)|(N^{1-p}+N^{-q})\,.$$

Moreover for each $j > i$

$$\ln\left|\tilde\lambda_i-\tilde\lambda_j\right| = \mathbb{E}\ln\left|\tilde\lambda_i+u_i-\tilde\lambda_j-u_j\right| + O(N^{-q+p})\,.$$

Hence, we deduce that on $|\lambda_i| \leq M$ for all i, there exists a finite constant C such that

$$\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} \leq \exp\left(-N^2\beta\left(\mathcal{E}(\tilde\mu_N) - \mathcal{E}(\mu_V^{\mathrm{eq}})\right) + CN\ln N + CN^{2-q+p} + CN^{3-p}\right)$$

As we chose $q = p+1$, $p > 2$, the error is at most of order $N\ln N$. We now use the fact that

$$\mathcal{E}(\tilde\mu_N) - \mathcal{E}(\mu_V^{\mathrm{eq}}) = D(\tilde\mu_N,\mu_V^{\mathrm{eq}})^2 + \int (V_{\mathrm{eff}})(x)d(\tilde\mu_N-\mu_V^{\mathrm{eq}})(x)$$

where the last term is non-negative, and Theorem 3.11, to conclude

$$P_N^{\beta,V}\left(\{D(\tilde\mu_N,\mu_V^{\mathrm{eq}} \geq t\} \cap \{\max|\lambda_i| \leq M\}\right) \leq e^{CN\ln N - \beta N^2t^2}\left(\int e^{-N\beta V_{\mathrm{eff}}(x)}dx\right)^N$$

where the last integral is bounded by a constant as V_{eff} is non-negative and goes to infinity at infinity faster than logarithmically. We finally remove the cutoff by M thanks to Theorem 3.11.

3.3 The Dyson-Schwinger equations

3.3.1 Goal and strategy

We want to show that for sufficiently smooth functions f that

- $$\mathbb{E}\left[\frac{1}{N}\sum f(\lambda_i)\right] = \mu_V^{\mathrm{eq}}(f) + \sum_{g=1}^{K} \frac{1}{N^g} c_g(f) + o(\frac{1}{N^K})$$

- $\sum f(\lambda_i) - \mathbb{E}[\sum f(\lambda_i)]$ converges to a centered Gaussian.

We will provide two approaches, one which deals with general functions and a second one, closer to what we will do for discrete β ensembles, where we will restrict ourselves to Stieltjes transform $f(x) = (z-x)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, which in fact gives these results for all analytic function f by Cauchy formula. The present approach allows to consider sufficiently smooth functions but we will not try to get the optimal smoothness. We will as well restrict ourselves to $K = 2$, but the strategy is similar to get higher order expansion. The strategy is similar to the case of the GUE :

- We derive a set of equations, the Dyson-Schwinger equations, for our observables (the correlation functions, that are moments of the empirical measure, or the moments of Stieltjes transform) : it is an infinite system of equations, a priori not closed. However, it will turn out that asymptotically it can be closed.

- We linearize the equations around the limit. It takes the form of a linearized operator acting on our observables being equal to observables of smaller magnitudes. Inverting this linear operator is then the key to improve the concentration bounds, starting from the already known concentration bounds of Corollary 3.16.

- Using optimal bounds on our observables and the inversion of the master operator, we recursively obtain their large N expansion.

- As a consequence, we derive the central limit theorem.

3.3.2 Dyson-Schwinger Equation

Hereafter we set $M_N = N(\hat{\mu}_N - \mu_V)$. We let Ξ be defined on the set of $C_b^1(\mathbb{R})$ functions by

$$\Xi f(x) = V'(x)f(x) - \int \frac{f(x)-f(y)}{x-y} d\mu_V(y)\,.$$

Ξ will be called the master operator. The Dyson-Schwinger equations are given in the following lemma.

Lemma 3.17. *Let $f_i : \mathbb{R} \to \mathbb{R}$ be C^1_b functions, $0 \le i \le K$. Then,*

$$
\begin{aligned}
\mathbb{E}[M_N(\Xi f_0)\prod_{i=1}^K N\hat\mu_N(f_i)] &= (\frac{1}{\beta}-\frac{1}{2})\mathbb{E}[\hat\mu_N(f_0')\prod_{i=1}^K N\hat\mu_N(f_i)] \\
&+ \frac{1}{\beta}\sum_{\ell=1}^K \mathbb{E}[\hat\mu_N(f_0 f_\ell')\prod_{i\neq\ell} N\hat\mu_N(f_i)] \\
&+ \frac{1}{2N}\mathbb{E}[\int \frac{f_0(x)-f_0(y)}{x-y} dM_N(x)dM_N(y)\prod_{i=1}^p N\hat\mu_N(f_i)]
\end{aligned}
$$

Proof. This lemma is a direct consequence of integration by parts which implies that for all j

$$
\begin{aligned}
\mathbb{E}[f_0'(\lambda_j)\prod_{i=1}^K N\hat\mu_N(f_i)] &= \beta\mathbb{E}\left[f_0(\lambda_j)\left(NV'(\lambda_j)-\sum_{k\neq j}\frac{1}{\lambda_j-\lambda_k}\right)\prod_{i=1}^K N\hat\mu_N(f_i)\right] \\
&\quad -\sum_{\ell=1}^K \mathbb{E}[f_0(\lambda_j)f_\ell'(\lambda_j)\prod_{i\neq\ell} N\hat\mu_N(f_i)]
\end{aligned}
$$

Summing over $j \in \{1,\ldots,N\}$ and dividing by N yields

$$
\begin{aligned}
&\beta N\mathbb{E}\left[\left(\hat\mu_N(V'f_0)-\frac{1}{2}\int\int\frac{f_0(x)-f_0(y)}{x-y}d\hat\mu_N(x)d\hat\mu_N(y)\right)\prod_{i=1}^K N\hat\mu_N(f_i)\right] \\
= \quad &(1-\frac{\beta}{2})\mathbb{E}[\hat\mu_N(f_0')\prod_{i=1}^K N\hat\mu_N(f_i)] + \sum_{\ell=1}^K \mathbb{E}[\hat\mu_N(f_0 f_\ell')\prod_{i\neq\ell} N\hat\mu_N(f_i)]
\end{aligned}
$$

where we used that $(x-y)^{-1}(f(x)-f(y))$ goes to $f'(x)$ when y goes to x. We first take $f_\ell = 1$ for $\ell \in \{1,\ldots,K\}$ and f_0 with compact support and deduce that as $\hat\mu_N$ goes to μ_V almost surely as N goes to infinity, we have

$$
\mu_V(f_0V') - \frac{1}{2}\int\frac{f_0(x)-f_0(y)}{x-y}d\mu_V(x)d\mu_V(y) = 0\,. \tag{31}
$$

This implies that μ_V has compact support and hence the formula is valid for all f_0. We then linearize around μ_V to get the announced lemma. $\diamond$

The central point is therefore to invert the master operator Ξ. We follow a lemma from [6]. For a function $h : \mathbb{R} \to \mathbb{R}$, we recall that $\|h\|_{C^j(\mathbb{R})} := \sum_{r=0}^j \|h^{(r)}\|_{L^\infty(\mathbb{R})}$, where $h^{(r)}$ denotes the r-th derivative of h.

Lemma 3.18. *Given $V : \mathbb{R} \to \mathbb{R}$, assume that μ_V^{eq} has support given by $[a,b]$ and that*

$$
\frac{d\mu_V}{dx}(x) = S(x)\sqrt{(x-a)(b-x)}
$$

with $S(x) \geq \bar{c} > 0$ a.e. on $[a, b]$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a C^k function and assume that V is of class C^p. Then there exists a unique constant c_g such that the equation

$$\Xi f(x) = g(x) + c_g$$

has a solution of class $C^{(k-2)\wedge(p-3)}$. More precisely, for $j \leq (k-2) \wedge (p-3)$ there is a finite constant C_j such that

$$\|f\|_{C^j(\mathbb{R})} \leq C_j \|g\|_{C^{j+2}(\mathbb{R})}, \tag{32}$$

where, for a function h, $\|h\|_{C^j(\mathbb{R})} := \sum_{r=0}^{j} \|h^{(r)}\|_{L^\infty(\mathbb{R})}$.

This solution will be denoted by $\Xi^{-1}g$. It is C^k if g is C^{k+2} and $p \geq k+1$. It decreases at infinity like $|V'(x)x|^{-1}$.

Remark 3.19. *The inverse of the operator Ξ can be computed, see [6]. For $x \in [a, b]$ we have that $\Xi^{-1}g(x)$ equals*

$$\frac{1}{\beta(x-a)(b-x)S(x)} \left(\int_a^b \sqrt{(y-a)(b-y)} \frac{g(y)-g(x)}{y-x} dy - \pi \left(x - \frac{a+b}{2} \right) (g(x)+c_g) + c_2 \right),$$

where c_g and c_2 are chosen so that $\Xi^{-1}g$ converges to finite constants at a and b. We find that for $x \in S$

$$\begin{aligned} \Xi^{-1}g(x) &= \frac{1}{\beta S(x)} PV \int_a^b g(y) \frac{1}{(y-x)\sqrt{(y-a)(b-y)}} dy \\ &= \frac{1}{\beta S(x)} \int_a^b (g(y)-g(x)) \frac{1}{(y-x)\sqrt{(y-a)(b-y)}} dy, \end{aligned}$$

and outside of S f is given by (see Remark 3.10).

$$f(x) = \left(V'(x) - \int (x-y)^{-1} d\mu_V^{\mathrm{eq}}(y) \right)^{-1} \int \frac{f(y)}{x-y} d\mu_V^{\mathrm{eq}}(y) \,.$$

Remark 3.20. *Observe that by Remark 3.7, the density of μ_V^{eq} has to vanish at the boundary like $|x-a|^{q/2}$ for some $q \in \mathbb{N}$. Hence the only case when we can invert this operator is when $q = 1$. Moreover, by the same remark,*

$$S(x)\sqrt{(x-a)(b-x)} = \sqrt{V'(x)^2 - f(x)} = V'(x) - PV \int (x-y)^{-1} d\mu_V^{eq}(y)$$

so that S extends to the whole real line. Assuming that S is positive in $[a, b]$ we see that it is positive in a open neighborhood of $[a, b]$ since it is smooth. We can assume without loss of generality that it is smooth everywhere by the large deviation principle for the support.

We will therefore assume hereafter that

Assumption 3.21. *$V : \mathbb{R} \to \mathbb{R}$ is of class C^p and μ_V^{eq} has support given by $[a, b]$ and that*

$$\frac{d\mu_V}{dx}(x) = S(x)\sqrt{(x-a)(b-x)}$$

with $S(x) \geq \bar{c} > 0$ a.e. on $[a, b]$. Moreover, we assume that $(|V'(x)x| + 1)^{-1}$ is integrable.

The first condition is necessary to invert Ξ on all test functions (in critical cases, Ξ is may not be surjective). The second implies that for $\Xi^{-1}f$ decays fast enough at infinity so that it belongs to L^1 (for f smooth enough) so that we can use the Fourier inversion theorem.

We then deduce from Lemma 3.17 the following :

Corollary 3.22. *Assume that 3.21 with $p \geq 4$. Take f_0 C^k, $k \geq 3$ and f_i C^1. Let $g = \Xi^{-1} f_0$ be the C^{k-2} function such that there exists a constant c_g such that $\Xi f_0 = g + c_g$. Then*

$$\begin{aligned}
&\mathbb{E}[\prod_{i=0}^{K} M_N(f_i)] = (\frac{1}{\beta} - \frac{1}{2})\mathbb{E}[\hat{\mu}_N((\Xi^{-1} f_0)')\prod_{i=1}^{K} M_N(f_i)] \\
&+\frac{1}{\beta}\sum_{\ell=1}^{K}\mathbb{E}[\hat{\mu}_N(\Xi^{-1} f_0 f_\ell')\prod_{i\neq \ell} M_N(f_i)] \\
&+\frac{1}{2N}\mathbb{E}[\int \frac{\Xi^{-1} f_0(x) - \Xi^{-1} f_0(y)}{x-y} dM_N(x) dM_N(y) \prod_{i=1}^{K} M_N(f_i)]
\end{aligned}$$

3.3.3 Improving concentration inequalities

We are now ready to improve the concentration estimates we obtained in the previous section. We could do that by using the Dyson-Schwinger equations (this is what we will do in the discrete case) but in fact there is a quicker way to proceed by infinitesimal change of variables in the continuous case :

Lemma 3.23. *Take $g \in C^4$ and assume $p \geq 4$. Then there exists universal finite constants C_V and $c > 0$ such that for all $M > 0$*

$$P_N^{\beta,V}\left(N|\int g(x) d(\hat{\mu}^N - \mu_V^{\text{eq}})(x)| \geq C_V \|g\|_{C^4} \ln N + M \ln N\right) \leq e^{-cN} + N^{-M}.$$

Proof. Take f compactly supported on a compact set K. Making the change of variable $\lambda_i = \lambda_i' + \frac{1}{N} f(\lambda_i')$, we see that $Z_N^{\beta,V}$ equals

$$\int \prod |\lambda_i - \lambda_j + \frac{1}{N}(f(\lambda_i) - f(\lambda_j))|^\beta \prod e^{-N\beta V(\lambda_i + \frac{1}{N} f(\lambda_i))}(1 + \frac{1}{N} f'(\lambda_i)) d\lambda_i \quad (33)$$

Observe that by Taylor's expansion there are $\theta_{ij} \in [0, 1]$ such that

$$\prod |\lambda_i - \lambda_j + \frac{1}{N}(f(\lambda_i) - f(\lambda_j))| = \prod |\lambda_i - \lambda_j| \exp\{\frac{1}{N}\sum_{i<j} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}$$

$$-\frac{1}{N^2}\sum_{i<j}\theta_{ij}\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)^2\}$$

where the last term is bounded by $\|f'\|_\infty^2$. Similarly there exists $\theta_i \in [0,1]$ such that

$$V(\lambda_i+\frac{1}{N}f(\lambda_i)) = V(\lambda_i)+\frac{1}{N}f(\lambda_i)V'(\lambda_i)+\frac{1}{N^2}f(\lambda_i)^2V''(\lambda_i+\frac{\theta_i}{N}f(\lambda_i))$$

where the last term is bounded for N large enough by $C_K(V)\|f\|_\infty^2$ with $C_K = \sup_{d(x,K)\le 1}|V''(x)|$. We deduce by expanding the right hand side of (33) that

$$\int \exp\{\frac{\beta}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}-\beta\sum_i V'(\lambda_i)f(\lambda_i)\}dP_N^{\beta,V} \le e^{\beta C_K\|f\|_\infty^2+\beta\|f'\|_\infty^2+\|f'\|_\infty}$$

Using Chebychev inequality we deduce that if f is C^1 and compactly supported

$$P_N^{\beta,V}\left(\left|\frac{1}{2N}\sum_{i,j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}-\sum V'(\lambda_i)f'(\lambda_i)\right|\ge M\ln N\right)\le N^{-M}e^{C(f)} \tag{34}$$

with $C(f)=C_K\|f\|_\infty^2+(\beta+1)(1+\|f'\|_\infty)^2$. But

$$\frac{1}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}-\sum V'(\lambda_i)f'(\lambda_i)$$

$$=-N(\hat\mu^N-\mu)(\Xi f)+\frac{1}{2}N\int\int\frac{f(x)-f(y)}{x-y}d(\hat\mu^N-\mu_V^{\mathrm{eq}})(x)d(\hat\mu^N-\mu_V^{\mathrm{eq}})(y)$$

where if f is C^2 the last term is bounded by $C\|f\|_{C^2}\ln N$ with probability greater than $1-e^{-cN}$ by Corollary 3.16. Hence, we deduce from (35) that

$$P_N^{\beta,V}\left(N\left|(\hat\mu^N-\mu)(\Xi f)\right|\ge M\ln N\right)\le N^{C\|f\|_{C^2}-M}e^{C\|f\|_{C^1}^2}+e^{-cN}$$

and inverting f by putting $g=\Xi f$ concludes the proof for f with compact support. Again using Theorem 3.11 allows to extend the result for f with full support. $\diamond$

Exercise 3.24. *Concentration estimates could as well be improved by using Dyson-Schwinger equations. However, using the Dyson-Schwinger equations necessitates to loose in regularity at each time, since it requires to invert the master operator. Hence, it requires stronger regularity conditions. Prove that if Assumption 3.21 holds with $p\ge 12$, for any f be C^k with $k\ge 11$. Then for $\ell=1,2$, there exists C_ℓ such that*

$$\left|\mathbb{E}[(N(\hat\mu_N-\mu_V^{\mathrm{eq}})(f))^\ell]\right|\le C_\ell\|f\|_{C^{3+4\ell}}\|f\|_{C^1}^{1_{\ell=2}}(\ln N)^{\frac{\ell+1}{2}}\,.$$

Hint : Use the DS equations, concentration, invert the master operator and bootstrap if you do not get the best estimates at once.

Theorem 3.25. *Suppose that Assumption 3.21 holds with $p \geq 10$. Let f be C^k with $k \geq 9$. Then*

$$m_V(f) = \lim_{N\to\infty} \mathbb{E}[N(\hat{\mu}_N - \mu_V^{\mathrm{eq}})(f)] = (\frac{1}{\beta} - \frac{1}{2})\mu_V^{\mathrm{eq}}[(\Xi^{-1}f)'] \,.$$

Let f_0, f_1 be C^k with $k \geq 9$ and $p \geq 12$. Then

$$C_V(f_0, f_1) = \lim_{N\to\infty} \mathbb{E}[M_N(f_0)M_N(f_1)] = m_V(f_0)m_V(f_1) + \frac{1}{\beta}\mu_V^{\mathrm{eq}}(f_1'\Xi^{-1}f_0) \,.$$

Remark 3.26. *Notice that as C is symmetric, we can deduce that for any f_0, f_1 in C^k with $k \geq 9$,*

$$\mu_V^{\mathrm{eq}}(f_1'\Xi^{-1}f_0) = \mu_V^{\mathrm{eq}}(f_0'\Xi^{-1}f_1) \,.$$

Proof. To prove the first convergence observe that

$$\begin{aligned} \mathbb{E}[M_N(f_0)] &= (\frac{1}{\beta} - \frac{1}{2})\mathbb{E}[\hat{\mu}_N((\Xi^{-1}f_0)')] \\ &\quad + \frac{1}{2N}\mathbb{E}[\int \frac{\Xi^{-1}f_0(x) - \Xi^{-1}f_0(y)}{x-y} dM_N(x)dM_N(y)] \,. \end{aligned} \tag{35}$$

The first term converges to the desired limit as soon as $(\Xi^{-1}f_0)'$ is continuous. For the second term we can use the previous Lemma and the basic concentration estimate 3.16 to show that it is neglectable. The arguments are very similar to those used in the proof of Corollary 3.16 but we detail them for the last time. First, not that if χ_M is the indicator function that all eigenvalues are bounded by M, we have by Theorem 3.11 that

$$|\mathbb{E}[(1-\chi_M)\int \frac{\Xi^{-1}f_0(x) - \Xi^{-1}f_0(y)}{x-y} dM_N(x)dM_N(y)]| \leq \|\Xi^{-1}f_0\|_{C^1} N^2 e^{-cN} \,.$$

We therefore concentrate on the other term, up to modify $\Xi^{-1}f_0$ outside $[-M, M]$ so that it decays to zero as fast as wished and is as smooth as the original function (it is enough to multiply it by a smooth cutoff function). In particular we may assume it belongs to L^2 and write its decomposition in terms of Fourier transform. With some abuse of notations, we still denote $(\widehat{\Xi^{-1}f_0})_t$ the Fourier transform of this eventually modified function. Then, we have

$$\begin{aligned} & |\mathbb{E}[\chi_M \int \frac{\Xi^{-1}f_0(x) - \Xi^{-1}f_0(y)}{x-y} dM_N(x)dM_N(y)]| \\ \leq & \int |t(\widehat{\Xi^{-1}f_0})_t| \int_0^1 \mathbb{E}[\chi_M |M_N(e^{i\alpha t.})|^2] d\alpha dt \end{aligned}$$

To bound the right hand side under the weakest possible hypothesis over f_0, observe that by Corollary 3.16 applied on only one of the M_N we have

$$\mathbb{E}[\chi_M |M_N(e^{i\alpha t.})|^2] \leq C\sqrt{N \ln N}|t|\mathbb{E}[|M_N(e^{i\alpha t.})|] + N^2 e^{-cN} \tag{36}$$

where again we used that even though $e^{i\alpha t\cdot}$ has infinite 1/2 norm, we can modify this function outside $[-M, M]$ into a function with 1/2 norm of order $|t|$. We next use Lemma 3.23 to estimate the first term in (36) (with $\|e^{i\alpha t\cdot}\|_{C^4}$ of order $|\alpha t|^4 + 1$) and deduce that :

$$
\begin{aligned}
& |\mathbb{E}[\chi_M \int \frac{\Xi^{-1} f_0(x) - \Xi^{-1} f_0(y)}{x-y} dM_N(x) dM_N(y)]| \\
\leq \;& C(\ln N)^{3/2} \sqrt{N} \int |t(\widehat{\Xi^{-1} f_0})_t||t|^5 dt \\
\leq \;& C(\ln N)^{3/2} \sqrt{N} \|\Xi^{-1} f_0\|_{C^7} \\
\leq \;& C(\ln N)^{3/2} \sqrt{N} \|f_0\|_{C^9}
\end{aligned}
$$

Hence, we deduce that

$$
\frac{1}{N} |\mathbb{E}[\int \frac{\Xi^{-1} f_0(x) - \Xi^{-1} f_0(y)}{x-y} dM_N(x) dM_N(y)]| \leq C(\ln N)^{3/2} \sqrt{N}^{-1} \|f_0\|_{C^9}
$$

goes to zero if f_0 is C^9. This proves the first claim. Similarly, for the covariance, we use Corollary 3.22 with $p = 1$ to find that for $f_0, f_1 C^k$,

$$
\begin{aligned}
C_N(f_0, f_1) &= \mathbb{E}[(N(\hat{\mu}^N - \mu_V^{\mathrm{eq}}))(f_0) M_N(f_1)] \\
&= (\frac{1}{\beta} - \frac{1}{2}) \mu_V^{\mathrm{eq}}((\Xi^{-1} f_0)') \mathbb{E}[M_N(f_1)] + \frac{1}{\beta} \mu_V^{eq}(\Xi^{-1} f_0 f_1') \\
&\quad + (\frac{1}{\beta} - \frac{1}{2}) \mathbb{E}[(\hat{\mu}_N - \mu_V^{\mathrm{eq}})((\Xi^{-1} f_0)') M_N(f_1)] \\
&\quad + \frac{1}{\beta} \mathbb{E}[(\hat{\mu}_N - \mu_V^{eq})(\Xi^{-1} f_0 f_1')] \qquad (37) \\
&\quad + \frac{1}{2N} \mathbb{E}[\int \frac{\Xi^{-1} f_0(x) - \Xi^{-1} f_0(y)}{x-y} dM_N(x) dM_N(y) M_N(f_1)]
\end{aligned}
$$

The first line converges towards the desired limit. The second goes to zero as soon as $(\Xi^{-1} f_0)'$ is $C^1$and f_1 is C^4, as well as the third line. Finally, we can bound the last term by using twice Lemma 3.23, Cauchy-Schwartz and the basic concentration estimate once

$$
\begin{aligned}
& |\mathbb{E}[\int \frac{\Xi^{-1} f_0(x) - \Xi^{-1} f_0(y)}{x-y} dM_N(x) dM_N(y) M_N(f_1)]| \\
\leq \;& C(\ln N)^{5/2} \sqrt{N} \int dt |(\widehat{\Xi^{-1} f_0})_t| \|f_1\|_{C^4} |t|^6 \\
\leq \;& C(\ln N)^{5/2} \sqrt{N} \|\Xi^{-1} f_0\|_{C^7} \|f_1|_{C^4}^{1/2}
\end{aligned}
$$

which once plugged into (37) yields the result. $\diamond$

3.3.4 Central limit theorem

Theorem 3.27. *Suppose that Assumption 3.21 holds with $p \geq 10$. Let f be C^k with $k \geq 9$. Then $M_N(f) := \sum_{i=1}^N f(\lambda_i) - N\mu_V^{\mathrm{eq}}(f)$ converges in law under*

$P^N_{\beta,V}$ *towards a Gaussian variable with mean* $m_V(f)$ *and covariance* $\sigma(f) = \mu_V^{eq}(f'\Xi^{-1}f)$ *.*

Observe that we have weaker assumptions on f than in Lemma 3.25. This is because when we use the Dyson-Schwinger equations, we have to invert the operator Ξ several times, hence requiring more and more smoothness of the test function f. Using the change of variable formula instead allows to invert it only once, hence lowering our requirements on the test function.

Proof. We can take f compactly supported by Theorem 3.11. We come back to the proof of Lemma 3.23 but go one step further in Taylor expansion to see that the function

$$\Lambda_N(f) := \frac{\beta}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} - \frac{\beta}{2N^2}\sum_{i<j}\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)^2 - \beta\sum V'(\lambda_i)f'(\lambda_i)$$
$$-\frac{\beta}{2N}\sum V''(\lambda_i)(f(\lambda_i))^2 + \frac{1}{N}\sum_{i=1}^N f'(\lambda_i)$$

satisfies

$$\left|\ln\int e^{\Lambda_N(f)}dP_N^{\beta,V}\right| \le C\frac{1}{N}\|f\|^3_{C^1}$$

where the constant C may depend on the support of f. For any $\delta > 0$, with probability greater than $1 - e^{-C(\delta)N^2}$ for some $C(\delta) > 0$, the empirical measure $\hat{\mu}^N$ is at Vasershtein distance smaller than δ from μ_V^{eq}. On this set, for f C^1

$$\frac{1}{2N^2}\sum_{i,j}\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)^2 + \frac{1}{N}\sum V''(\lambda_i)(f(\lambda_i))^2 = C(f) + O(\delta)$$

where

$$C(f) = \frac{1}{2}\int\left(\frac{f(x)-f(y)}{x-y}\right)^2 d\mu_V^{eq}(x)d\mu_V^{eq}(y) + \int V''(x)f(x)^2 d\mu_V^{eq}(x)$$

whereas

$$\frac{1}{N}\sum_{i=1}^N f'(\lambda_i) = M(f) + o(1), \quad \text{if } M(f) = \int f'(x)d\mu_V^{eq}(x)\,.$$

As $\Lambda_N(f)$ is at most of order N, we deduce by letting N and then δ going to zero that

$$Z_N(f) := \frac{\beta}{2N}\sum_{i,j=1}^N\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} - \beta\sum V'(\lambda_i)f'(\lambda_i)$$

satisfies for any $f\,C^1$

$$\lim_{N\to\infty}\int e^{Z_N(f)}dP_N^{\beta,V} = e^{(\frac{\beta}{2}-1)M(f)+\frac{\beta}{2}C(f)}\,.$$

In the line above we took into account that we added a diagonal term to $Z_N(f)$ which contributed to the mean. We can now replace f by tf for real numbers f and conclude that $Z_N(f)$ converges in law towards a Gaussian variable with mean $(\frac{\beta}{2}-1)M(f)$ and covariance $\beta C(f)$. On the other hand we can rewrite $Z_N(f)$ as

$$Z_N(f) = \beta M_N(\Xi f) + \varepsilon_N(f)$$

where

$$\varepsilon_N(f) = \frac{\beta N}{2}\int \frac{f(x)-f(y)}{x-y} d(\hat{\mu}^N - \mu_V^{eq})(x) d(\hat{\mu}^N - \mu_V^{eq})(y)$$

Now, we can use Lemma 3.23 to bound the probability that $\varepsilon_N(f)$ is greater than some small δ. We again use the Fourier transform to write :

$$\varepsilon_N(f) = \frac{\beta N}{2}\int (it\hat{f}_t) \int_0^1 (\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t} (\widehat{\hat{\mu}^N - \mu_V^{eq}})_{\alpha t} dt\,.$$

We can bound the L^1 norm of $\varepsilon_N(f)$ by Cauchy-Schwartz inequality by

$$\mathbb{E}[|\varepsilon_N(f)|] \le \frac{\beta N}{2}\int |t\hat{f}_t| \int_0^1 \mathbb{E}[|(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}|^2]^{1/2} \mathbb{E}[|(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}|^2]^{1/2} dt d\alpha\,.$$

Finally, Lemma 3.23 implies that

$$\mathbb{E}[|(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}|^2]^{1/2} \le C|t|^4 \frac{\ln N}{N} + N^{-C}$$

from which we deduce that there exists a finite constant C

$$\mathbb{E}[|\varepsilon_N(f)|] \le C\int |t|^5 |\hat{f}_t| dt \frac{\ln N^2}{N}\,.$$

Thus, the convergence in law of $Z_N(f)$ implies the convergence in law of $M_N(\Xi f)$ towards a Gaussian variable with covariance $C(f)$ and mean $(\frac{1}{2}-\frac{1}{\beta})M(f)$. If f is C^9, we can invert Ξ and conclude that $M_N(f)$ converges towards a Gaussian variable with mean $m(f) = (\frac{1}{2}-\frac{1}{\beta})M(\Xi^{-1}(f))$ and covariance $C(\Xi^{-1}(f))$. To identify the covariance, it is enough to show that $C(f) = \mu_V^{eq}((\Xi f)' f)$. But on the support of μ_V^{eq}

$$(\Xi f)'(x) = V'' f(x) + PV\int \frac{f(x)-f(y)}{(x-y)^2} d\mu_V^{eq}(y)$$

from which the result follows.

$\diamond$

3.4 Expansion of the partition function

Theorem 3.28. *1. For f C^{17} and V C^{20},*

$$\mathbb{E}_{P^N_{\beta,V}}[\hat{\mu}^N(f)] = \mu_V^{\mathrm{eq}}(f) + \frac{1}{N} m_V(f) + \frac{1}{N^2} K_V(f) + o(\frac{1}{N^2})\,,$$

with $m_V(f)$ as in Theorem 3.25 and

$$K_V(f) = (\frac{1}{\beta}-\frac{1}{2})m_V(((\Xi^{-1}f_0)')+\frac{1}{2}\int itdt\int \widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha C_V(e^{it\alpha\cdot},e^{it(1-\alpha)\cdot})\,.$$

2. *Assume V C^{20}, then*

$$\ln Z^N_{\beta,V} = C^0_\beta N\ln N + C^1_\beta \ln N + N^2F_0(V) + NF_1(V) + F_2(V) + o(1)$$

with $C^0_\beta = \frac{\beta}{2}$, $C^1_\beta = \frac{3+\beta/2+2/\beta}{12}$ and

$$\begin{aligned} F_0(V) &= -\mathcal{E}(\mu^{\rm eq}_V) \\ F_1(V) &= -(\frac{\beta}{2}-1)\int \ln\frac{d\mu^{\rm eq}_V}{dx}d\mu^{\rm eq}_V + f_1 \\ F_2(V) &= -\beta\int_0^1 K_{V_\alpha}(V-V_0)d\alpha + f_2 \end{aligned} \tag{38}$$

where f_1, f_2 only depends on $b-a$, the width of the support of $\mu^{\rm eq}_V$.

Proof. The first order estimate comes from Theorem 3.25. To get the next term, we notice that if $\Xi^{-1}f$ belongs to L^1 we can use the Fourier transform of $\Xi^{-1}f$ (which goes to infinity to zero faster than $(|t|+1)^{-3}$ as $\Xi^{-1}f$ is C^6) so that

$$\begin{aligned} &\mathbb{E}[\int \frac{\Xi^{-1}f(x)-\Xi^{-1}f(y)}{x-y}dM_N(x)dM_N(y)] \\ = &\int dtit\widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha \mathbb{E}[\widehat{M_N}(e^{it\alpha\cdot})\widehat{M_N}(e^{it(1-\alpha)\cdot})] \\ \simeq &\int dtit\widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha C_V(e^{it\alpha\cdot},e^{it(1-\alpha)\cdot}) \end{aligned}$$

We can therefore use (35) to conclude that

$$\begin{aligned} N(\mathbb{E}[M_N(f)]-m(f)) \simeq\ & (\frac{1}{\beta}-\frac{1}{2})m_V(((\Xi^{-1}f)') \\ & +\frac{1}{2}\int dtit\int \widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha C_V(e^{it\alpha\cdot},e^{it(1-\alpha)\cdot}) \end{aligned}$$

which proves the first claim. We used that f is C^{12} so that $(\Xi^{-1}f)'$ is C^9 and Theorem 3.25 for the convergence of the first term. For the second we notice that the covariance is uniformly bounded by $C(|t|^{12}+1)$, so we can apply monotone convergence theorem when $\int dt|\widehat{\Xi^{-1}f}(t)||t|^{13}$ is finite, so f C^{16+}.

To prove the second point, the idea is to proceed by interpolation from a case where the partition function can be explicitly computed, that is where V is quadratic. We interpolate V with a potential $V_0(x) = c(x-d)^2/4$ so that the limiting equilibrium measure $\mu_{c,d}$, which is a semi-circle law shifted by d and enlarged by a factor $\sqrt{c}$, has support $[a,b]$ (so $d=(a+b)/2$ and $c=(b-a)^2/16$). The advantage of keeping the same support is that the potential $V_\alpha=\alpha V+(1-\alpha)V_0$ has equilibrium measure $\mu_\alpha=\alpha\mu_V^{\mathrm{eq}}+(1-\alpha)\mu_{c,d}$ since it satisfies the characterization of Lemma 3.5. We then write

$$\begin{aligned}\ln\frac{Z^N_{\beta,V}}{Z^N_{\beta,V_0}} &= \int_0^1 \partial_\alpha \ln Z^N_{\beta,V_\alpha}d\alpha\\ &= -\beta N^2\int_0^1 \mathbb{E}_{P^N_{\beta,V_\alpha}}[\hat\mu^N(V-V_0)]d\alpha\end{aligned}$$

It is not hard to see that if μ_V^{eq} satisfy hypotheses 3.2, so does μ_α and that the previous expansion can be shown to be uniform in α. Hence, we obtain the expansion from the first point if V is C^{20} with

$$\begin{aligned}F_0(V) &= -\beta\int_0^1 \mu_{V_\alpha}(V-V_0)d\alpha+f_0\\ F_1(V) &= -\beta\int_0^1 m_{V_\alpha}(V-V_0)d\alpha+f_1\\ F_2(V) &= -\beta\int_0^1 K_{V_\alpha}(V-V_0)d\alpha+f_2\end{aligned}$$

where f_0,f_1,f_2 are the coefficients in the expansion of Selberg integrals given in [76] :

$$Z^N_{V_0,\beta}=N^{\frac{\beta N}{2}}N^{\frac{3+\beta/2+2/\beta}{12}}e^{N^2f_0+Nf_1+f_0+o(1)}$$

with f_0,f_1,f_2 only depending on $b-a$:

$$\begin{aligned}f_0 &= (\beta/2)\Big[-\frac{3}{4}+\ln\Big(\frac{b-a}{4}\Big)\Big]\\ f_1 &= (1-\beta/2)\ln\Big(\frac{b-a}{4}\Big)-1/2-\beta/4+(\beta/2)\ln(\beta/2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\\ f_2 &= \chi'(0;2/\beta,1)+\frac{\ln(2\pi)}{2}\end{aligned}$$

The first formula of Theorem 3.28 is clear from the large deviation principle and the last is just what we proved in the first point. Let us show that the first order correction is given in terms of the relative entropy as stated in (46). Indeed, by integration by part and Remark 3.26 we have

$$\begin{aligned}(\frac{1}{\beta}-\frac{1}{2})^{-1}m_V(f) &= \mu_V^{\mathrm{eq}}[(\Xi^{-1}f)'] \\ &= -\int \Xi^{-1}f(\frac{d\mu_V^{\mathrm{eq}}}{dx})'dx \\ &= -\int \Xi^{-1}f(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx})'d\mu_V^{\mathrm{eq}} \\ &= -\int f'\Xi^{-1}(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx})d\mu_V^{\mathrm{eq}}\end{aligned}$$

To complete our proof, we will first prove that if g is C^{10} ,

$$\lim_{s\to 0} s^{-1}(\mu_{V-sf}^{\mathrm{eq}}-\mu_V^{\mathrm{eq}})(g) = \mu_V^{\mathrm{eq}}(\Xi^{-1}gf')\,. \tag{39}$$

which implies the key estimate

$$(\frac{1}{\beta}-\frac{1}{2})^{-1}m_V(f) = -\int f'\Xi^{-1}(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx})d\mu_V^{\mathrm{eq}} = \partial_t\mu_{V+tf}(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx})|_{t=0}\,. \tag{40}$$

To prove (39), we first show that $m_V(f)=\int f(x)d\mu_V(x)$ is continuous in V in the sense that

$$D(\mu_V,\mu_W)\le\sqrt{\|V-W\|_\infty}\,. \tag{41}$$

Indeed, by Lemma 3.5 applied to $\mu=\mu_W$ and since $\int V_{\mathrm{eff}}d(\mu_W-\mu_V)\ge 0$, we have

$$\begin{aligned}D(\mu_W,\mu_V)^2 &\le \mathcal{E}(\mu_W)-\mathcal{E}(\mu_V) \\ &\le \inf\{\int Wd\mu+\frac{1}{2}\Sigma(\mu)\}-\inf\{\int Vd\mu+\frac{1}{2}\Sigma(\mu)\} \\ &\le \|W-V\|_\infty\end{aligned}$$

As a consequence $(\mu_{V-sf}^{\mathrm{eq}}-\mu_V^{\mathrm{eq}})(g)$ goes to zero like $\sqrt{s}$ for g Lipschitz and f bounded. We can in fact get a more accurate estimate by using the limiting Dyson-Schwinger equation (31) to μ_{V-sf}^{eq} and μ_V^{eq} and take their difference to get :

$$\begin{aligned}(\mu_{V-sf}^{\mathrm{eq}} &- \mu_V^{\mathrm{eq}})(\Xi_V g) = s\mu_{V-\frac{s}{\beta N}f}^{\mathrm{eq}}(gf') \\ &+ \frac{1}{2}\int\frac{g(x)-g(y)}{x-y}d(\mu_{V-sf}-\mu_V^{\mathrm{eq}})(x)d(\mu_{V-sf}-\mu_V^{\mathrm{eq}})(y)\,.\end{aligned} \tag{42}$$

The last term is at most of order s if g is C^2 by (41) (see a similar argument in (29)), and so is the first. Hence we deduce from (42) that $(\mu_{V-sf}^{\mathrm{eq}}-\mu_V^{\mathrm{eq}})(g)$ is of order s if $g\in C^4$ and f is C^5. Plugging back this estimate into the last term in (42) together with (41), we get (39) for $g\in C^8$ and $f\in C^9$.

From (40), we deduce that

$$\begin{aligned} F_1(V) - f_1 &= -\beta \int_0^1 m_{V_\alpha}(V - V_0) d\alpha \\ &= (\frac{\beta}{2} - 1) \int_0^1 (\partial_\alpha \mu_{V_\alpha})(\ln \frac{d\mu_{V_\alpha}^{\mathrm{eq}}}{dx}) d\alpha - (\frac{\beta}{2} - 1) \int_0^1 \mu_{V_\alpha} (\partial_\alpha \ln \frac{d\mu_{V_\alpha}^{\mathrm{eq}}}{dx}) d\alpha \\ &= (\frac{\beta}{2} - 1) \int_0^1 \partial_\alpha [\mu_{V_\alpha} (\ln \frac{d\mu_{V_\alpha}^{\mathrm{eq}}}{dx})] d\alpha \end{aligned}$$

wich yields the result. Above in the second line the last term vanishes as $\mu_{V_\alpha}^{\mathrm{eq}}(1) = 1$.

$\diamond$

3.5 The Stieltjes transforms approach

Another common approach is to study the fluctuations of the Stieltjes transform, namely moments of :

$$Y_z = N(G_N(z) - \mathbb{E}[G_N(z)]) = \sum_{i=1}^N \frac{1}{z - \lambda_i} - \mathbb{E}\left[\sum_{i=1}^N \frac{1}{z - \lambda_i}\right]$$

for $z \in \mathbb{C} \backslash \mathbb{R}$. This requires that V is real analytic in order to get closed equations for correlators of this functional. Namely, we will assume that

Assumption 3.29. *-V is real analytic,*
-μ_V^{eq} has a connected support $[a, b]$,
-V_{eff} is strictly positive outside the support of μ_V^{eq}.

Hereafter we will therefore assume that V is analytic in an open neighborhood $\mathcal{R}$ of the real line. All our contours and complex numbers will be taken in this neighborhood.

First notice that by integration by parts we have the so-called loop equations

Lemma 3.30. *Let $G_N(z) = \frac{1}{N} \sum \frac{1}{z - \lambda_i}$ and $G(z) = \int \frac{1}{z-x} d\mu_V^{\mathrm{eq}}(x)$ for $z \in \mathbb{C} \backslash \mathbb{R}$*

$$\begin{aligned} &\mathbb{E}\Big[\left(\frac{1}{N} \partial_z G_N(z) \left[-1 + \frac{\beta}{2}\right] + \frac{\beta}{2} G_N(z)^2 - \frac{\beta}{2\pi i} \oint \frac{V'(\xi)}{z - \xi} G_N(\xi) d\xi\right) \prod_k Y_{z_k}\Big] \\ &= -\frac{1}{N} \mathbb{E}\Big[\sum_{j=1}^p \partial_{z_j} \frac{G_N(z) - G_N(z_j)}{z - z_j} \prod_{\ell \neq j} Y_{z_\ell}\Big] \end{aligned} \tag{43}$$

Proof. We start with

$$\int \mathrm{d}\lambda \sum_{i=1}^N \partial_{\lambda_i} \left[\frac{1}{z - \lambda_i} \frac{\mathrm{d}P_N^{\beta, V}}{\mathrm{d}\lambda} \prod_{j=1}^k Y_{z_j}\right] = 0$$

(This follows by integration by parts formula $\int_{-\infty}^{\infty} \partial_x f(x) \mathrm{d}x = 0$ for functions f vanishing at infinity) On the other hand, if we expand out this derivative we have :

$$\int \mathrm{d}P_N^{\beta,V} \Bigg\{ \sum_{i=1}^{N} \frac{1}{(z-\lambda_i)^2} + \frac{1}{z-\lambda_i} \left\{ \beta \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} - \beta N V'(\lambda_i) \right\}$$

$$+ \sum_j \sum_i \frac{1}{z-\lambda_i} \frac{1}{(z_j-\lambda_i)^2} \frac{1}{Y_{z_j}} \Bigg\} \prod_{k=1}^{p} Y_{z_k} = 0$$

We now use the fact that V is real analytic so that Cauchy formula implies that

$$\sum \frac{V'(\lambda_i)}{z-\lambda_i} = -\frac{1}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} \sum \frac{1}{\xi - \lambda_i} \mathrm{d}\xi = -\frac{1}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} N G_N(\xi) \mathrm{d}\xi$$

where the contour encircles the λ_i's. We have seen in theorem 3.11 that when V_{eff} is positive outside the support of μ_V^{eq}, for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ so that

$$P_N^{\beta,V} \left(\exists i : \lambda_i \in [a-\varepsilon, b+\varepsilon]^c \right) \leq e^{-c(\varepsilon)N} .$$

This entitles us to change the probability measure to have support in $[a-\varepsilon, b+\varepsilon]$ up to exponentially small errors everywhere. We then can simply take a contour around $[a-\varepsilon, b+\varepsilon]$. $\diamond$

3.5.1 Analysis of the Dyson-Schwinger equation : heuristics

We know by Lemma 3.14 that G_N converges to G and hence (43) yields with $p=0$ that

$$\frac{1}{2} G(z)^2 - \frac{1}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} G(\xi) \mathrm{d}\xi = 0 . \tag{44}$$

We next guess the corrections to this limit.

- *First order correction.* Setting $\Delta G_N := G_N - G$, (43) yields with $p=0$ that

$$\mathbb{E}\left[\frac{1}{N} \partial_z G_N(z) \left[-1 + \frac{\beta}{2} \right] + K[\Delta G_N](z) + \frac{\beta}{2} (\Delta G_N(z))^2 \right] = 0 \tag{45}$$

  where

$$K f(z) = \beta G(z) f(z) - \frac{\beta}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} f(\xi) \mathrm{d}\xi .$$

  By Lemma 3.14, we know that $\mathbb{E}[(\Delta G_N(z))^2]$ is smaller than $(\ln N)/N$. Assume that K is invertible with bounded inverse. Then, we deduce from (45) that $\mathbb{E}[\Delta G_N]$ is at most of order $\ln N/N$. If we can prove that $\mathbb{E}[(\Delta G_N(z))^2]$ is $o(N^{-1})$, then we deduce from (45) that

$$\lim_{N\to\infty} N \mathbb{E}[\Delta G_N(z)] = (1 - \frac{\beta}{2}) K^{-1}[\partial_z G](z) =: G_1(z) . \tag{46}$$

- *Limiting covariance.* To get the limiting covariance, let us take $p = 1$. Let $c(z, z') = \mathbb{E}[Y_z Y_{z'}]$. The Dyson-Schwinger equation then reads

$$\begin{aligned} K(c(., z'))(z) &= -\frac{\beta N}{2}\mathbb{E}[(\Delta G_N(z))^2 Y_{z'}] \\ &\quad -(\frac{\beta}{2} - 1)\partial_z \mathbb{E}[G_N(z) Y_{z'}] - \partial_{z'}\mathbb{E}[\frac{G_N(z) - G_N(z')}{z - z'}] \end{aligned} \tag{47}$$

  The concentration estimates imply that $\mathbb{E}[(\Delta G_N(z))^2 Y_{z'}] = O((\ln N)^{3/2}/\sqrt{N})$ and $\mathbb{E}[G_N(z) Y_{z'}]$ is of order $\ln N$, hence if K is reversible, we deduce that c is of order $O((\ln N)^{3/2}\sqrt{N})$. As we have shown in the previous point that $\mathbb{E}[\Delta G_N]$ is at most of order $\ln N/N$, we deduce that $\mathbb{E}[|\Delta G_N(z)|^2]$ is at most of order $O((\ln N/N)^{3/2})$ which completes the proof of (46).

  To improve our estimate on $\mathbb{E}[(\Delta G_N(z))^2 Y_{z'}]$ we next use the concentration estimates on $Y_{z'}$ and our new bound on the covariance to obtain a bound of order $(\ln N/N)^{3/2} \times \sqrt{N \ln N} = (\ln N)^2/N$. This allows to improve our estimate thanks to (47) and a bound on c of order $(\ln N)^2$. Proceeding once more, we deduce that the last term in (47) is neglectable. Note also that $\mathbb{E}[G_N(z) Y_{z'}] = \mathbb{E}[(G_N(z) - G(z)) Y_{z'}]$ goes also to zero. As this is an analytic function, its derivative goes as well to zero. Then, we deduce from (45) that

$$\lim_{N\to\infty} \mathbb{E}[N G_N(z) Y_{z'}] = -K^{-1}[\partial_{z'}\frac{G(.) - G(z')}{. - z'}](z) =: W(z, z') .$$

- *Second order correction.* Going back to (45) with $p = 0$ we have

$$K[\mathbb{E}[N(N\Delta G_N - G_1)]](z) = -\frac{\beta}{2}(\mathbb{E}[Y_z^2] + \mathbb{E}[N\Delta G_N(z)]^2) - (\frac{\beta}{2} - 1)\partial_z \mathbb{E}[N\Delta G_N(z)]$$

  and we can go to the limit $N \to \infty$ to deduce

$$\lim_{N\to\infty} K[\mathbb{E}[N(N\Delta G_N - G_1)]](z) = -\frac{\beta}{2}(W(z, z) + G_1(z)^2) - (\frac{\beta}{2} - 1)\partial_z G_1(z)$$

  so that taking the inverse of K yields the desired limit :

$$\lim_{N\to\infty} \mathbb{E}[N(N\Delta G_N - G_1)](z) = K^{-1}(-\frac{\beta}{2}(W(., .) + G_1(.)^2) - (\frac{\beta}{2} - 1)\partial_z G_1)(z) .$$

The above heuristics can be made rigorous provided we invert the operator K (and show its inverse is continuous to neglect error terms after we inverted it). This is what we do next.

3.5.2 Inverting the master operator

Observe that we want to apply K to functions which are differences of Stieltjes transforms of probability measures and therefore going to infinity like $1/z^2$. We therefore search for f with such a decay satisfying $g(z) = Kf(z)$ for a given g. As a consequence g goes to infinity like $1/z$ at best. We can rewrite

$$Kf(z) = \beta(G(z) - V'(z))f(z) - \beta \oint \frac{V'(\xi) - V'(z)}{2\pi i(z - \xi)} f(\xi)d\xi \, .$$

We make the following crucial assumption of off-criticality :

Assumption 3.31. *There exists a real analytic function S which does not vanish on a complex neighborhood of $[a - \varepsilon, b + \varepsilon]$ so that*

$$\frac{d\mu_V^{\mathrm{eq}}}{dx} = S(x)\sqrt{(x-a)(b-x)} \, .$$

This implies that

$$G(z) - V'(z) = \pi\sqrt{(z-a)(z-b)}S(z)$$

with S real analytic and not vanishing on $[a-\varepsilon, b+\varepsilon]$. Indeed,(44) implies that

$$G(z)^2 - 2V'(z)G(z) + Q(z) = 0$$

with $Q(z) = 2\oint \frac{V'(\xi)-V'(z)}{2\pi i(z-\xi)} G(\xi)d\xi$. Solving this equation yields

$$G(z) = V'(z) - \sqrt{V'(z)^2 - Q(z)} \tag{48}$$

Remember that $G(z)$ is analytic outside of the support of μ_V^{eq} where its imaginary part jumps by $\pi d\mu_V^{\mathrm{eq}}/dx$. As V' is analytic, and $V'(z)^2 - Q(z)$ is analytic, we see that the discontinuity of G can only come from the square root term in (48) when it is complex, that is when $V'(z)^2 - Q(z)$ is negative on the real line. Hence, this square root becomes as z goes to the real line the density of μ_V^{eq}. The conclusion follows.

This behaviour is essential to invert K, in the spirit of Tricomi airfoil equation. Indeed, we write with $\sigma(z) = \sqrt{(z-a)(z-b)}$, for $g = Kf$, that is

$$g(z) = -\pi\beta S(z)\sigma(z)f(z) - \beta Q_f(z)$$

with $Q_f(z) = 2\oint \frac{V'(\xi)-V'(z)}{2\pi i(z-\xi)} f(\xi)d\xi$.

$$\begin{aligned}
\sigma(z)f(z) &= -\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\sigma(\xi)f(\xi)d\xi \\
&= -\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi) - Q_f(\xi))d\xi \\
&= -\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}g(\xi)d\xi
\end{aligned}$$

where in the first line we took a contour around z and used Cauchy formula, in the second line we passed the contour around $[a, b]$ and used the definition of

$Kf = g$, using that the residue at infinity vanishes because $\sigma(z)f(z)$ goes like $1/z$, and in the last line we used that Q_f/S is analytic. Hence, we deduce that

$$K^{-1}g(z) = -\frac{1}{\sigma(z)}\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi))d\xi\,,$$

where the contour surrounds $[a-\varepsilon, b+\varepsilon]$. We note that away from $[a,b]$, K^{-1} is bounded. Also it maps holomorphic to holomorphic functions so that bounds on functions translate into bounds on its derivatives up to take slightly smaller imaginary part of the argument, thus giving continuity of the inverse.

3.5.3 Central limit theorem

To prove the central limit theorem we show by induction over p that

$$\lim_{N\to\infty}\mathbb{E}[\prod_{i=1}^{p} Y_{z_i}] = \sum_{i=2}^{p} W(z_1,z_i)\lim_{N\to\infty}\mathbb{E}[\prod_{\substack{\ell\neq i\\ \ell\geq 2}} Y_{z_\ell}]$$

We already showed that this is true for $p=1,2$ and assume we have proved it for $p\leq K$. This in particular imply that $\mathbb{E}[|Y_z|^{\frac{\lfloor 2K\rfloor}{2}}]$ is bounded uniformly in N for z away from the real axis (since we can take half of the $z_j = z$ and the other half to be its conjugate). We then use the Dyson-Schwinger equation with $p=K$ in order to obtain the result for $K+1$:

$$\mathbb{E}\Big[(N\beta K(G_N-G)(z))\prod_{k=1}^{p} Y_{z_k}\Big] = \mathbb{E}\Big[\sum_{j=1}^{p}\partial_{z_j}\Big[\frac{G_N(z)-G_N(z_j)}{z-z_j}\Big]\prod_{\ell\neq j} Y_{z_\ell}\Big]+\varepsilon_N(z) \tag{49}$$

where

$$\varepsilon_N(z) = -\mathbb{E}\Big[\Big(\partial_z G_N(z)\Big[-1+\frac{\beta}{2}\Big]+\frac{\beta N}{2}(G_N(z)-G(z))^2\Big)\prod_{k=1}^{p} Y_{z_k}\Big].$$

We then use that by concentration inequality Lemma 3.14 and the induction bound

$$|\varepsilon_N(z)| \leq \frac{\ln N}{N}(\sqrt{N}\ln N)^n + \frac{1}{(\Im z)^2\prod|\Im z_{k|}}e^{-N\ln N}$$

where $n=1$ if p is odd and 0 if p even. Indeed, we know by the induction bound that $\mathbb{E}[\prod_{k=1}^{p}|Y_{z_k}|]$ is of order one if p is even, but of order $\sqrt{N}\ln N$ if p is odd. Let us consider the case p odd which is slightly more complicated. Plugging back this estimate and inverting K yields that $|\mathbb{E}[N(G_N(z)-G(z))\prod_{k=1}^{p} Y_{z_k}]|$ is at most of order $\ln N\sqrt{N}$. Because we have already seen that $\mathbb{E}[N(G_N(z)-G(z))]$ is bounded, we deduce that $\mathbb{E}[\prod_{k=1}^{p+1} Y_{z_k}]$ is at most of order $\ln N\sqrt{N}$ and therefore we improve the previous bound (note $p+1$ is even) to

$$|\varepsilon_N(z)| = O(\frac{1}{N}(\frac{\ln N}{N})^{1/2}\ln N\sqrt{N}) = O(\frac{(\ln N)^{3/2}}{N}$$

which in turns yields a better bound on $\mathbb{E}[\prod_{k=1}^{p+1} Y_{z_k}]$ of order $(\ln N)^{3/2}$. Bootstrapping this new bound once more shows that

$$|\varepsilon_N(z)| = O(\frac{1}{N}(\frac{\ln N}{N})^{1/2} \ln N^{3/2}) = O(\frac{(\ln N)^2}{N^{3/2}})$$

which now implies with the Dyson Schwinger equation and (46) (to take into account that we recenter with respect to the expectation instead of the limit) that

$$\mathbb{E}[\prod_{i=1}^{p} Y_{z_i} Y_z] = \sum_{j=1}^{p} K^{-1}\left[\partial_{z_j}[\frac{G(.) - G(z_j)}{. - z_j}]\right](z)\mathbb{E}\left[\prod_{\ell \neq j} Y_{z_\ell}\right] + o(1)$$

which provides the desired estimate.

4 Discrete Beta-ensembles

We will consider discrete ensembles which are given by a parameter θ and a weight function w :

$$P_N^{\theta,w}(\vec{\ell}) = \frac{1}{Z_N^{\theta,\omega}} \prod_{i<j} I_\theta(\ell_j - \ell_i) \prod_i w(\ell_i, N)$$

where for $x \geq 0$ we have set

$$I_\theta(x) = \frac{\Gamma(x+1)\Gamma(x+\theta)}{\Gamma(x)\Gamma(x+1-\theta)}$$

where Γ is the usual Γ-function, $\Gamma(n+1) = n\Gamma(n)$. The coordinates $\ell_1, \ldots, \ell_N$ are discrete and belong to the set $\mathbb{W}_\theta$ such that

$$\ell_{i+1} - \ell_i \in \{\theta, \theta+1, \ldots\}$$

and $\ell_i \in (a(N), b(N))$ with $w(a(N), N) = w(b(N), N) = 0$ and $\ell_1 - a(N) \in \mathbb{N}, b(N) - \ell_N \in \mathbb{N}$.

Example 4.1. *When $\theta = 1$ this probability measure arises in the setting of lozenge tilings of the hexagon. More specifically, if one looks at a "slice" of the hexagon with sides of size A, B, C, then the number of lozenges of a particular orientation is exactly N and the locations of these lozenges are distributed according the $P_N^{1,\omega}$. Along the vertical line at distance t of the vertical side of size A (see Figure 2), the distribution of horizontal lozenges corresponds to a potential of the form*

$$w(\ell, N) = \left[(A+B+C+1-t-\ell)_{t-B}\,(\ell)_{t-C}\right],$$

where $(a)_n$ is the Pochhammer symbol, $(a)_n = a(a+1)\cdots(a+n-1)$.

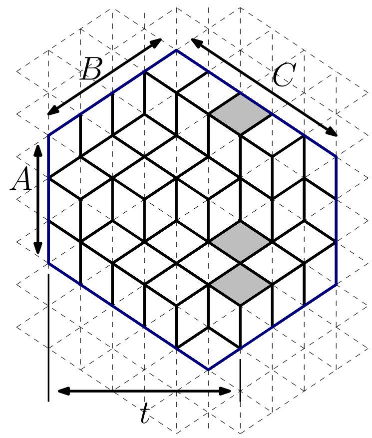

Figure 1: Lozenge tilings of a hexagon

More generally, as $x \to +\infty$ the interaction term scales like :

$$I_\theta(x) \approx |x|^{2\theta} \text{ as } x \to \infty$$

so the model for θ should be compared to the β ensemble model with $\beta \leftrightarrow 2\theta$. Note however that when $\theta \neq 1$, the particles configuration do not live on $\mathbb{Z}^{\mathbb{N}}$. These discrete β-ensembles were studied in [14]. Large deviation estimates can be generalized to the discrete setting but Dyson-Schwinger equations are not easy to establish. Indeed, discrete integration by parts does not give closed equations for our observables this time. A nice generalization was proposed by Nekrasov that allows an analysis similar to the analysis we developed for continuous β models. It amounts to show that some functions of the observables are analytic, in fact thanks to the fact that its possible poles cancel due to discrete integration by parts. We present this approach below.

4.1 Large deviations, law of large numbers

Let $\hat{\mu}_N$ be the empirical measure :

$$\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i/N}$$

Assumption 4.2. *Assume that $a(N) = \hat{a}N + O(\ln N), b(N) = \hat{b}N + O(\ln N)$ for some finite $\hat{a}, \hat{b}$ and the weight $w(x,N)$ is given for $x \in (a(N), b(N))$ by :*

$$w(x,N) = \exp\left(-NV_N\left(\frac{x}{N}\right)\right)$$

where $V_N(u) = 2\theta V_0(u) + \frac{1}{N}e_N(Nu)$. V_0 is continuous on $[\hat{a}, \hat{b}]$ and twice con-

tinuously differentiable in $(\hat{a}, \hat{b})$. It satisfies

$$|V_0''(u)| \leq C(1 + \frac{1}{|u - \hat{a}|} + \frac{1}{|\hat{b} - u)|}) .$$

e_N is uniformly bounded on $[a(N) + 1, b(N) - 1]/N$ by $C \ln N$ for some finite constant C independent of N.

For the sake of simplicity, we define V_0 to be constant outside of $[\hat{a}, \hat{b}]$ and continuous at the boundary.

Example 4.3. *In the setting of lozenge tilings of the hexagon of Example 4.1 we assume that for large N*

$$A = \hat{A}N + O(1), B = \hat{B}N + O(1), C = \hat{C}N + O(1), t = \hat{t}N + O(1)$$

with $\hat{t} > \max\{\hat{B}, \hat{C}\}$. Then $a(N) = 0, b(N) = A + B + C + 1 - t$ obey $\hat{a} = 0, \hat{b} = \hat{A} + \hat{B} + \hat{C} - \hat{t}$. Moreover, the potential satisfies our hypothesis with

$$\begin{aligned} V_0(u) &= u \ln u + (\hat{A} + \hat{B} + \hat{C} - \hat{t} - u) \ln(\hat{A} + \hat{B} + \hat{C} - \hat{t} - u) \\ &\quad -(\hat{A} + \hat{C} - u) \ln(\hat{A} + \hat{C} - u) - (\hat{t} - \hat{C} + u) \ln(\hat{t} - \hat{C} + u) \end{aligned}$$

Notice that V_N is infinite at the boundary since w vanishes. However, particles stay at distance at least $1/N$ of the boundary and therefore up to an error of order $1/N$, we can approximate V_N by V_0.

Theorem 4.4. *If Assumption 4.2 holds, the empirical measure converges almost surely :*

$$\hat{\mu}_N \to \mu_{V_0}$$

where μ_{V_0} is the equilibrium measure for V_0. It is the unique minimizer of the energy

$$\mathcal{E}(\mu) = \int \left(\frac{1}{2} V_0(x) + \frac{1}{2} V_0(y) - \frac{1}{2} \ln |x - y| \right) d\mu(x) d\mu(y)$$

subject to the constraint that μ is a probability measure on $[\hat{a}, \hat{b}]$ with density with respect to Lebesgue measure bounded by θ^{-1}.

Remark 4.5. *We have already seen that $\mathcal{E}$ is a strictly convex good rate function on the set of probability measures on $[\hat{a}, \hat{b}]$, see (13). To see that it achieves its minimal value at a unique minimizer, it is therefore enough to show that we are minimizing this function on a closed convex set. But the set of probability measures on $[\hat{a}, \hat{b}]$ with density bounded by $1/\theta$ is clearly convex. It can be seen to be closed as it is characterized as the countable intersection of closed sets given as the set of probability measures on $[\hat{a}, \hat{b}]$ so that*

$$\left| \int f(x) d\mu(x) \right| \leq \frac{\|f\|_1}{\theta}$$

for bounded continuous function f on $[\hat{a}, \hat{b}]$ so that $\|f\|_1 = \int |f(x)| dx < \infty$.

The case where $\hat{a}, \hat{b}$ are infinite can also be considered [14]. This result can be deduced from a large deviation principle similar to the continuous case [39] :

Theorem 4.6. *If Assumption 4.2 holds, the law of $\hat{\mu}^N$ under $P_N^{\theta,w}$ satisfies a large deviation principle in the scale N^2 with good rate function I which is infinite outside of the set $\mathcal{P}_\theta$ of probability measures on $[\hat{a}, \hat{b}]$ absolutely continuous with respect to the Lebesgue measure and with density bounded by $1/\theta$, and given on $\mathcal{P}_\theta$ by*

$$I(\mu) = 2\theta(\mathcal{E}(\mu) - \inf_{\mathcal{P}_\theta} \mathcal{E}) .$$

Proof. The proofs are very similar to the continuous case, we only sketch the differences. In this discrete framework, because the particles have spacings bounded below by θ, we have, for all $x < y$,

$$\theta \# \{i : \ell_i \in N[x, y]\} \leq (y - x)N + \theta$$

so that

$$\hat{\mu}_N ([x, y]) \leq \frac{|y - x|}{\theta} + \frac{1}{N} .$$

In particular, $\hat{\mu}^N$ can only deviate towards probability measures in $\mathcal{P}_\theta$. The proof of the large deviation upper bound is then exactly the same as in the continuous case. For the lower bound, the proof is similar and boils down to concentrate the particles very close to the quantiles of the measure towards which the empirical measure deviates : one just need to find such a configuration in $\mathbb{W}_\theta$. We refer the reader to [39].

In particular in the limit we will have :

$$\frac{\mathrm{d}\mu_V^{\mathrm{eq}}}{\mathrm{d}x} \leq \frac{1}{\theta}$$

The variational problem defining μ_V^{eq} in this case takes this bound into account. Noticing that $\mathcal{E}(\mu_V^{\mathrm{eq}} + t\nu) \geq \mathcal{E}(\mu_V^{\mathrm{eq}})$ for all ν with zero mass, non-negative outside the support of μ_V^{eq} and non-positive in the region where $d\mu_V^{\mathrm{eq}} = \theta^{-1}dx$, the characterization of the equilibrium measure is that $\exists C_V$ s.t. if we define :

$$V_{\mathrm{eff}}(x) = V_0(x) - \int \ln(|x - y|)\mathrm{d}\mu_V^{\mathrm{eq}}(y) - C_V$$

and V_{eff} satisfies :

$$\begin{cases} V_{\mathrm{eff}}(x) = 0 & \text{on } 0 < \frac{\mathrm{d}\mu_V^{\mathrm{eq}}}{\mathrm{d}x} < \frac{1}{\theta} \\ V_{\mathrm{eff}}(x) \geq 0 & \text{on } \frac{\mathrm{d}\mu}{\mathrm{d}x} = 0 \\ V_{\mathrm{eff}}(x) \leq 0 & \text{on } \frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{\theta} \end{cases}$$

The analysis of the large deviation principle and concentration are the same as in the continuous β ensemble case otherwise. ◇

4.2 Concentration of measure

As in the continuous case we consider the pseudo- distance D (3.12) and the regularization of the empirical measure $\tilde{\mu}_N$ given by the convolution of $\hat{\mu}^N$ with a uniform variable on $[0, \frac{\theta}{N}]$ (to keep measures with density bounded by $1/\theta$). We then have as in the continuous case

Lemma 4.7. *Assume V_0 is C^1. There exists C finite such that for all $t \geq 0$*

$$P_N^{\theta,\omega}\left(D(\tilde{\mu}_N, \mu_{V_0}) \geq t\right) \leq e^{CN \ln N - N^2 t^2}$$

As a consequence, for any $N \in \mathbb{N}$, any $\varepsilon > 0$

$$P_N^{\theta,\omega}\left(\sup_{z:\Im z \geq \varepsilon} \left| \int \frac{1}{z-x} d(\hat{\mu}^N - \mu_{V_0})(x) \right| \geq \frac{t}{\varepsilon^2} + \frac{1}{\varepsilon^2 N}\right) \leq e^{CN \ln N - N^2 t^2}$$

Proof. We set $Q_N^{\theta,\omega}(\ell) = N^{-\theta N^2} Z_N^{\theta,\omega} P_N^{\theta,\omega}(\ell)$ and set for a configuration ℓ, $\mathcal{E}(\ell) := \mathcal{E}(\hat{\mu}_N)$,

$$\mathcal{E}(\ell) = \frac{1}{N}\sum_{i=1}^N V_0(\frac{\ell_i}{N}) - \frac{2\theta}{N^2}\sum_{i<j} \ln|\frac{\ell_i}{N} - \frac{\ell_j}{N}|\,.$$

- *We first show that $Q_N^{\theta,\omega}(\ell) = e^{-N^2 2\theta \mathcal{E}(\ell) + O(N \ln N)}$.* Indeed, Stirling formula shows that $\ln \Gamma(x) = x \ln x - x - \ln \sqrt{2\pi x} + O(\frac{1}{x})$, which implies that

$$\prod_{i<j} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} = \prod_{i<j} |\ell_j - \ell_i|^{2\theta} e^{O(\sum_{i<j} \frac{1}{\ell_j - \ell_i})}$$

  with $\sum_{i<j} \frac{1}{(\ell_j - \ell_i)} = O(N \ln N)$ as $\ell_j - \ell_i \geq \theta(j-i)$. Similarly, by our assumption on V_N, for all configuration ℓ so that $\ell_1 \neq a(N)$ and $\ell_N \neq a(N)$, we have :

$$\frac{1}{N}\sum_{i=1}^N V_N(\frac{\ell_i}{N}) = \frac{1}{N}\sum_{i=1}^N V_0(\frac{\ell_i}{N}) + O(\frac{\ln N}{N})\,.$$

  Hence we deduce that for any configuration with positive probability :

$$Q_N^{\theta,\omega}(\ell) = e^{-N^2 2\theta \mathcal{E}(\ell) + O(N \ln N)} \tag{50}$$

- *We have the lower bound $N^{-\theta N^2} Z_N^{\theta,\omega} \geq e^{-N^2 2\theta \mathcal{E}(\mu_{V_0}) + CN \ln N}$.* To prove this bound we simply have to choose a configuration matching this lower bound. We let $(q_i)_{1 \leq i \leq N}$ be the quantiles of μ_{V_0} so that

$$\mu_{V_0}([\hat{a}, q_i]) = \frac{i - 1/2}{N}\,.$$

Then we set

$$Q_i = a(N) + \theta(i-1) + \lfloor Nq_i - a(N) - (i-1)\theta \rfloor$$

Because the density of μ_{V_0} is bounded by $1/\theta$, $q_{i+1} - q_i \geq \theta$ and therefore $Q_{i+1} - Q_i \geq \theta$. Moreover, $Q_1 - a(N)$ is an integer. Hence, Q is a configuration. We have by the previous point that

$$N^{-\theta N^2} Z_N^{\theta,\omega} \geq e^{-N^2 2\theta \mathcal{E}(Q) + O(N \ln N)} \tag{51}$$

We finally can compare $\mathcal{E}(Q)$ to $\mathcal{E}(\mu_{V_0})$. Indeed, by definition $Q_i \in [Nq_i, Nq_i + 1]$ and $Q_i - Q_j \geq \theta(i-j)$, so that

$$\begin{aligned}
\sum_{i<j} \ln |\frac{Q_i - Q_j}{N}| &\geq \sum_{i+[\frac{2}{\theta}]<j} \ln |\frac{Q_i - Q_j}{N}| + O(N \ln N) \\
&\geq \sum_{i+[\frac{2}{\theta}]<j} \ln |q_j - q_i - \frac{1}{N}| + O(N \ln N) \\
&= \sum_{i+[\frac{2}{\theta}]<j} \ln |q_j - q_i| + O(N \ln N) \\
&\geq N^2 \sum_{i+[\frac{2}{\theta}]<j} \int_{q_{j-1}}^{q_j} \int_{q_i}^{q_{i+1}} \ln |x-y| d\mu_{V_0}(x) d\mu_{V_0}(y) + O(N \ln N) \\
&\geq N^2 \int_{x<y} \ln |x-y| d\mu_{V_0}(x) d\mu_{V_0}(y) + O(N \ln N)
\end{aligned}$$

where we used that the logarithm is monotone and the density of μ_{V_0} uniformly bounded by $1/\theta$.

Moreover

$$|\sum_i \left(\frac{1}{N} V_0(\frac{Q_i}{N}) - \int_{q_i}^{q_{i+1}} V_0(x) d\mu_{V_0}(x) \right)| \leq C \sum_i \int_{q_i}^{q_{i+1}} (|\frac{Q_i}{N} - q_i| + |q_{i+1} - q_i|) d\mu_{V_0}$$

is bounded by C'/N.

We conclude that

$$\mathcal{E}(Q) \leq \mathcal{E}(\mu_{V_0}) + O(\frac{\ln N}{N})$$

so that we deduce the announced bound from (51).

- *We then show that $Q_N^{\theta,\omega}(\ell) = e^{-N^2 2\theta \mathcal{E}(\tilde{\mu}_N) + O(N \ln N)}$.* We start from (50) and need to show we can replace the empirical measure of $\hat{\mu}^N$ by $\tilde{\mu}_N$ and then add the diagonal term $i = j$ up to an error of order $N \ln N$. Indeed, if u, v are two independent uniform variables on $[0, \theta]$, independent of ℓ,

$$\sum_{i \neq j} \ln |\frac{\ell_i}{N} - \frac{\ell_j}{N}| - \sum_{i,j} \mathbb{E}[\ln |\frac{\ell_i}{N} - \frac{\ell_j}{N} + \frac{u-v}{N}|]$$

$$= -\sum_i \mathbb{E}[\ln|\frac{u-v}{N}|] + O(\sum_{i<j}\frac{1}{\ell_j-\ell_i}) = O(N\ln N)$$

whereas

$$\frac{1}{N}\sum_{i=1}^{N}(V_0(\frac{\ell_i}{N}) - \mathbb{E}[V_0(\frac{\ell_i}{N}+\frac{u}{N})]) = O(\frac{\ln N}{N})$$

- $P_N^{\theta,\omega}(\ell) \leq e^{-N^2 2\theta D^2(\tilde{\mu}_N,\mu_{V_0})+O(N\ln N)}$.

  We can now write

  $$\mathcal{E}(\tilde{\mu}_N) = \mathcal{E}(\mu_{V_0}) + \int V_{eff}(x)d(\tilde{\mu}_N - \mu_{V_0})(x) + D^2(\tilde{\mu}_N,\mu_{V_0})$$

  D^2 is indeed positive as $\tilde{\mu}_N$ and μ_{V_0} have the same mass. $V_{eff}(x)$ vanishes on the liquid regions of μ_{V_0}, is non-negative on the voids where $\tilde{\mu}_N - \mu_{V_0}$ is non-negative, and non positive on the frozen regions where $\tilde{\mu}_N - \mu_{V_0}$ is non-negative since $\tilde{\mu}_N$ has density bounded by $1/\theta$. Hence we conclude that

  $$\int_{[\hat{a},\hat{b}]} V_{eff}(x)d(\tilde{\mu}_N - \mu_{V_0})(x) \geq 0\,.$$

  On the other hand the effective potential is bounded and so our assumption on $a(N) - N\hat{a}$ implies

  $$N^2 \int_{[\hat{a},\hat{b}]^c} V_{eff}(x)d(\tilde{\mu}_N - \mu_{V_0})(x) = O(N\ln N)\,.$$

  Hence, we can conclude by the previous two points.

$\diamond$

4.3 Nekrasov's equations

The analysis of the central limit theorem is a bit different than for the continuous β ensemble case. Introduce :

$$\begin{aligned} G_N(z) &= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\frac{\ell_i}{N}} \\ G(z) &= \int \frac{1}{z-x}\mathrm{d}\mu_V^{\mathrm{eq}}(x)\,. \end{aligned}$$

We want to study the fluctuations of $\{N(G_N(z) - G(z))\}$. To this end, we would like an analogue of Dyson-Schwinger equations in this discrete setting. The candidate given by discrete integration by parts is not suited to asymptotic analysis as it yields densities which depend on $\prod(1+(\ell_i-\ell_j)^{-1})$ which is not a function of $\hat{\mu}^N$. In this case the analysis goes by the **Nekrasov's equations** which Nekrasov calls "non-perturbative" Dyson-Schwinger equations. Assume that we can write :

Assumption 4.8.

$$\frac{w(x,N)}{w(x-1,N)} = \frac{\phi_N^+(x)}{\phi_N^-(x)}$$

where $\phi_N^\pm$ are analytic functions in some subset $\mathcal{M}$ of the complex plane which includes $[a(N), b(N)]$ and independent of N.

Example 4.9. *In the example of random lozenge tilings of Example 4.1 we can take*

$$\phi_N^+(z) = \frac{1}{N^2}(t-C+z)(A+B+C-t-z), \quad \phi_N^-(z) = \frac{1}{N^2}z(A+C-z).$$

With these defined, Nekrasov's equation is the following statement.

Theorem 4.10. *If Assumption 4.8 holds*

$$R_N(\xi) = \phi_N^-(\xi)\mathbb{E}_{P_N^{\theta,w}}\left[\prod_{i=1}^N \left(1-\frac{\theta}{\xi-\ell_i}\right)\right] + \phi_N^+(\xi)\mathbb{E}_{P_N^{\theta,w}}\left[\prod_{i=1}^N \left(1+\frac{\theta}{\xi-\ell_i-1}\right)\right]$$

is analytic in $\mathcal{M}$.

Proof. In fact this can be checked by looking at the poles of the right hand side and showing that the residues vanish. Noting that there is a residue when $\xi = \ell_i$ or $\ell_i - 1$ we find that the residue at $\xi = m$ is

$$-\theta\phi_N^-(m)\sum_i \sum_{\ell_i=m} P_N^{\theta,w}(\ell_1,..,\ell_{i-1},m,\ell_{i+1},\ldots,\ell_N)\left[\prod_{j\neq i}^N\left(1-\frac{\theta}{m-\ell_j}\right)\right]$$

$$+\theta\phi_N^+(m)\sum_i \sum_{\ell_i=m-1} P_N^{\theta,w}(\ell_1,..,\ell_{i-1},m-1,\ell_{i+1},\ldots,\ell_N)\left[\prod_{j\neq i}^N\left(1+\frac{\theta}{m-\ell_j-1}\right)\right].$$

If $m = a(N)+1$ the second term vanish since the configuration space is such that $\ell_i > a(N)$ for all i, whereas $\phi_N^-(a(N)+1) = 0$. Hence both term vanish. The same holds at $b(N)$ and therefore we now consider $m \in (a(N)+1, b(N))$. Similarly, a configuration where $\ell_i = m$ implies that $\ell_{i-1} \le m-\theta$ whereas $\ell_i = m-1$ implies $\ell_{i-1} \le m-1-\theta$. However, the first term vanishes when $\ell_{i-1} = m-\theta$. Hence, in both sums we may consider only configurations where $\ell_{i-1} \le m-1-\theta$. The same holds for $\ell_{i+1} \ge m+\theta$. Then notice that if ℓ is a configuration such that when we shift ℓ_i by one we still have a configuration, our specific choice of weight w and interaction with the function Γ imply that

$$\phi_N^-(m)P_N^{\theta,w}(\ell_1,..,m,\ell_{i+1},\ldots,\ell_N)\left[\prod_{j\neq i}^N\left(1-\frac{\theta}{m-\ell_j}\right)\right]$$

$$= \phi_N^+(m) P_N^{\theta,w}(\ell_1, .., m-1, \ell_{i+1}, \dots, \ell_N) \left[\prod_{j \neq i}^{N} \left(1 + \frac{\theta}{m - \ell_j - 1} \right) \right] .$$

On the other hand a configuration such that when we shift the ith particle by one we do not get an admissible configuration has residue zero. Hence, we find that the residue at $\xi = \ell_i$ and $\ell_i - 1$ vanishes. $\diamond$

Nekrasov's equation a priori still contains the analytic function R_N as an unknown. However, we shall see that it can be asymptotically determined based on the sole fact that it is analytic, provided the equilibrium measure is off-critical.

Assumption 4.11. *Uniformly in $\mathcal{M}$,*

$$\phi_N^{\pm}(z) =: \phi^{\pm}(\frac{z}{N}) + \frac{1}{N}\phi_1^{\pm}(\frac{z}{N}) + O(\frac{1}{N^2})$$

Observe here that $\phi_1^{\pm}$ may depend on N and be oscillatory in the sense that it may depend on the boundary point. For instance, in the case of binomial weights, $\phi_N^+(x) = (\frac{M+1}{N} - x), \phi_N^-(x) = x$, we see that if M/N goes to $\mathbf{m}$, $\phi^-(x) = x$ and $\phi_1^-(x) = 0$, but

$$\phi^+(x) = m - x, \phi_1^+(x) = M + 1 - N\mathbf{m}$$

where the latter may oscillate, even if it is bounded. We will however hide this default of convergence in the notations. The main point is to assume the functions in the expansion are bounded uniformly in N and $z \in \mathcal{M}$.

Example 4.12. *With the example of lozenge tiling, we have*

$$\phi^+(z) = (\hat{t} - \hat{C} + z)(\hat{A} + \hat{B} + \hat{C} - t - z), \quad \phi^-(z) = z(\hat{A} + \hat{C} - z).$$

whereas if $\Delta D = D - L\hat{D}$,

$$\phi_1^+(x) = x(\Delta t - \Delta C + \Delta A + \Delta B + \Delta C - \Delta t), \phi_1^-(x) = \frac{N}{L} x(\Delta A + \Delta C) .$$

To analyze the asymptotics of G_N, we expand the Nekrasov's equations around the equilibrium limit. We set $\xi = Nz$ for $z \in \mathbb{C} \backslash \mathbb{R}$. Since we know by Lemma 4.7 that $\Delta G_N(z) = G_N(z) - G(z)$ is small (away from $[a, b]$), we can expand the Nekrasov's equation of Lemma 4.10 to get :

$$R_N(\xi) = R_\mu(z) - \theta Q_\mu(z) \mathbb{E}\left[\Delta G_N(z)\right] + \frac{1}{N} E_\mu(z) + \Gamma_\mu(z) \tag{52}$$

where we have set :

$$\begin{aligned}
R_\mu(z) &:= \phi^-(z) e^{-\theta G(z)} + \phi^+(z) e^{\theta G(z)} \\
Q_\mu(z) &:= \phi^-(z) e^{-\theta G(z)} - \phi^+(z) e^{\theta G(z)} \\
E_\mu(z) &:= \phi^-(z) e^{-\theta G(z)} \frac{\theta^2}{2} \partial_z G(z) + \phi^+(z) e^{\theta G(z)} \left(\frac{\theta^2}{2} - \theta \right) \partial_z G(z) \\
&\quad + \phi_1^-(z) e^{-\theta G(z)} + \phi_1^+(z) e^{\theta G(z)} .
\end{aligned}$$

Γ_μ is the reminder term given by (52) which basically is bounded on $\{\Im z \geq \varepsilon\} \cap \mathcal{M}$ by

$$|\Gamma_\mu(z)| \leq C(\varepsilon)\left(\mathbb{E}[|\Delta G_N(z)|^2] + \frac{1}{N}|\partial_z \mathbb{E}[\Delta G_N(z)]| + o(\frac{1}{N})\right)$$

The a priori concentration inequalities of Lemma 4.7 show that $\Gamma_\mu(z) = O(\ln N/N)$. We deduce by taking the large N limit that R_μ is analytic in $\mathcal{M}$ and we set $\tilde{R}_\mu = R_N - R_\mu$.

Let us assume for a moment that we have the stronger control on Γ_μ

Lemma 4.13. *For any $\varepsilon > 0$,*

$$\mathbb{E}[|\Delta G_N(z)|^2] + \frac{1}{N}|\partial_z \mathbb{E}[\Delta G_N(z)]| = o(\frac{1}{N})$$

uniformly on $\mathcal{M} \cap \{|\Im z| \geq \varepsilon\}$.

Let us deduce the asymptotics of $N\mathbb{E}[\Delta G_N(z)]$. To do that let us assume we are in a off-critical situation in the sense that

Assumption 4.14.

$$\theta Q_\mu(z) = \sqrt{(z-a)(b-z)}H(z) =: \sigma(z)H(z)$$

where H does not vanish in $\mathcal{M}$.

Remark 4.15. *Observe that if ρ is the density of the equilibrium measure,*

$$e^{2i\pi\theta\rho(E)} = \frac{R_\mu(E) + Q_\mu(E-i0)}{R_\mu(E) + Q_\mu(E+i0)}\,.$$

Our assumption implies therefore that $\rho(E) = 0$ or $1/\theta$ outside $[a,b]$ and goes to these values as a square root. There is a unique liquid region, where the density takes values in $(0, 1/\theta)$, it is exactly $[a,b]$.

We now proceed with similar techniques as in the β ensemble case, to take advantage of equation (52) as we used the Dyson-Schwinger equation before.

Lemma 4.16. *If Assumption 4.14 holds, for any $z \in \mathcal{M}\backslash\mathbb{R}$,*

$$\mathbb{E}[N\Delta G_N(z)] = m(z) + o(1) \tag{53}$$

with $m(z) = K^{-1}E_\mu(z)$ where

$$K^{-1}f(z) = \frac{1}{2i\pi\sigma(z)}\oint_{[a,b]} \frac{1}{\xi - z}\frac{1}{H(\xi)}f(\xi)d\xi\,.$$

Remark 4.17. *If we compare to the continuous setting, K is the operator of multiplication by $\theta Q_\mu(z)$ whereas in the continuous case it was multiplication by $\beta\frac{d\mu}{dx} = G(z) - V'(z)$. Choosing $\phi^+(z) = e^{-V'(z)/2}, \phi^-(z) = e^{+V'(z)/2}$ we see that $Q_\mu(z) = \sinh(\theta G_\mu - V'(z)/2)$ is the hyperbolic sinus of the density. Hence, the discrete and continuous master operators can be compared up to take a* sinh.

Proof. To get the next order correction we look at (52) :

$$\theta Q_\mu(z)\mathbb{E}\left[\Delta G_N(z)\right] = \frac{1}{N}E_\mu(z) - \tilde{R}_\mu(z) + \Gamma_\mu(z)$$

We can then rewrite as a contour integral for $z \in \mathcal{M}$:

$$\begin{aligned}
\sigma(z)\mathbb{E}\left[\Delta G_N(z)\right] &= \frac{1}{2i\pi}\oint_z \frac{1}{\xi - z}\left[\sigma(\xi)\mathbb{E}\left[\Delta G_N(\xi)\right]\right]\mathrm{d}\xi \\
&= \frac{1}{2i\pi}\oint_{[a,b]} \frac{1}{\xi - z}\frac{1}{H(\xi)}\left[\frac{1}{N}E_\mu(\xi) - \tilde{R}_\mu(\xi) + \Gamma_\mu(\xi)\right]\mathrm{d}\xi \\
&= \frac{1}{2i\pi}\oint_{[a,b]} \frac{1}{\xi - z}\frac{1}{H(\xi)}\left[\frac{1}{N}E_\mu(\xi)\right]\mathrm{d}\xi + o\left(\frac{1}{N}\right)
\end{aligned}$$

where we used that $\sigma\Delta G_N$ goes to zero like $1/z$ to deduce that there is no residue at infinity so that we can move the contour to a neighborhood of $[a,b]$, that $\tilde{R}_\mu/H$ is analytic in a neighborhood of $[a,b]$ to remove its contour integral, and assumed Lemma 4.13 holds to bound the reminder term, as the integral is bounded independently of N. $\diamond$

Remark 4.18. *The previous proof shows, without Lemma 4.13, that* $\mathbb{E}[\Delta G_N(z)]$ *is at most of order* $\ln N/N$ *sin* Γ_μ *is at most of this order by basic concentration estimates.*

We finally prove Lemma 4.13. To do so, it is enough to bound $\mathbb{E}[|\Delta G_N(z)|^2]$ by $o(1/N)$ uniformly on $\mathcal{M} \cap \{|\Im z| \geq \epsilon/2\}$ by analyticity. Note that Lemma 4.7 already implies that this is of order $\ln N/N$. To improve this bound, we get an equation for the covariance. To get such an equation we replace the weight $w(x,N)$ by

$$w_t(x,N) = w(x,N)\left(1 + \frac{t}{z' - x/N}\right)$$

for t very small. This changes the functions $\phi_N^\pm$ by

$$\phi_N^{+,t}(x) = \phi_N^+(x)\left(z' - x/N + t\right)\left(z' - x/N + \frac{1}{N}\right),$$

$$\phi_N^{-,t}(x) = \phi_N^-(x)\left(z' - x/N\right)\left(z' - x/N + t + \frac{1}{N}\right).$$

We can apply the Nekrasov's equations to this new measure for t small enough (so that the new weights w_t does not vanish for $z' \in \mathcal{M}$) to deduce that

$$R_N^t(\xi) = \phi_N^{-,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^N\left(1 - \frac{\theta}{\xi - \ell_i}\right)\right] + \phi_N^{+,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^N\left(1 + \frac{\theta}{\xi - \ell_i - 1}\right)\right] \tag{54}$$

is analytic. We start expanding with respect to N by writing

$$\phi_N^{\pm,t}(Nx)/(z'-x)(t+z'-x) = (\phi^{\pm}(x) + \frac{1}{N}\phi_1^{\pm,t}(x) + o(\frac{1}{N}))$$

with

$$\phi_1^{+,t}(x) = \phi_1^{+}(x) + \frac{\phi^{+}(x)}{z'-x},\ \phi_1^{-,t}(x) = \phi_1^{-}(x) + \frac{\phi^{-}(x)}{t+z'-x}\,.$$

We set

$$\tilde{R}_N^t(x) = (R_N^t(Nx) - R_\mu(x))/(z'-x)(t+z'-x)$$

which is analytic up to a correction which is $o(1/N)$ and analytic away from z' in a neighborhood of which it has two simple poles. We divide both sides of Nekrasov equation by $(z'-x)(t+z'-x)$, and take $\xi = Nz$ and again using Lemma 4.7, we deduce that

$$\theta Q_\mu(z)\mathbb{E}_{P_N}^{\theta,w_t}\left[\Delta G_N(z)\right] = \tilde{R}_N^t(z) + \frac{1}{N}E_\mu^t(z) + \Gamma_\mu^t(z) \tag{55}$$

where

$$E_\mu^t(z) = E_\mu(z) + \frac{\phi^{+}(z)}{z'-z}e^{\theta G(z)} + \frac{\phi^{-}(z)}{t+z'-z}e^{-\theta G(z)}$$

and $\Gamma_\mu^t(z)$ is a reminder term. It is the sum of the reminder term coming from (52) and the error term coming from the expansion of $\phi^{\pm,t}$. The latter has single poles at z' and $z'+t$ and is bounded by $1/N^2$. We can invert the multiplication by Q_μ as before to conclude (taking a contour which does not include z' so that $\tilde{R}_N^t$ stays analytic inside) that

$$\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right] = K^{-1}[\frac{1}{N}E_\mu^t + \Gamma_\mu^t](z) + o(\frac{1}{N})\,,$$

where we noticed that the residues of ε_N are of order one.

We finally differentiate with respect to t and take $t=0$ (note therefore that we need no estimates under the tilted measure P_N^{θ,w_t}, but only those take at $t=0$ where we have an honest probability measure). Noticing that the operator K does not depend on t, we obtain, with $\bar{\Delta}G_N(z') = G_N(z') - \mathbb{E}[G_N(z')]$:

$$N^2\mathbb{E}_{P_N^{\theta,w}}\left[\Delta G_N(z)\bar{\Delta}G_N(z')\right] = -K^{-1}[\frac{\phi^{-}(.)}{(z'-.)^2}e^{-\theta G(.)}](z) + NK^{-1}[\partial_t\Gamma_\mu^t|_{t=0}](z) \tag{56}$$

It is not difficult to see by a careful expansion in Nekrasov's equation (54) that

$$\begin{aligned}|\partial_t\Gamma_\mu^t(z)|_{t=0}| \quad &\leq \quad C(\varepsilon)\Big(N\mathbb{E}[|\Delta G_N(z)|^2|\bar{\Delta}G_N(z')|] \\ &\quad +|\partial_z\mathbb{E}[\Delta G_N(z)\bar{\Delta}G_N(z')]| + \frac{1}{N}\mathbb{E}[|\bar{\Delta}G_N(z')|]\Big)\end{aligned} \tag{57}$$

By Lemma 4.7, it is at most of order $(\ln N)^3/\sqrt{N}$ so that we proved

$$N^2\mathbb{E}_{P_N^{\theta,w}}\left[\Delta G_N(z)\bar{\Delta}G_N(z')\right] = -K^{-1}[\frac{\phi^-(.)}{(z'-.)^2}e^{-\theta G(.)}](z) + O((\ln N)^3\sqrt{N}) \tag{58}$$

This shows by taking $z' = \bar{z}$ that for $\Im z \geq \varepsilon$

$$\mathbb{E}[|N\Delta G_N(z)|^2] \leq (\ln N)^3\sqrt{N}\,. \tag{59}$$

We note here that $\Delta G_N(z)$ and $\bar{\Delta}G_N(z)$ only differ by $\ln N/N$ by Remark 4.18. This completes the proof of Lemma 4.13.

We derive the central limit theorem in the same spirit.

Theorem 4.19. *If Assumption 4.14 holds, for any $z_1,\ldots,z_k \in \mathcal{M}\backslash\mathbb{R}$, $(N\Delta G_N(z_1)-m(z_1),\ldots,N\Delta G_N(z_k)-m(z_k))$ converges in distribution towards a centered Gaussian vector with covariance*

$$C(z,z') = -K^{-1}[\frac{\phi^-(.)}{(z'-.)^2}e^{-\theta G(.)}](z)$$

Remark 4.20. *It was shown in [14] that the above covariance is the same than for random matrices and is given by*

$$C(z,w) = \frac{1}{(z-w)^2}\left(1-\frac{zw-\frac{1}{2}(a+b)(z+w)+ab}{\sqrt{(z-a)(z-b)}\sqrt{(w-a)(w-b)}}\right).$$

It only depends on the end points and therefore is the same than for continuous β ensembles with equilibrium measure with same end points. However notice that the mean given in (62) *is different.*

Proof. We first prove the convergence of the covariance by improving the estimates on the reminder term in (56) by a bootstrap procedure. It is enough to improve the estimate on $\partial_t\Gamma_\mu$ according to (56). But already, our new bound on the covariance (59) and Lemma 4.7 allow to bound the right hand side of (57) by $(\ln N)^4/N$. This allows to improve the estimate on the covariance as in the previous proof and we get :

$$\mathbb{E}[|N\Delta G_N(z)|^2] \leq C(\epsilon)(\ln N)^4\,. \tag{60}$$

In turn, we can again improve the estimate on $|\partial_t\Gamma_\mu(z)|$ since we now can bound the right hand side of (57) by $(\ln N)^5 N^{-1/2}$, which implies the desired convergence of $\mathbb{E}[\Delta G_N(z)\Delta G_N(z')]$ towards $C(z,z')$.

To derive the central limit theorem it is enough to show that the cumulants of degree higher than two vanish. To do so we replace the weight $w(x,N)$ by

$$w_t(x,N) = w(x,N)\prod_{i=1}^{p}\left(1+\frac{t_i}{z_i-x/N}\right).$$

The cumulants are then given by

$$N\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right]|_{t_1=t_2=\cdots=t_p=0}\,.$$

Indeed, recall that the cumulant of $N\bar{\Delta}G_N(z_1),\ldots N\bar{\Delta}G_N(z_p)$ is given by

$$\partial_{t_1}\cdots\partial_{t_p}\ln\mathbb{E}_{P_N^{\theta,w_t}}[\exp\{N\sum_{i=1}^{p}t_iG_N(z_i)\}]|_{t_1=t_2=\cdots=t_p=0}$$

which is also given by

$$\partial_{t_2}\cdots\partial_{t_p}\ln\mathbb{E}_{P_N^{\theta,w_t}}[N\bar{\Delta}G_N(z_1)]|_{t_1=t_2=\cdots=t_p=0}\,.$$

Noticing that $\mathbb{E}_{P_N^{\theta,w_t}}[\bar{\Delta}G_N(z)-\Delta G_N(z)]$ is independent of t, we conclude that it is enough to show that

$$N\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right]|_{t_1=t_2=\cdots=t_p=0}$$

goes to zero for $p\geq 2$. In fact, we can perform an analysis similar to the previous one. This changes the functions $\phi_N^{\pm}$ by

$$\phi_N^{+,t}(x)=\phi_N^{+}(z)\prod_{i=1}^{p}(z_i-x/N+t_i)\,,\phi_N^{-,t}(x)=\phi_N^{-}(z)\prod_{i=1}^{p}(z_i-x/N)\,.$$

We can apply the Nekrasov's equations to this new measure for t_i small enough (so that the new weights do not vanish) to deduce that

$$R_N^t(\xi)=\phi_N^{-,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^{N}\left(1-\frac{\theta}{\xi-\ell_i}\right)\right]+\phi_N^{+,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^{N}\left(1+\frac{\theta}{\xi-\ell_i-1}\right)\right] \tag{61}$$

is analytic. Expanding in N we deduce that

$$\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right]=K^{-1}[\frac{1}{N}E_\mu^t(z)+\Gamma_\mu^t(z)]$$

where

$$E_\mu^t(z)=E_\mu(z)+\sum_{i=1}^{p}\frac{\phi^+(x)}{z_i-x}e^{\theta G(z)}+\sum_{i=1}^{p}\frac{\phi^-(x)}{t_i+z_i-x}e^{-\theta G(z)}$$

and

$$|\partial_{t_1}\cdots\partial_{t_p}\Gamma_\mu^t|_{t_i=0}(z)|\leq C(\epsilon)\bigg(\mathbb{E}\left[(|\Delta G_N(z)|^2+\frac{1}{N^2})\prod|N\bar{\Delta}G_N(z_i)||\right]$$

$$+\frac{1}{N}|\partial_z\mathbb{E}[\Delta G_N(z)\prod N\bar{\Delta}G_N(z_i)]|\bigg)\,.$$

The contour in the definition of K^{-1} includes z and $[a,b]$ but not the z_i's. Taking the derivative with respect to $t_1,\dots,t_p$ at zero we see that for $p \geq 1$

$$\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}\mathbb{E}_{P_N^{\theta,w_t}}\left[N\Delta G_N(z)\right] = K^{-1}[\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}N\Gamma^t_\mu(z)]$$

where we used that the operator K is independent of t. We finally need to show that the right hand side goes to zero. It will, provided we show that for all $p \in \mathbb{N}$, all $z_1,\dots,z_p \in \mathcal{M}\backslash[A,B]$ there exists C depending only on $\min d(z_i,[A,B])$ and p such that

$$\left|\mathbb{E}[\prod_{i=1}^{p} N\Delta G_N(z_i)]\right| \leq C(\ln N)^{3p}\,.$$

This provides also bounds on $\mathbb{E}[|\Delta G_N(z)|^p]$ when p is even. Indeed $\partial_{t_2}\cdots\partial_{t_p}N\Gamma^t_\mu(z)$ can be bounded by a combination of such moments. We can prove this by induction over p. By our previous bound on the covariance, we have already proved this result for $p=2$ by (60). Let us assume we obtained this bound for all $\ell \leq p$ for some $p \geq 2$. To get bounds on moments of correlators of order $p+1$, let us notice that $|\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}N\Gamma^t_\mu|_{t=0}$ is at most of order $(\ln N)^{3p+2}$ if p is even by the induction hypothesis and Lemma 4.7(by bounding uniformly the Stieltjes functions depending on z). This is enough to conclude. If p is odd, we can only get bounds on moments of modulus of the Stieltjes transform of order $p-1$. We do that and bound also the Stieltjes transform depending on the argument z_1 by using Lemma 4.7. We then get a bound of order $(\ln N)^{3p+3}\sqrt{N}$ for $|\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}N\Gamma^t_\mu|_{t=0}$. This provides a similar bound for the correlators of order $p+1$, which is now even. Using Hölder inequality back on the previous estimate and Lemma 4.7 on at most one term, we finally bound $|\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}N\Gamma^t_\mu|_{t=0}$ by $(\ln N)^{3(p+1)}$ which concludes the argument.

$\diamond$

4.4 Second order expansion of linear statistics

In this section we show how to expand the expectation of linear statistics one step further. To this end we need to assume that $\phi_N^\pm$ expands to the next order.

Assumption 4.21. *Uniformly in $\mathcal{M}$,*

$$\phi^\pm_N(z) =: \phi^\pm(z) + \frac{1}{N}\phi^\pm_1(z) + \frac{1}{N^2}\phi^\pm_2(z) + O(\frac{1}{N^3})$$

Lemma 4.22. *Suppose Assumption 4.21 holds. Then,*

$$\lim_{N\to\infty}\mathbb{E}[N^2\Delta G_N(z) - Nm(z)] - r(z) = 0 \tag{62}$$

with $r(z) = K^{-1}F_\mu(z)$ *where*

$$\begin{aligned}
F_\mu(z) &= \phi^-(z)e^{-\theta G(z)}\left(\frac{\theta^2}{2}\partial_z m(z) - \frac{\theta^3}{3}\partial_z^2 G(z) + \frac{\theta^2}{2}[C(z,z) + (\frac{\theta}{2}\partial_z G(z) + m(z))^2]\right) \\
&\quad + \phi_1^-(z)e^{-\theta G(z)}\left(\frac{\theta^2}{2}\partial_z G(z) - \theta m(z)\right) + \phi_2^-(z)e^{-\theta G(z)} \\
&\quad + \phi^+(z)e^{\theta G(z)}\Bigg((\frac{\theta^2}{2} - \theta)\partial_z m(z) + (\frac{\theta^3}{3} + \theta - \frac{\theta^2}{2})\partial_z^2 G(z) \\
&\quad + \frac{\theta^2}{2}[(m(z) - \frac{2-\theta}{2}\partial_z G(z))^2 + C(z,z)]\Bigg) \\
&\quad + \phi_1^+(z)e^{\theta G(z)}[\theta m(z) + (\frac{\theta^2}{2} - \theta)\partial_z G(z)] + \phi_2^-(z)e^{\theta G(z)}
\end{aligned}$$

Proof. The proof is as before to show that

$$\theta Q_\mu(z)\mathbb{E}\left[\Delta G_N(z)\right] = \frac{1}{N}E_\mu(z) + \frac{1}{N^2}F_\mu(z) + \tilde{R}_\mu^N(z) + o\left(\frac{1}{N^2}\right)$$

by using Nekrasov's equation of Theorem 4.10, expanding the exponentials and using Lemmas 4.19 and 4.16. We then apply K^{-1} on both sides to conclude.

◇

5 Generalizations

In the last lecture, we will discuss generalizations and applications of the previous considerations:

1. The continuous Beta-models with several cuts [18]
2. The discrete Beta models with several cuts [17]
3. Beta ensembles with complex potentials [64]

While we give a precise description of the first question, we only outline the last two.

5.1 The several cut case: the continuous case

In this section we consider again the continuous β-ensembles, but in the case where the equilibrium measure has a disconnected support. The strategy has to be modified since in this case the master operator Ξ is not invertible. In fact, the central limit theorem is not true as if we consider a smooth function f which equals one on one connected piece of the support but vanishes otherwise, and if we expect that the eigenvalues stay in the vicinity of the support of the equilibrium measure, the linear statistic $\sum f(\lambda_i)$ should be an integer and therefore can not fluctuate like a Gaussian variable. It turns out however that the previous strategy works as soon as we fix the filling fractions, the number of eigenvalues in a neighborhood of each connected piece of the support. The idea will therefore be to obtain central limit theorems conditionally to filling fractions. We will as well expand the partition functions for such fixed filling fractions. The latter expansion will allow to estimate the distribution of the filling fractions and to derive their limiting distribution, giving a complete picture of the fluctuations. These ideas were developed in [13, 19]. [13] also includes the case of hard edges. After this work, a very special case (two connected components and a polynomial potential) could be treated in [32] by using Riemann Hilbert. I will here follow the strategy of [13], but will use general test functions instead of Stieltjes functionals as in Section 3. So as in Section 3, we consider the probability measure

$$\mathrm{d}P_N^{\beta,V}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{\beta,V}}\Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)}\prod_{i=1}^N \mathrm{d}\lambda_i\,.$$

By Theorems 3.4 and 3.3, if V satisfies Assumption 3.2, we know that the empirical measure of the λ's converges towards the equilibrium measure μ_V^{eq}. We shall hereafter assume that $\mu_V^{\mathrm{eq}} = \mu_V$ has a disconnected support but a off-critical density in the following assumption.

Assumption 5.1. *$V:\mathbb{R}\to\mathbb{R}$ is of class C^p and μ_V^{eq} has support given by $S=\cup_{i=1}^K[a_i,b_i]$ with $b_i<a_{i+1}<b_{i+1}<a_{i+2}$ and*

$$\frac{d\mu_V}{dx}(x) = H(x)\sqrt{\prod_{i=1}^K (x-a_i)(b_i-x)}$$

where H is a continuous function such that $H(x)\ge \bar{c}>0$ a.e. on S.

We discuss this assumption in Lemma 5.5. Let us notice that the fact that the support μ_V has a finite number of connected components is guaranteed when V is analytic. Also, the fact that the density vanishes as a square root at the boundary of the support is generic, cf [71]. Remember, see Lemma 3.5, that μ_V is described by the fact that the effective potential V_{eff} is non-negative outside of the support of μ_V.We will also assume hereafter that Assumption 3.2 holds and that V_{eff} is strictly positive outside S. By Theorem 3.8, we therefore know that the eigenvalues will remain in $S_\varepsilon=\cup_{i=1}^p S_\varepsilon^i, S_\varepsilon^i:=[a_i-\varepsilon,b_i+\varepsilon]$ with probability greater than $1-e^{-C(\varepsilon)N}$ with some $C(\varepsilon)>0$ for all $\varepsilon>0$. We take ε small enough so that S_ε is still the union of p disjoint connected components $S_i, 1\le i\le p$. Moreover, we will assume that V is C^1 so that the conclusions of Theorem 3.14 and Corollary 3.16 still hold. In particular

Corollary 5.2. *Assume V is C^1. There exists $c>0$ and C finite such that*

$$P_N^{\beta,V}\left(\max_{1\le i\le p}|\#\{j:\lambda_j\in[a_i-\varepsilon,b_i+\varepsilon]\}-N\mu_V([a_i,b_i])|\ge C\sqrt{N}\ln N\right)\le e^{-cN}$$

We can therefore restrict our study to the probability measure given, if we denote by $N_i = \#\{j : \lambda_j \in [a_i - \varepsilon, b_i + \varepsilon]\}$, $\hat{n}_i = N_i/N$ and $\hat{n} = (\hat{n}_1, \ldots, \hat{n}_K)$, by

$$\mathrm{d}P_{N,n}^{\beta,V}(\lambda_1, \ldots, \lambda_N) = 1_{\max_i |N_i - N\mu([a_i,b_i])| \le C\sqrt{N}\ln N} \frac{1_{S_\varepsilon}}{Z_{N,\varepsilon}^{\beta,V}} \Delta(\lambda)^\beta e^{-N\beta \sum V(\lambda_i)} \prod_{i=1}^N \mathrm{d}\lambda_i$$

since exponentially small corrections do not affect our polynomial expansions. As $\varepsilon > 0$ is kept fixed we forget it in the notations and denote

$$\mathrm{d}P_{N,\hat{n}}^{\beta,V}(\lambda_1, \ldots, \lambda_N) = \frac{1_{N_i = \bar{n}_i N} 1_{S_\varepsilon}}{Z_{N,\hat{n}}^{\beta,V}} \Delta(\lambda)^\beta e^{-N\beta \sum V(\lambda_i)} \prod_{i=1}^N \mathrm{d}\lambda_i$$

the probability measure obtained by conditioning the filling fractions to be equal to $\hat{n} = (n_1, \ldots, n_p)$. Clearly, we have

$$Z_N^{\beta,V} = \sum_{|N_i - N\mu([a_i,b_i])| \le C\sqrt{N}\ln N} \frac{N!}{N_1! \cdots N_p!} Z_{N,\hat{n}}^{\beta,V} \tag{63}$$

$$P_N^{\beta,V} = \sum_{|N_i - N\mu([a_i,b_i])| \le C\sqrt{N}\ln N} \frac{N!}{N_1! \cdots N_K!} \frac{Z_{N,\hat{n}}^{\beta,V}}{Z_N^{\beta,V}} P_{N,\hat{n}}^{\beta,V} \tag{64}$$

where the combinatorial term $\frac{N!}{N_1! \cdots N_K!}$ comes from the ordering of the eigenvalues to be distributed among the cuts. Hence, we will retrieve large N expansions of the partition functions and linear statistics of the full model from those of the fixed filling fraction models.

5.1.1 The fixed filling fractions model

To derive central limit theorems and expansion of the partition function for fixed filling fractions we first need to check that we have the same type of results that before we fix the filling fractions. We leave the following Theorem as an exercise, its proof is similar to the proof of Theorem 3.4. Recall the notation :

$$\mathcal{E}(\mu) = \int \int [\frac{1}{2}V(x) + \frac{1}{2}V(y) - \frac{1}{2}\ln|x-y|]\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Theorem 5.3. *Fix $n_i \in (0,1)$ so that $\sum n_i = 1$. Under the above assumptions*

- *Assume that $(\hat{n}_i)_{1 \le i \le K}$ converges towards $(n_i)_{1 \le i \le K}$. The law of the vector of p empirical measures $\hat{\mu}_i^N = \frac{1}{N_i} \sum_{j=N_1+\cdots+N_{i-1}+1}^{N_1+\cdots+N_i} \delta_{\lambda_j}$ under $P_{N,\hat{n}}^{\beta,V}$ satisfies a large deviation principle on the space of p tuples of probability measures on $S_i = [a_i - \varepsilon, b_i + \varepsilon]$, $1 \le i \le p$, in the scale N^2 with good rate function $I_n = J_n - \inf J_n$ where*

$$J_n(\mu_1, \ldots, \mu_p) = \beta\mathcal{E}(\sum_{i=1}^K n_i \mu_i).$$

- *J_n achieves its minimal value uniquely at $(\mu_i^n)_{1\le i\le p}$. Besides there exists p constants C_i^n such that*

$$V_{eff}^n(x) = V(x) - \int \ln|x-y| d(\sum n_i \mu_i^n(y)) - C_i^n \tag{65}$$

is greater or equal to 0 on S_i and equal to 0 on the support of μ_i^n.

- *The conclusions of Lemma 3.14 and Corollary 3.16 hold in the fixed filling fraction case in the sense that for $\hat{n} = N_i/N, \sum N_i = N$ we can smooth $\sum \hat{n}_i \hat{\mu}_i^N = \hat{\mu}^N$ into $\tilde{\mu}_N$ (by pulling appart eigenvalues and taking the convolution by a small uniform variable), so that there exists $c > 0, C_{p,q} < \infty$ such that for $t > 0$*

$$P_{N,\hat{n}}^{\beta,V}\left(D(\tilde{\mu}_N, \sum \hat{n}_i \mu_i^{\hat{n}}) \ge t\right) \le e^{C_{p,q}N\ln N - \beta N^2 t^2} + e^{-cN}$$

Note above that the filling fractions N_i/N may vary when N grows : the first two statements hold if we take the limit, and the last with $\hat{n}_i = N_i/N$ exactly equal to the filling fractions (the measures μ_i^n are defined for any given n_i such that $\sum n_i = 1$). The last result does not hold if $\hat{n}$ is replaced by its limit n, unless $\hat{n}$ is close enough to n. To get the expansion for the fixed filling fraction model it is essential to check that they are off critical if the $\hat{n}_i$ are close to $\mu(S_i)$:

Lemma 5.4. *Assume V is analytic. Fix $\varepsilon > 0$. There exists $\delta > 0$ so that if* $\max_i |n_i - \mu_V(S_i)| \le \delta$, *$(\mu_i^n)_{1\le i\le p}$ are off-critical in the sense that there exists $a_i^n < b_i^n$ in S_ε^i and H_i^n uniformly bounded below by a positive constant on S_ε^i such that*

$$d\mu_i^n(x) = H_i^n(x)\sqrt{(x-a_i^n)(b_i^n-x)}dx\,.$$

Proof. We first observe that $n \to \int f d\mu_i^n$ is smooth for all smooth functions f. Indeed, take two filling fractions n, m and denote in short by $\mu^n = \sum n_i \mu_i^n$. Recall that μ^n minimizes $\mathcal{E}$ on the set of probability measures with filling fractions n. We decompose $\mathcal{E}$ as

$$\mathcal{E}(\nu) = \beta \int V_{\mathrm{eff}}^n(x) d(\nu - \mu^n)(x) + \frac{\beta}{2} D^2(\nu, \mu^n) - \beta \sum C_h^n(\nu([\hat{a}_h, \hat{b}_h]) - n_h) \tag{66}$$

where V_{eff}^n is the effective potential for the measure μ^n. Note here that we used that as $\nu - \mu^n$ has zero mass to write

$$\int \ln|x-y| d(\nu - \mu^n)(x)(\nu - \mu^n)(y) = -D^2(\nu, \mu^n)\,.$$

We then take ν a measure with filling fractions m and since μ^m minimizes $\mathcal{E}$ among such measures,

$$\mathcal{E}(\mu^m) \le \mathcal{E}(\nu)\,. \tag{67}$$

We choose ν to have the same support than μ^n so that $\int V^n_{\rm eff}(x)d(\nu-\mu^n)(x)=0$ and notice that $\int V^n_{\rm eff}(x)d(\mu^m-\mu^n)(x)\ge 0$. Hence, we deduce from (66) and (67) that

$$D^2(\mu^m,\mu^n)\le D^2(\nu,\mu^n)\,.$$

Finally we choose $\nu=\mu^n+\sum_i(m_i-n_i)\frac{1_{B_i}}{|B_i|}dx$ with B_i is an interval in the support of μ^n_i where its density is bounded below by some fixed value. For $\max|m_i-n_i|$ small enough it is a probability measure. Then, it is easy to check that

$$D^2(\mu^m,\mu^n)\le D^2(\mu,\mu^n)\le C\|m-n\|^2_\infty$$

from which the conclusion follows from (27).

Next, we use the Dyson-Schwinger equation with the test function $f(x)=(z-x)^{-1}$ to deduce that $G^n_i(z)=\int(z-x)^{-1}d\mu^n_i(x)$ satisfies the equation

$$G^n_i(z)(\sum n_jG^n_j(z))=\int\frac{V'(x)}{z-x}d\mu^n_i(x)=V'(z)G^n_i(z)+f^n_i(z)$$

where $f^n_i(z)=-\int(V'(y)-V'(z))(y-z)^{-1}d\mu^n_i(y)$. Hence we deduce that

$$G^n_i(z)=\frac{1}{2n_i}\left(V'(z)-\sum_{j\neq i}n_jG^n_j(z)-\sqrt{(V'(z)-\sum_{j\neq i}n_jG^n_j(z))^2-4n_if^n_i(z)}\right)\,.$$

The imaginary part of G^n_i gives the density of μ^n_i in the limit where z goes to the real axis. Since the first term in the above right hand side is obviously real, the latter is given by the square root term and therefore we want to show that

$$F(z,n)=(V'(z)-\sum_{j\neq i}n_jG^n_j(z))^2-4n_if^n_i(z)$$

vanishes only at two points a^n_i,b^n_i for $z\in S_i$. The previous point shows that F is Lipschitz in the filling fraction n as V is C^3 (since then f^i_n is the integral of a C^1 function under μ^n_i) whereas Assumption 5.1 implies that at $n^*_i=\mu_V(S_i)$, F vanishes at only two points and has non-vanishing derivative at these points. This implies that the points where $F(z,n)$ vanishes in S_i are at distance of order at most $\max|n_i-m_i|$ of a_i,b_i. However, to guarantee that there are exactly two such points, we use the analyticity of V which guarantees that $F(.,n)$ is analytic for all n so that we can apply Rouché theorem. As $F(z,n^*)$ does not vanish on the boundary of some compact neighborhood K of a_i, for n close enough to n^*, we have $|F(z,n)-F(z,n^*)|\le|F(z,n^*)|$ for $z\in\partial K$. This guarantees by Rouché's theorem, since $F(.,n)$ is analytic in neighborhood of S_i as V is, that $F(.,n)$ and $F(.,n^*)$ have the same number of zeroes inside K. $\diamond$

To apply the method of Section 3, we can again use the Dyson-Schwinger equations and in fact Lemma 3.17 still holds true : Let $f_i:\mathbb{R}\to\mathbb{R}$ be C^1_b

functions, $0 \leq i \leq p$. Then, taking the expectation under $P^{\beta,V}_{N,\hat{n}}$, we deduce

$$
\begin{aligned}
\mathbb{E}[M_N(\Xi f_0)\prod_{i=1}^{p} N\hat{\mu}_N(f_i)] &= (\frac{1}{\beta}-\frac{1}{2})\mathbb{E}[\hat{\mu}_N(f_0')\prod_{i=1}^{p} N\hat{\mu}_N(f_i)] \\
&+ \frac{1}{\beta}\sum_{\ell=1}^{p}\mathbb{E}[\hat{\mu}_N(f_0 f_\ell')\prod_{i\neq\ell} N\hat{\mu}_N(f_i)] \\
&+ \frac{1}{2}\mathbb{E}[\int \frac{f_0(x)-f_0(y)}{x-y} dM_N(x)dM_N(y)\prod_{i=1}^{p} N\hat{\mu}_N(f_i)] \\
&+O(e^{-cN})
\end{aligned}
$$

where the last term comes from the boundary terms which are exponentially small by the large deviations estimates of Theorem 3.8. We still denoted $M_N(f) = \sum f(\lambda_i) - N\sum \hat{n}_i \mu_i^{\hat{n}}(f)$ but this time the mass in each S_i is fixed so this quantity is unchanged if we change f by adding a piecewise constant function on the S_i's. We therefore have this time to find for any sufficiently smooth function g a function f such that there are constants C_j so that

$$\Xi^{\hat{n}} f(x) = V'(x)f(x) - \sum_{i=1}^{p} \hat{n}_i \int \frac{f(x)-f(y)}{x-y} d\mu_i^{\hat{n}}(y) = g(x) + C_j,\, x \in S_j\,.$$

By the characterization of $\mu^{\hat{n}}$, if $S_j^{\hat{n}} = [a_j^{\hat{n}}, b_j^{\hat{n}}]$ denotes the support of $\mu^{\hat{n}}$ inside S_ε^i, this question is equivalent to find f so that on every $[a_j^{\hat{n}}, b_j^{\hat{n}}]$,

$$\Xi^{\hat{n}} f(x) := PV\int \frac{f(y)}{x-y} H_j^{\hat{n}}(y)\sqrt{(y-a_j^{\hat{n}})(b_j^{\hat{n}}-y)}dy = g(x) + C_j$$

This question was solved in [77] under the condition that g, f are Hölder with some positive exponent. Once one gets existence of these functions, the property of the inverse are the same as before since inverting the operator on one S_i will correspond to the same inversion. For later use, we prove a slightly stronger statement :

Lemma 5.5. *Let $\theta \in [0,1]$ and set for $n_i \in (0,1), \sum n_i = 1$. Let S_i^n denote the support of μ_i^n. We set, for $i \in \{1,\ldots,K\}$, all $x \in S_i^n$*

$$\Xi_\theta^n f(x) := V'(x)f(x) - n_i\int \frac{f(x)-f(y)}{x-y} d\mu_i^n(y) + \theta\sum_{j\neq i} n_j \int \frac{f(x)-f(y)}{x-y} d\mu_j^n(y)\,.$$

Then for all $g \in C^k$, $k > 2$, there exist constants $C_j, 1 \leq j \leq p$, so that the equation

$$\Xi_\theta^n f(x) = g(x) + C_j, x \in S_j^n$$

has a unique solution which is Hölder for some exponent $\alpha > 0$. We denote by $(\Xi_\theta^n)^{-1} g$ this solution. There exists finite constant D_j such that

$$\|(\Xi_\theta^n)^{-1} g\|_{C^j} \leq D_j \|g\|_{C^{j-2}}\,.$$

Proof. Let us first recall the result from [77, section 90] which solves the case $\theta = 1$. Let $S_k^n = [a_k^n, b_k^n]$. Because of the characterization of the equilibrium measure, inverting Ξ_1 is equivalent to seek for f Hölder such that there are K constants $(C_k)_{1\le k\le K}$ such that on S_k

$$K_1 f(t) = PV\int_{\cup S_k} \frac{f(x)}{t-x}dx = g(t) + C_k$$

for all $k \in \{1, \ldots, K\}$. Then, by [77, section 90], if g is Hölder, there exists a unique solution and it is given by

$$K_1^{-1} g(x) := f(x) = \frac{\sigma(x)}{\pi} \sum_k PV\int_{S_k} \frac{dy}{\sigma(y)} \frac{1}{y-x}(g(y) + C_k)$$

where $\sigma(x) = \sqrt{\prod(x-a_i^n)(x-b_i^n)}$. The proof shows uniqueness and then exhibits a solution. To prove uniqueness we must show that $K_1 f = C_k$ has a unique solution, namely zero. To do so one remarks that

$$\Phi(z) = \int_{\cup S_k} \frac{f(x)}{x-z}dx$$

is such that $\Psi(z) = (\Phi(z) - C_k/2)\sqrt{(x-a_k^n)(x-b_k^n)}$ is holomorphic in a neighborhood of S_k^n and vanishes at a_k^n, b_k^n. Indeed, $K_1 f = C_k$ is equivalent to
$\Phi^+(x) + \Phi^-(x) = C_k$ implies that $\Psi^+(x) = \Psi^-(x)$ on the cuts. Hence

$$\Phi(z) - C_k/2 = [(z-a_k^n)(z-b_k^n)]^{1/2}\Omega(z) \tag{68}$$

with Ω holomorphic in a neighborhood of S_k^n, and so $\Phi'(z)\sigma(z)$ is holomorphic everywhere. Hence, since Φ' goes to zero at infinity like $1/z^2$, $P(z) = \Phi'(z)\sigma(z)$ is a polynomial of degree at most $K-2$. We claim that this is a contradiction with the fact that then the periods of Φ vanish, see [42, Section II.1] for details. Let us roughly sketch the idea. Indeed, because $\Phi = u + iv$ is analytic outside the cuts, if $\Lambda = \cup\Lambda_k$ is a set of contours surrounding the cuts and Λ^c the part of the imaginary plan outside Λ, we have by Stockes theorem

$$J = \int_{\Lambda^c} \left((\partial_x u)^2 + (\partial_y u)^2\right) dxdy = \int_\Lambda u d\bar{v}$$

Letting Λ going to S we find

$$\int_\Lambda u d\bar{v} = \int_S u^+ dv^+ - \int_S u^- dv^-$$

But by the condition $\Phi^+ + \Phi^- = C_k$ we see that $u^+ + u^- = \Re(C_k), d(v^+ + v^-) = 0$ and hence

$$J = \sum_k \Re(C_k)\int_{S_k} dv^+ = \sum_k \Re(C_k)(v^+(b_k^n) - v^+(a_k^n))\,.$$

On the other hand $\Phi(z) = \int_{-\infty}^{z} P(\xi)/\sigma(\xi) d\xi$ for any path avoiding the cuts and hence converges towards finite values on the cuts. But since $\Phi'(\xi) = \frac{P(\xi)}{\sigma(x)}$ is analytic outside the cuts, going to zero like $1/z^2$ at infinity,

$$0 = \int_{\Lambda_k} \Phi'(\xi) d\xi = 2 \int_{S_k^n} \Phi'(x) dx = 2(\Phi(b_k^n) - \Phi(a_k^n))$$

Thus, $v(b_k^n) - v(a_k^n) = 0$ and we conclude that $J = 0$. Therefore Φ vanishes, and so does f.

Next, we consider the general case $\theta \in (0,1)$. We show that Ξ_θ^n is injective on the space of Hölder functions. Again, it is sufficient to consider the homogeneous equation

$$K_\theta f(x) = (1-\theta) K_0 f(x) + \theta K_1 f(x) = C_k \tag{69}$$

on S_k for all k. Here $K_0 f(x) = \int_{S_k} \frac{f(y)}{y-x} dy$ on S_k for all k. If K_θ is injective, so is Ξ_θ^n by dividing the function f on S_k by $\sigma_k(x) S_k(x) = d\mu_k^n/dx$. Recall that Tricomi airfol equation shows that K_0 is invertible, see Lemma 3.18, and we have just seen that K_1 is injective. To see that K_θ is still injective for $\theta \in [0,1]$ we notice that we can invert K_1 to deduce that we seek for an Hölder function f $(= K_1 g)$ and a piecewise constant function C so that

$$f(x) = -\frac{1-\theta}{\theta} K_1^{-1}(K_0 f - C)$$

Let us consider this equation for $x \in S_k$ and put $f = K_0^{-1} g$. By the formula for K_1^{-1} and K_0^{-1} we deduce that we seek for constants d, D and a function g so that on S_k :

$$\frac{1}{\sigma_k(x)} PV \int_{S_k} \frac{g(y) + d_k}{x - y} \sigma_k(y) dy = -\frac{1-\theta}{\theta} \frac{1}{\sigma(x)} \sum_{\ell} PV \int_{S_\ell} \frac{g(y) + D_\ell}{y - x} \sigma(y) dy \,.$$

Here, we used a formula for K_0^{-1} where σ_k was replaced by σ_k^{-1} : this alternative formula is due to Parseval formula [90, (2) p.174], see (16) and (18) in [90]. Note here that both side vanish at the end points of S_k by the choices of the constants. As a consequence

$$\frac{1}{\sigma_k(x)} \int_{S_k} \frac{g(y) + d_k}{x - y} \sigma_k(y) dy + \frac{1-\theta}{\theta} \frac{1}{\sigma(x)} \sum_{\ell} \int_{S_\ell} \frac{g(y) + D_\ell}{y - x} \sigma(y) dy$$

is analytic in a neighborhood of S_k.We next integrate over a contour $\mathcal{C}_k$ around

S_k to deduce that

$$
\begin{aligned}
\int_{S_k} \frac{g(y)+d_k}{x-y}\sigma_k(y)dy &= \frac{1}{2\pi i}\int_{\mathcal{C}_k}\frac{dz}{z-x}\int_{S_k}\frac{g(y)+d_k}{z-y}\sigma_k(y)dy \\
&= -\frac{1-\theta}{\theta}\frac{1}{2\pi i}\int_{\mathcal{C}_k}\frac{dz}{z-x}\frac{\sigma_k(z)}{\sigma(z)}\sum_\ell\int_{S_\ell}\frac{g(y)+D_\ell}{y-z}\sigma(y)dy \\
&= -\frac{1-\theta}{\theta}\int_{S_k}\frac{\sigma_k(y)}{\sigma(y)}\frac{g(y)+D_k}{x-y}\sigma(y)dy \\
&= -\frac{1-\theta}{\theta}\int_{S_k}\frac{g(y)+D_k}{x-y}\sigma_k(y)dy
\end{aligned}
$$

where we used that σ_k/σ is analytic in a neighborhood of S_k, as well as the terms coming from the other cuts. Hence we seek for g satisfying

$$\frac{1}{\theta}\int_{S_k}\frac{g(y)+d_k}{x-y}\sigma_k(y)dy = 0$$

for some constant d_k. Tricomi airfol equation shows that this equation has a unique solution which is when $g+d_k$ is a multiple of $1/(\sigma_k)^2$. By our smoothness assumption on g, we deduce that $g+d_k$ must vanish. This implies that $f = K_0^{-1}g$ vanishes by Tricomi. Hence, we conclude that K_θ, and therefore Ξ_θ^n is injective on the space of Hölder continuous functions.

To show that Ξ_θ^n is surjective, it is enough to show that it is surjective when composed with the inverse of the single cut operators $\Xi^n = (\Xi^{n_1}, \ldots, \Xi^{n_p})$, that is that

$$L_\theta f(x) := n_i f(x) + \theta R f(x), Rf(x) = \sum_{j\neq i} n_j \int \frac{(\Xi_j^n)^{-1}f(y)}{x-y}d\mu_j^n(y), x\in S_i = [a_i^n, b_i^n]$$

is surjective. But R is a kernel operator and in fact it is Hilbert-Schmidt in $L^2(\sigma^{-\epsilon}dx)$ for any $\epsilon > 0$ (here $\sigma(x) = \prod\sqrt{(x-a_i^n)(b_i^n-x)}$). Indeed, on $x\in S_i$, R is a sum of terms of the form

$$\int \frac{(\Xi_j^n)^{-1}f(y)}{x-y}d\mu_j^n(y) = \int \frac{1}{x-y}\frac{1}{S_j(y)}PV\left(\int_{a_j^n}^{b_j^n}\frac{f(t)}{(y-t)}\sigma_j(t)dt\right)d\mu_j^n(y)$$

by Remark 3.19. Even though we have a principal value inside the (smooth) integral we can apply Fubini and notice that

$$
\begin{aligned}
PV\int\frac{1}{(x-y)(y-t)}\frac{1}{S_j(y)}d\mu_j^n(y) &= \frac{1}{x-t}PV\int\left(\frac{1}{(x-y)}+\frac{1}{(y-t)}\right)d\sigma_j(y) \\
&= 1-\frac{1}{x-t}\sigma_j(x)
\end{aligned}
$$

where we used that t belongs to S_j but not x to compute the Hilbert transport of σ_j at t and x. Hence, the above term yields

$$\int\frac{(\Xi_j^n)^{-1}f(y)}{x-y}d\mu_j^n(y) = \int_{S_j}\frac{f(t)}{\sigma_j(t)}(1-\frac{1}{x-t}\sigma_j(x))dt, x\in S_i$$

from which it follows that R is a Hilbert-Schmidt operator in $L^2(\sigma^{-\epsilon}dx)$. Hence, R is a compact operator in $L^2(\sigma^{-\epsilon}dx)$. But L_θ is injective in this space. Indeed, for $f \in L^2(\sigma^{-\epsilon}dx)$, $L_\theta f = 0$ implies that $f = \theta n_i^{-1} Rf$ is analytic. Writing back $h = (\Xi^n)^{-1} f$, we deduce that $\Xi^n_\theta h = 0$ with h Hölder, hence h must vanish by the previous consideration. Hence L_θ is injective. Therefore, by the Fredholm alternative, L_θ is surjective. Hence L_θ is a bijection on $L^2(\sigma^{-\epsilon}dx)$. But note that the above identity shows that R maps $L^2(\sigma^{-\epsilon})$ onto analytic functions, therefore K^{-1} maps Hölder functions with exponent α onto Hölder functions with exponent α. We thus conclude that $\Xi^n_\theta = L_\theta \circ \Xi^n$ is invertible onto the space of Hölder functions. We also see that the inverse has the announced property since for $x \in [a^n_j, b^n_j]$

$$(\Xi^n_\theta)^{-1} g(x) = \Xi^{-1}[g - h], \quad h(x) = \theta \sum_{j \neq i} n_j \int \frac{(\Xi^n_\theta)^{-1} g(y)}{x - y} d\mu^n_j(y)]$$

where h is C^∞. The announced bound follows readily from the bound on one cut as on L^k we have

$$(\Xi^n_\theta)^{-1} f(x) = (\Xi^n_0)^{-1}(f - \theta \sum_{\ell \neq k} \int \frac{(\Xi^{n_\theta})^{-1} f}{x - y} d\mu^n_\ell(y))$$

is such that

$$\|(\Xi^n_\theta)^{-1} f\|_{C^s} \leq c_s \|f - \theta \sum_{\ell \neq k} \int \frac{(\Xi^{n_\theta})^{-1} f}{x - y} d\mu^n_\ell(y))\|_{C^{s+2}} \leq \tilde{c}_s(\|f\|_{C^{s+2}} + \|(\Xi^n_\theta)^{-1} f\|_\infty) .$$

$\diamond$

As Ξ^n_1 is invertible with bounded inverse we can apply exactly the same strategy as in the one cut case to prove the central limit theorem :

Theorem 5.6. *Assume V is analytic and the previous hypotheses hold true. Then there exists $\epsilon > 0$ so that for $\max |n_i - \mu([a_i, b_i])| \leq \epsilon$, for any f C^k with $k \geq 11$, the random variable $M_N(f) := \sum_{i=1}^N f(\lambda_i) - N\mu^n(f)$ converges in law under $P^{\beta,V}_{N,n}$ towards a Gaussian variable with mean $m^n_V(f)$ and covariance $C^n_V(f,f)$, which are defined as in Theorem 3.27 but with μ^n instead of μ and Ξ^n instead of Ξ.*

We can also obtain the expansion for the partition function

Theorem 5.7. *Assume V is analytic and the previous hypotheses hold true. Then there exists $\epsilon > 0$ so that for $\max |\hat{n}_i - \mu([a_i, b_i])| \leq \epsilon$, $\hat{n}_i = N_i/N$, we have*

$$\begin{aligned} \ln \left(\frac{N!}{(\hat{n}_1 N)! \cdots (\hat{n}_K N)!} Z^{N,\hat{n}}_{\beta,V} \right) &= C^0_\beta N \ln N + C^1_\beta \ln(N) \\ &+ N^2 F^{\hat{n}}_0(V) + N F^{\hat{n}}_1(V) + F^{\hat{n}}_2(V) + o(1) \end{aligned} \tag{70}$$

with $C^0_\beta = \frac{\beta}{2}$, $C^1_\beta = -(K-1)/2 + \frac{3+\beta/2+2/\beta}{12}$ *and for* $n_i > 0$, $\sum n_i = 1$,

$$\begin{aligned} F^n_0(V) &= -\mathcal{E}(\mu^n_V) \\ F^n_1(V) &= (\frac{\beta}{2}-1)\int \ln(\frac{d\mu^n_V}{dx})d\mu^n_V - \frac{\beta}{2}n_i \ln n_i + f_1 \end{aligned}$$

where f_1 *depends only on the boundary points of the support.* $F^n_2(V)$ *is a continuous function of* n. *Above the error term is uniform on* n *in a neighborhood of* n^*.

Proof. The proof is again by interpolation. We first remove the interaction between cuts by introducing for $\theta \in [0,1]$

$$\mathrm{d}P^{\beta,\theta,V}_{N,\hat{n}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z^{\beta,\theta,V}_{N,\hat{n}}}\prod_{h\neq h'} e^{N^2\frac{\beta}{2}\theta\int \ln|x-y|d(\hat{\mu}^N_h-\mu^{\hat{n}}_h)(x)d(\hat{\mu}^N_{h'}-\mu^{\hat{n}}_{h'})(y)}\prod dP^{\beta,V^{\hat{n}}_{eff}}_{N,\hat{n}_h}$$

where $P^{\beta,V^{\hat{n}}_{eff}}_{N,\hat{n}_h}$ is the β ensemble on S_h with potential given by the effective potential. We still have a similar large deviation principle for the $\hat{\mu}^N_h$ under $P^{\beta,\theta,V}_{N,\hat{n}}$ and the minimizer of the rate function is always $\mu^{\hat{n}}_h$. Hence we are always in a off critical situation. Moreover, we can write the Dyson-Schwinger equations for this model : it is easy to see that the master operator is Ξ^n_θ of Lemma 5.5 which we have proved to be invertible. Therefore, we deduce that the covariance and the mean of linear statistics are in a small neighborhood of $C^{\theta,\hat{n}}_V$ and $m^{\theta,\hat{n}}_V$. It is not hard to see that this convergence is uniform in θ.

Hence, we can proceed and compute

$$\ln\frac{Z^{\beta,1,V}_{N,\hat{n}}}{Z^{\beta,0,V}_{N,\hat{n}}} = N^2\int_0^1 P^{\beta,\theta,V}_{N,\hat{n}}\left(\sum_{h<h'}\beta\int \ln|x-y|d(\hat{\mu}^N_h-\mu^{\hat{n}}_h)(x)d(\hat{\mu}^N_{h'}-\mu^{\hat{n}}_{h'})(y)\right)d\theta$$

Indeed, using the Fourier transform of the logarithm we have

$$\begin{aligned} &N^2 P^{\beta,\theta,V}_{N,\hat{n}}\left(\int \ln|x-y|d(\hat{\mu}^N_h-\mu^{\hat{n}}_h)(x)d(\hat{\mu}^N_{h'}-\mu^{\hat{n}}_{h'})(y)\right) \\ = &\int \frac{1}{t}P^{\beta,\theta,V}_{N,\hat{n}}\left((N\int e^{itx}d(\hat{\mu}^N_h-\mu^{\hat{n}}_h)(x)(N\int e^{-ity}d(\hat{\mu}^N_{h'}-\mu^{\hat{n}}_{h'})(y))\right)dt \end{aligned}$$

where the above RHS is close to

$$[C^{\theta,\hat{n}}_V(e^{it.},e^{-it.}) + |m^{\theta,\hat{n}}_V(e^{it.})|^2]$$

Hence, decoupling the cuts in this way only provides a term of order one in the partition function. It is not hard to see that it will be a continuous function of the filling fraction (as the inverse of $\Xi^{\hat{n}}_\theta$ is uniformly continuous in n). Finally we can use the expansion of the one cut case of Theorem 3.28 to expand $Z^{\beta,0,V}_{N,\hat{n}}$ to conclude. ◇

5.1.2 Central limit theorem for the full model

To tackle the model with random filling fraction, we need to estimate the ratio of the partition functions according to (63). Recall that $n_i^* = \mu([a_i, b_i])$. We can now extend the definition of the partition function to non-rational values of the filling fractions by using Theorem 5.7. Then we have

Theorem 5.8. *Under the previous hypotheses, for* $\max|n_i - n_i^*| \le \epsilon$*, there exists a positive definite form* Q *and a vector* v *such that*

$$\begin{aligned} D(n) &:= \frac{(Nn_1^*)! \cdots (Nn_K^*)!}{(Nn_1! \cdots (Nn_K)!} \frac{Z_{\beta,V}^{N,n}}{Z_{\beta,V}^{N,n^*}} \\ &= \exp\{-\frac{1}{2}Q(N(n-n^*))(1+O(\epsilon)) + \langle N(n-n^*), v\rangle + o(1)\}\,, \end{aligned}$$

where $Z_{\beta,V}^{N,n^*}/(Nn_1^*)! \cdots (Nn_K^*)!$ *is defined thanks to the expansion of* (70) *whenever* $n^* N$ *takes non-integer values (note here the right hand side makes sense for any filling fraction* n*).* $O(\epsilon)$ *is bounded by* $C\epsilon$ *uniformly in* N*. We have* $Q = -D^2 F_0^n(V)|_{n=n^*}$ *and* $v_i = \partial_{n_i} F_1^n(V)|_{n=n^*}$*. As a consequence, since the probability that the filling fractions* $\hat{n}$ *are equal to* n *is proportional to* $D(n)$*, we deduce that the distribution of* $N(\hat{n}-n^*) - Q^{-1}v$ *is equivalent to a centered discrete Gaussian variable with values in* $-Nn^* - Q^{-1}v + \mathbb{Z}$ *and covariance* Q^{-1}*.*

Note here that Nn^* is not integer in general so that $N(\hat{n}-n^*) - Q^{-1}v$ does not live in a fixed space : this is why the distribution of $N(\hat{n}-n^*) - Q^{-1}v$ does not converge in general. As a corollary of the previous theorem, we immediatly have that

Corollary 5.9. *Let* f *be* C^{11}*. Then*

$$\mathbb{E}_{P_{\beta,V}^N}[e^{\sum f(\lambda_i) - N\mu(f)}] = \exp\{\frac{1}{2}C_V^{n^*}(f,f) + m_V^{n^*}(f)\}$$
$$\times \sum_n \frac{\exp\{-\frac{1}{2}Q(N(n-n^*)) + \langle N(n-n^*), v + \partial_n \mu^n|_{n=n^*}(f)\rangle\}}{\sum_n \exp\{-\frac{1}{2}Q(N(n-n^*)) + \langle N(n-n^*), v\rangle\}}(1+o(1))$$

We notice that we have a usual central limit theorem as soon as $\partial_n \mu^n|_{n=n^*}(f)$ vanishes (in which case the second term vanishes), but otherwise the discrete Gaussian variations of the filling fractions enter into the game. This term comes from the difference $N\mu(f) - N\mu^n(f)$.

As is easy to see, the last thing we need to show to prove these results is that

Lemma 5.10. *Assume* V *analytic, off-critical. Then*

- $n \to \mu^n(f)$ *is* C^1 *and* $C_V^n(f,f), m_V^n(f)$ *are continuous in* n*,*
- $n \to F_i^n(V)$ *is* C^{2-i} *in a neighborhood of* n^**,*
- $Q = -D^2 F_0^n(V)|_{n=n^*}$ *is definite positive.*

Let us remark that this indeed implies Corollary 5.9 and Theorem 5.8 since by Theorem 5.7 we have for $|n_i - n_i^*| \le \epsilon$

$$\begin{aligned}
\ln \frac{(Nn_1^*! \cdots (Nn_K^*)!}{(Nn_1)! \cdots (Nn_K)!} \frac{Z_{\beta,V}^{N,n}}{Z_{\beta,V}^{N,n^*}} &= N^2\{F_0^n(V) - F_0^{n^*}(V)\} + N(F_1^n(V) - F_1^{n^*}(V)) \\
&\quad + (F_2^n(V) - F_2^{n^*}(V)) + o(1) \\
&= -\frac{1}{2} Q(N(n-n^*), N(n-n^*))(1+O(\epsilon)) \\
&\quad + \partial_n F_1^n(V)|_{n=n^*}.(N(n-n^*))
\end{aligned}$$

where we noticed that $\partial_n F_0^n(V)$ vanishes at n^* since n^* minimizes F_0^n and Q is definite positive. Hence we obtain the announced estimate on the partition function. About Corollary 5.9 we have by (63) and by conditionning on filling fractions

$$\begin{aligned}
\mathbb{E}_{P_{\beta,V}^N}[e^{\sum f(\lambda_i) - N\mu_V(f)}] &= \mathbb{E}\left[e^{N(\mu^{\hat{n}}(f) - \mu^{n^*}(f))} \mathbb{E}_{P_{\beta,V}^{N,\hat{n}}}[e^{\sum f(\lambda_i) - N\mu^{\hat{n}}(f)}]\right] \\
&\simeq \mathbb{E}\left[e^{N\langle \hat{n} - n^*, \partial_n \mu^n(f)|_{n=n^*}\rangle} \mathbb{E}_{P_{\beta,V}^{N,\hat{n}}}[e^{\sum f(\lambda_i) - N\mu^{\hat{n}}(f)}]\right](1+o(1))
\end{aligned}$$

So we only need to prove Lemma 5.10.

Proof. $n \to \mu^n$ is twice continuously differentiable. We have already seen in the proof of Lemma 5.4 that $n \to \mu^n$ is Lipschitz for the distance D for n in a neighborhood of n^*. This implies that $\nu_\epsilon = \epsilon^{-1}(\mu^{n+\epsilon\kappa} - \mu^n)$ is tight (for the distance D and hence the weak topology). Let us consider a limit point ν and its Stieltjes transform $G_\nu(z) = \int (z-x)^{-1} d\nu(x)$. Along this subsequence, the proof of Lemma 5.4 also shows that $\epsilon^{-1}(a_i^{n+\epsilon\kappa} - a_i^n)$ has a limit (and similarly for b^n, as well as H_i^n). Hence, we see that ν is absolutely continuous with respect to Lebesgue measure, with density blowing up at most like a square root at the boundary. By (65) in Theorem 5.3 we deduce that

$$G_\nu(E+i0) + G_\nu(E-i0) = 0$$

for all E inside the support of μ^n. This implies that $\sqrt{\prod(z-a_i^n)(b_i^n - z)}G_\nu(z)$ has no discontinuities in the cut, hence is analytic. Finally, G_ν goes to zero at infinity like $1/z^2$ so that $\sqrt{\prod(z-a_i^n)(b_i^n - z)}G_\nu(z)$ is a polynomial of degree at most $p-2$. Its coefficients are uniquely determined by the $p-1$ equations fixing the filling fractions since for a contour C_i^n around $[a_i^n, b_i^n]$

$$\int_{C_i^n} G_{\mu^n}(z)dz = n_i \Rightarrow \int_{C_i^n} G_\nu(z)dz = \kappa_i\,.$$

There is a unique solution to such equations. As it is linear in κ, it is given by

$$G_\nu(z) = \sum \kappa_i \omega_i^n(z) \tag{71}$$

where $\omega_i^n(z) = P_i^n(z)/\sqrt{\prod(z-a_i^n)(b_i^n-z)}$ satisfy

$$\int_{C_j^n} \omega_i^n(z)dz = \delta_{i,j} \tag{72}$$

and P_i^n are polynomials of degree smaller or equal than $p-2$. Hence G_ν is uniquely determined as well as ν, we conclude that $n \to \mu^n$ is differentiable, as well as a_i^n, b_i^n. The latter implies that $n \to \omega_i^n$ is as well differentiable and hence $n \to G_{\mu^n}$ is twice continuously differentiable. In turn, we conclude that a_i^n, b_i^n, H_i^n are twice continuously differentiable with respect to n, and therefore so is the density of μ^n.

$C_V^n(f,f), m_V^n(f)$ *are continuous in* n. From the continuity of $d\mu^n/dx$ we deduce that Ξ^n is continuous, and since Ξ^n has uniformly bounded inverse (provided we take sufficiently smooth functions), we deduce that $(\Xi^n)^{-1}$ is continuous in n, from which the continuity of $C_V^n(f,f), m_V^n(f)$ follows for smooth enough f.

$n \to F_i^n(V)$ *is* C^{2-i}, $i = 0,1,2$. For $i = 1$, by the formulas of Theorem 5.6, it is a straightforward consequence of the fact that $\mathrm{d}\mu^n/dx$ is continuously differentiable and its differential is integrable. It amounts to show that the inverse of the operators Ξ_θ^n are continuous in n, but again this is due to the continuity of the endpoints and the explicit formulas we have.

$D^2F_0^n(V)$ *is well defined and definite negative at* $n = n^*$. Set

$$\nu_\eta = \lim_{t\to 0} \frac{\mu^{n+t\eta} - \mu^n}{t} \tag{73}$$

By the formula for J in terms of the effective potential

$$\begin{aligned} F_0^{n^*+t\eta}(V) - F_0^{n^*}(V) &= \left(J[\mu^{n^*+t\eta}] - J[\mu^{n^*}]\right) \\ &= -\frac{\beta}{2}\left(D^2[\mu^{n^*+t\eta}, \mu^{n^*}] - \iint V_{eff}^{n^*}(x) d(\mu^{n^*+t\kappa} - \mu^{n^*})(x)\right) \end{aligned}$$

where we used that at $n = n^*$ the constants in the effective potential are all equal and that $\sum \eta_i = 0$. Since $V_{eff}^{n^*}$ vanishes on $\cup[a_i, b_i]$ as well as its derivative and the derivatives of $\epsilon \to \mu_{n+\epsilon\eta}$ are smooth and supported in $\cup[a_i, b_i]$, we deduce that $F^0_{{}^*n+t\eta}$ is a $\mathcal{C}^2$ function of t and its Hessian is :

$$\partial_t^2 F_0^{n^*+t\eta}|_{t=0} = -\frac{\beta}{2} D^2[\nu^*, \nu^*] \tag{74}$$

where $\nu^* = \partial_t \mu^{n^*+t\eta}|_{t=0}$. D^2F_0 vanishes only when ν^* vanishes, which implies $\eta = 0$ by (71), since no non trivial combination of the ω_i^n can vanish uniformly by (72). Therefore, the Hessian is definite negative. $\diamond$

5.2 Several cut case: discrete Beta-ensembles and tiling

The same ideas can be developed for discrete Beta -ensembles but it becomes much more technical. This is the goal of a manuscript with G. Borot and V.

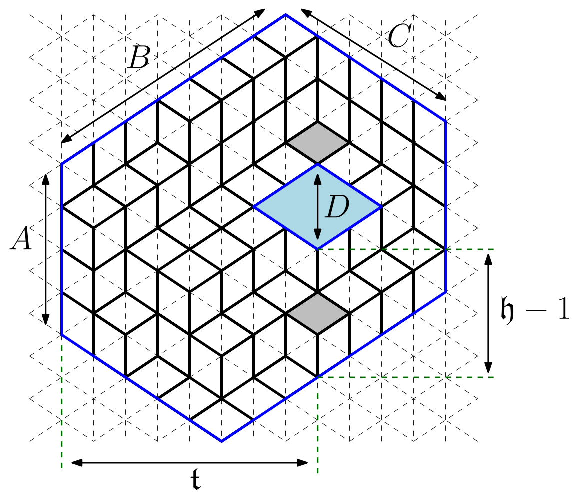

Figure 2: Oshaped domain: hexagon with a hole

Gorin that we should post by the summer in the form of a book. For instance, one of our goal is to derive fluctuations for tilings in domains which are not simply connected such as a 0-shaped domain, or a C shaped domain, and prove that fluctuations are still given by a Gaussian Free Field (as conjectured by Kenyon and Okounkov).

Let us simply outline the difficulties :

1. We first have to show that there are nor particles in void regions nor holes in saturated regions to check that these regions can be considered as frozen/deterministic. This can be done by large deviations arguments with rate functions given in terms of the effective potentials, but this is more technical than in the continuous case, in particular because the speed is N and the density is a product of N^2 terms, so errors have to be precisely estimated.

2. We then would like to fix filling fractions and extrapolate the partition function of our conditioned model with the one of independent models with one band per segment, each of these models being interpolated with a model for which we can compute the partition functions (generalizing the Gaussian ensembles and Selberg integrals in the continuous case). This raised several difficulties. It turns out that we can compute the partition function of Jack polynomials, see subsection 5.2.1, which plays the role of the Gaussian ensembles. They corresponds to one cut models with the four possible boundary conditions. To estimate the partition functions of more general one cut models, we need to proceed by interpolation which satisfy all our conditions to apply and solve Nekrasov's equations. This is much more technical than in the continuous case.

3. To estimate the original several cuts models, we need to condition by random variables such as filling fractions and interpolate the conditioned model with independent one cut models. Conditioning by filling fractions is not as clear as before because of the saturated regions and the fact that the lattice on which the particles live depends on the number of particles and their position. To this end we need to condition rather on positions of particles and filling fractions, and find the right random variables on which we can condition and be able to solve the Nekrasov equations. Interpolating between models become much more complicated to make sure that each steps of our interpolation satisfy the conditions to derive Nekrasov's equations.

4. One needs to solve these equations and identify the different terms: this uses algebraic geometry.

5.2.1 Expansion of the partition function

To expand the partition function in the spirit of what we did in the continuous case, we need to compare our partition function to one we know. In the continuous case, Selberg integrals were computed by Selberg. In the discrete case it turns out we can compute the partition function of binomial Jack measure [14] which corresponds to the choice of weight depending on two positive real parameters $\alpha, \beta > 0$ given by :

$$w_J(\ell) = (\alpha\beta\theta)^{\ell} \frac{\Gamma(M + \theta(N-1) + \frac{3}{2})}{\Gamma(\ell+1)\Gamma(M + \theta(N-1) + 1 - \ell)} \tag{75}$$

Then, the partition function can be computed explicitely and we find (see the work in progress with Borot and Gorin [17]) :

Theorem 5.11. *With summation going over $(\ell_1, \ldots, \ell_N)$ satisfying $\ell_1 \in \mathbb{Z}_{\geq 0}$ and $\ell_{i+1} - \ell_i \in \{\theta, \theta+1, \theta+2, \ldots\}$, $i \in \{1, \ldots, N-1\}$, we have*

$$\begin{aligned}
\mathbf{Z}_N^J &= \sum \prod_{1 \leq i < j \leq N} \frac{1}{N^{2\theta}} \frac{\Gamma(\ell_{i+1} - \ell_i + 1)\Gamma(\ell_{i+1} - \ell_i + \theta)}{\Gamma(\ell_{i+1} - \ell_i)\Gamma(\ell_{i+1} - \ell_i + 1 - \theta)} \prod_{i=1}^{N} w_J(\ell_i) \\
&= (1 + \alpha\beta\theta)^{MN} (\frac{\alpha\beta\theta}{N^2})^{\theta \frac{N(N-1)}{2}} \prod_{i=1}^{N} \frac{\Gamma(\theta(N+1-i))\Gamma(M + \theta(N-1) + \frac{3}{2})}{\Gamma(\theta)\Gamma(M + 1 + \theta(i-1))}.
\end{aligned}$$

On the other hand, the equilibrium measure μ_J for this model can be computed and we find that if $\frac{M}{N} \to (\mathbf{m} - \theta)$ and $q = \alpha\beta\theta$, there exists $\alpha, \beta \in (0, \mathbf{m})$ so that μ_J has density equal to 0 or $1/\theta$ outside (α, β), and in the liquid region (α, β) the density is given by :

$$\mu_J(x) = \frac{1}{\pi\theta} \operatorname{arccot}\left(\frac{x(1-q) + q\mathbf{m} - q\theta - \theta}{\sqrt{((x(1-q) + q\mathbf{m} - q\theta - \theta))^2 + 4xq(\mathbf{m} - x)}} \right),$$

where arccot is the reciprocal of the cotangent function. Therefore, depending on the choices of the parameters, the behavior of $\mu_J(x)$ as x varies from 0 to $\mathbf{m}$ is given by the following four scenarios (it is easy to see that all four do happen)

- Near zero $\mu_J(x) = 0$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = \theta^{-1}$ near $\mathbf{m}$;
- Near zero $\mu_J(x) = \theta^{-1}$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = \theta^{-1}$ near $\mathbf{m}$;
- Near zero $\mu_J(x) = 0$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = 0$ near $\mathbf{m}$;
- Near zero $\mu_J(x) = \theta^{-1}$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = 0$ near $\mathbf{m}$.

We want to interpolate our model with weight w with a Jack binomial model with weight w_J. To this end we would like to consider a model with the same liquid/frozen/void regions so that the model with weight $w^t w_J^{1-t}$, $t \in [0,1]$, corresponds to an equilibrium measure with the same liquid/frozen/void regions and an equilibrium measure given by the interpolation between both equilibrium measure. However, doing that we may have problems to satisfy the conditions of Nekrasov's equations if w/w_J may vanish or blow up. It is possible to circumvent this point by proving that the boundary points are frozen with overwhelming probability, hence allowing more freedom with the boundary point. However, constructing an interpolation that satisfy all our hypotheses is far from easy and complexifies quite a lot the analysis. In these lecture notes, we will not go to this technicality.

Theorem 5.12. *Assume there exists M, q so that $\ln(w/w_J)$ is approximated, uniformly on $[\hat{a}, \hat{b}]$ by*

$$\ln \frac{w}{w_j}(Nx) = -N(V - V_J)(x) + \Delta_1 V(x) + \frac{1}{N}\Delta_2 V(x) + o(\frac{1}{N}).$$

where $V - V_J$ and $\Delta_1 V$ are analytic in $\mathcal{M}$, whereas $\Delta_2 V$ is bounded continuous on $[\hat{a}, \hat{b}]$. Assume moreover that $\phi_N^{\pm}$ satisfies Assumption 4.14. Then, we have

$$\ln \frac{Z_N^{\theta,w}}{Z_N^J} = -N^2 F_0(\theta, V) + N F_1(\theta, w) + F_0(\theta, w) + o(1)$$

with

$$\begin{aligned}
F_0(\theta, V) &= -2\theta\mathcal{E}(\mu) + 2\theta\mathcal{E}(\mu_J) \\
F_1(\theta, V) &= \frac{1}{2\pi i}\int_0^1 \int_C (V_J - V)(z) m_t(z) dt + \frac{1}{2\pi i}\int_0^1 \int_C \Delta_1 V(z) G_t(z) dt \\
F_2(\theta, V) &= \frac{1}{2\pi i}\int_0^1 \int_C \left((V_J - V)(z) r_t(z) + \Delta_1 V(z) m_t(z) + \Delta_2 V(z) G_t(z)\right) dz dt
\end{aligned}$$

Proof. We consider P_N^{θ, w_t} the discrete β model with weight $w^t w_J^{1-t}$. We have

$$\begin{aligned}
\ln \frac{Z_N^{\theta,w}}{Z_N^J} &= \int_0^1 P_N^{\theta, w_t}(\sum_i \ln \frac{w}{w_J}(\ell_i)) dt \\
&= \int_0^1 P_N^{\theta, w_t}(\hat{\mu}^N \left(N^2(V_J - V) + N\Delta_1 V + \Delta_2 V\right)) dt + o(1).
\end{aligned}$$

Denote μ_t the equilibrium measure for $w^t w_J^{1-t}$. Clearly

$$\lim_{N\to\infty} \int_0^1 P_N^{\theta,w_t}(\hat{\mu}^N(\Delta_2 V))dt = \int_0^1 \mu_t(\Delta_2 V)\, dt\,.$$

For the first two terms we use the analyticity of the potentials and Cauchy formula to express everything in terms of Stieltjes functions

$$\int_0^1 P_N^{\theta,w_t}(\hat{\mu}^N\left(N^2(V_J - V) + N\Delta_1 V\right))dt$$

$$= \frac{1}{2\pi i}\int_0^1 \int_C \left(N^2(V_J - V) + N\Delta_1 V\right)(z) P_N^{\theta,w_t}(G_N(z))dzdt\,.$$

We then use Lemma 4.22 since all our assumptions are verified. This provides an expansion :

$$\ln \frac{Z_N^{\theta,w}}{Z_N^J} = -N^2 F_0(\theta,V) + NF_1(\theta,w) + F_0(\theta,w) + o(1)\,.$$

Again by taking the large N limit we can identify $F_0(\theta,V) = -\mathcal{E}(\mu_V)$. For F_1 we find

$$F_1(\theta,w) = \frac{1}{2\pi i}\int_0^1 \int_C (V_J - V)(z) m_t(z) dt + \frac{1}{2\pi i}\int_0^1 \int_C \Delta_1 V(z) G_t(z) dt$$

and

$$F_2(\theta,w) = \frac{1}{2\pi i}\int_0^1 \int_C \left((V_J - V)(z) r_t(z) + \Delta_1 V(z) m_t(z) + \Delta_2 V(z) G_t(z)\right) dzdt$$

$\diamond$

5.2.2 CLT for several cuts domains

Once the partition functions are expanded in terms of the filling fractions, or the more general random variables by which we condition, we can study the fluctuations of these variables and show they are governed by a discrete Gaussian variables as in the continuous case. Moreover, general linear statistics satisfy a standard central limit theorem conditionnally to these variables, so their fluctuations can be decomposed as the sum of a discrete Gaussian and a continuous one, as in the case of continuous Beta-ensembles. We state one of the main theorem of [17].

Theorem 5.13. *Consider a Beta discrete model satisfying some technical Assumptions and suppose that the equilibrium measure has one band per segment. Let $\epsilon > 0$ be small enough, so that the ϵ-neighborhoods of the bands are pairwise disjoint. Let $L \in \mathbb{Z}_{>0}$ and a L-tuple of integers $\boldsymbol{k} \in [H]^L$ both independent of $\mathbb{N}$, let a (possibly $\mathbb{N}$-dependent) L-tuple of functions $\boldsymbol{f}(z)$ such that $f_l(z)$*

is a holomorphic function of z in a $\mathbb{N}$-independent complex neighborhood of $[\hat{a}_{k_l} - \epsilon, \hat{b}_{k_l} + \epsilon]$ for any $l \in [L]$. Assume there exists a constant $C > 0$ such that $\max_l \sup_z |f_l(z)| \leq C$. Let $\textbf{\textsf{Gauß}}[\boldsymbol{f}]$ be a L-dimensional random Gaussian vector with certain covariance and mean Let $\textbf{\textsf{Gauß}}_{\mathbb{Z}}$ be a discrete Gaussian variable (with appropriate mean and covariance) Then, the distribution of $\left(\sum[f_l](\ell_i^{k_l}/\mathcal{N}) - \mathcal{N}\int_{\alpha_{k_l}}^{\beta_{k_l}} f_l(x)\mu_{h^{k_l}}(x)\mathrm{d}x\right)_{l=1}^{L}$ is asymptotically equal to the distribution of

$$\textbf{\textsf{Gauß}}[\boldsymbol{f}] + \Big(\big\langle(\textbf{\textsf{Gauß}}_{\mathbb{Z}} - \mathcal{N}\hat{\boldsymbol{n}}^{\mu}) \cdot \boldsymbol{\omega}_{k_l}[f_l]\big\rangle\Big)_{l=1}^{L}$$

where

$$\boldsymbol{\omega}_k[f] = \left(\int_{\alpha_k}^{\beta_k} f(x)\big(\partial_{\hat{p}_l}\mu_{h^k}^{\hat{\boldsymbol{p}}}(x)\big)|_{\hat{\boldsymbol{p}}=\hat{\boldsymbol{n}}^{\mu}}\, dx\right)_{l=1}^{K}.$$

Interestingly, tiling models generically satisfy these assumptions.

5.3 Beta-models with complex potentials

In this section, we are interested in estimating the partition function for β-ensembles with complex potentials, as in [64]. More precisely, we study

$$\mathcal{Z}_{V,\Gamma}(N,\beta) = \int_{\Gamma^N} \prod_{1\leq i<j\leq N} (z_i - z_j)^{\beta} e^{-N\beta\sum_{k=1}^{N} V(z_k)}\, d\mathbf{z}$$

where V is a polynomial with complex coefficients, $\beta \in 2\mathbb{N}_+$ is an even integer and $\Gamma \subset \mathbb{C}$ is an unbounded contour such that the integral converges. Because the potential has complex coefficients, the integrand is complex and the integral oscillatory. Therefore, the previous considerations do not hold. The natural questions we would like to answer are:

- Can we expand $\mathcal{Z}_{V,\Gamma}(N,\beta)$ as N goes to infinity ?
- Does the integral concentrate on a set where the empirical measure $\frac{1}{N}\sum \delta_{z_i}$ is close to some measure ν on $\mathbb{C}$? And then can we study the fluctuations ?

Let us roughly state the main theorem of [64], which introduces the notion of "one-cut regular" polynomial which generalizes that of one-cut off-critical potential of Section 3.

Theorem 5.14 (Expansion of the complex partition function)**.** *Let $\beta \in 2\mathbb{N}$. Let V be a "one cut-regular" polynomial with higher degree term z^{κ}/κ and let $\Gamma \subset \mathbb{C}$ be a simple (unbounded) contour consisting of a finite number of C^1 arcs, going out to infinity in the directions $e^{\frac{2\pi i\alpha}{\kappa}}$ and $e^{\frac{2\pi i\alpha'}{\kappa}}$, $\alpha, \alpha' \in [0, \kappa-1]$. Then there* *is a sequence of complex numbers $(F_k(\beta,V))_{k=-2}^{\infty}$ such that for every $K\in\mathbb{N}$ we have*

$$\ln \mathcal{Z}_{V,\Gamma}(N,\beta) = \frac{\beta}{2}N\ln N + \frac{3+\frac{\beta}{2}+\frac{2}{\beta}}{12}\ln N + \sum_{k=-2}^{K}\frac{F_k(\beta,V)}{N^k} + \mathcal{O}(N^{-K-1})\,.$$

Moreover, the definitions of the terms F_k extend the usual real case.

About linear statistics, we define the associated (complex) Gibbs measure

$$\mathbb{P}^{N,\beta}_{V,\Gamma}(dz) = \frac{1}{\mathcal{Z}_{V,\Gamma}(N,\beta)}1_{\Gamma^N}\prod_{1\le i<j\le N}(z_i-z_j)^{\beta}e^{-N\beta\sum_{k=1}^{N}V(z_k)}\,d\mathbf{z}$$

and prove

Theorem 5.15. *Let $\beta\in 2\mathbb{N}$. Let V be a "one cut-regular" polynomial and $\Gamma\subset\mathbb{C}$ be a smooth enough simple contour consisting going out to infinity in the directions $e^{\frac{2\pi i\alpha}{\kappa}}$ and $e^{\frac{2\pi i\alpha'}{\kappa}}$, $\alpha,\alpha'\in[0,\kappa-1]$. There exists a probability measure μ_V so that for every polynomial P,*

- $\lim_{N\to\infty}\int P(z_1)d\mathbb{P}^{N,\beta}_{V,\Gamma}(dz) = \int P(z)d\mu_V(z)\,.$
- *The cumulants of $Z_N(P) = \sum_{i=1}^{N}P(z_i) - N\int P(z_1)d\mathbb{P}^{N,\beta}_{V,\Gamma}(dz)$ converge towards zero except for the second one.*

Prior results for $\beta=2$ and V quartic or cubic were obtained in Barhoumi, Bleher, Deano, Yattselev '22, Bleher, Gharakhloo, and McLaughlin '24. Bertola, Bleher, Gharakhloo, McLaughlin, and Tovbis '22: set of indices of regular polynomials is open. Conjecture: almost surely true.

To obtain these results, there are two very useful properties :

- By Cauchy's theorem and the analyticity of the function we are integrating, for every simple contours $\Gamma,\Gamma'\subset\mathbb{C}$ consisting of a finite number of C^1 arcs and stretching out to infinity in the directions $e^{\frac{2\pi i\alpha}{\kappa}}$ and $e^{\frac{2\pi i\alpha'}{\kappa}}$.

  $$\mathcal{Z}_{V,\Gamma}(N,\beta) = \mathcal{Z}_{V,\Gamma'}(N,\beta)$$

- For every contour Γ and potential V as above,

  $$|\mathcal{Z}_{V,\Gamma}(N,\beta)| \le \mathcal{Z}^{\mathrm{Re}}_{V,\Gamma}(N,\beta)$$

  with, if $\varphi(z)=\Re(V(z))$,

  $$\mathcal{Z}^{\mathrm{Re}}_{V,\Gamma}(N,\beta) = \int_{\Gamma^N}\prod_{1\le i<j\le N}|z_i-z_j|^{\beta}e^{-N\beta\sum_{k=1}^{N}\varphi(z_k)}\,d|z|$$

Combining these two properties, we get

$$|\mathcal{Z}_{V,\Gamma}(N,\beta)| \leq \min_{\Gamma'} \mathcal{Z}^{\mathrm{Re}}_{V,\Gamma'}(N,\beta) \tag{76}$$

Heuristically we want to argue that this bound is approximately an equality in the sense that there exists a contour Γ_V so that $|\mathcal{Z}_{V,\Gamma}(N,\beta)| \sim \mathcal{Z}^{\mathrm{Re}}_{V,\Gamma_V}(N,\beta)$. In fact, we prove that

- We can find a contour Γ_V, the S-curve, with a smooth parametrization $\gamma : \mathbb{R} \to \Gamma_V$ so that

$$\begin{aligned} \mathcal{Z}_{V,\Gamma}(N,\beta) &= \int_{\Gamma_V^N} \prod_{1\leq i<j\leq N} (z_i - z_j)^\beta e^{-N\beta \sum_{k=1}^N V(z_k)}\, d\mathbf{z} \\ &= C_N \int_{\mathbb{R}^N} e^{iX_N} d\mathbb{P}_V^{\mathrm{Re,N}}(\mathbf{x}) \end{aligned}$$

  where C_N is a (complex) constant whose asymptotics can be computed and

$$d\mathbb{P}_V^{\mathrm{Re,N}}(\mathbf{x}) = \frac{1}{Z_V^{\mathrm{Re,N}}} \prod_{i<j} |\gamma(x_i) - \gamma(x_j)| e^{-N\beta \sum_{k=1}^N \mathrm{Re} V\circ\gamma(x_k)} \prod |\gamma'(x_k)| dx_k\, .$$

  Moreover, the real valued random variable X_N decomposes as

$$X_N = \frac{\beta}{2} N^2 \int a(x,y) d(L_N - \mu^*)(x) d(L_N-\mu^*)(y) + (1-\frac{\beta}{2}) N \int p(x) d(L_N - \mu^*)(x)$$

  with two smooth functions a, p and $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. Moreover, under $\mathbb{P}_V^{\mathrm{Re,N}}$, L_N converges towards some $\mu^* = C\sqrt{x(1-x)}dx$

- Under $\mathbb{P}_V^{\mathrm{Re,N}}$, $N\int f d(L_N - \mu^*)$ converges towards a Gaussian for smooth f. Hence X_N converges in law towards a quadratic form in Gaussian variables. Thus $\mathcal{Z}_{V,\Gamma}(N,\beta)/C_N$ converges towards a non zero constant.

We therefore need to analyze the real beta-ensembles

$$\mathcal{Z}^{\mathrm{Re}}_{V,\Gamma}(N,\beta) = \int_{\Gamma^N} \prod_{1\leq i<j\leq N} |z_i - z_j|^\beta e^{-N\beta \sum_{k=1}^N \varphi(z_k)}\, d|z|$$

By collecting the result of the previous section, we find that

Theorem 5.16. *Let*

$$\begin{aligned} I^\varphi[\mu] &\stackrel{\mathrm{def}}{=} \int_{\Gamma^2} \psi(z,w)\, d\mu(z) d\mu(w) \\ \psi(z,w) &\stackrel{\mathrm{def}}{=} \frac{\beta}{2} \ln \frac{1}{|z-w|} + \varphi(z) + \varphi(w), \quad \varphi \stackrel{\mathrm{def}}{=} \Re V \end{aligned}$$

Then, if φ is continuous, going to infinity like $|x|^\kappa/\kappa$,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathcal{Z}^{\mathrm{Re}}_{V,\Gamma}(N,\beta) = - \inf_{\mu\in\mathcal{P}(\Gamma)} I^\varphi[\mu] \,.$$

The infimum is achieved at a unique probability measure μ_φ called the equilibrium measure.

We next want to optimize over the choice of admissible contours in (76). Following [72], we define the set of admissible contours $\Gamma \in \mathcal{T}(\alpha,\alpha')$ is defined as follows

1. Γ is a finite union of C^1 Jordan arcs, Γ is connected.
2. Γ stretches out to infinity in the sectors $S_\alpha = \{z\in\mathbb{C} : |\arg(z)-\alpha| \le \pi/2\kappa\}$ and $S_{\alpha'}$.
3. For every $\ell \in \{1,\dots,\kappa\} \setminus \{\alpha,\alpha'\}$, there is an $R>0$ sufficiently large so that $\Gamma \cap (S_{2\pi(\ell-1)/\kappa} \setminus D_R) = \emptyset$.

We say that $\Gamma_\varphi \in \mathcal{T}(\alpha,\alpha')$ solves the max-min energy problem if

$$\sup_{\tilde\Gamma\in\mathcal{T}(\alpha,\alpha')} \inf_{\mu\in\mathcal{P}(\tilde\Gamma)} I^\varphi[\mu] = \inf_{\mu\in\mathcal{P}(\Gamma_\varphi)} I^\varphi[\mu] \,.$$

Theorem 5.17. *If V is a polynomial, there exists a contour $\Gamma_\varphi \in \mathcal{T}(\alpha,\alpha')$ solving the max-min energy problem.*

Denote $U^\mu(z) = \int \ln \frac{1}{|z-x|} d\mu(x)$ and $C^\mu(z) = \frac{1}{2\pi i}\int \frac{1}{x-z} d\mu(x)$.The S-curve is defined as follows [72]

Definition 5.18 (S-curve)**.** *$\Gamma \in \mathcal{T}(\alpha,\alpha')$ is an S-curve if there is a set of zero capacity E such that for any $z \in \operatorname{supp}\mu_\varphi \setminus E$ there is a neighbourhood $D \ni z$ such that $D \cap \operatorname{supp}\mu_\varphi$ is an analytic arc and*

$$\frac{\partial}{\partial n_+} \left(U^{\mu_\varphi} + \varphi\right)(z) = \frac{\partial}{\partial n_-} \left(U^{\mu_\varphi} + \varphi\right)(z) \forall z \in \operatorname{supp}\mu_\varphi \setminus E \qquad (77)$$

where $\frac{\partial}{\partial n_\pm}$ are the normal derivatives taken on either side of the contour.

Theorem 5.19. *Γ_φ is an S-curve in the external field $\varphi = \Re V$.*

We can finally define the notion of regularity following [72]

Definition 5.20. *Let $V(z) = \frac{z^\kappa}{\kappa} + \mathcal{O}(z^{\kappa-1})$ be a polynomial potential and $\Gamma_\varphi \in \mathcal{T}(\alpha,\alpha')$ be a solution of the associated max-min energy problem. Then we say that V is* one-cut regular *if*

1. *The support of the equilibrium measure μ_φ on Γ_φ is connected.*

2. *It is possible to choose the curve $\Gamma_\varphi \in \mathcal{T}(\alpha, \alpha')$ solving the max-min energy problem such that the polynomial R*

$$R(z) = \left(2\pi i \, \mathsf{C}^{\mu_\varphi}(z) + V'(z)\right)^2 z \in \mathbb{C} \setminus \operatorname{supp} \mu_\varphi \tag{78}$$

   with $C^\mu(z) = \frac{1}{2i\pi} \int \frac{1}{x-z} d\mu(x)$, has simple zeros at the endpoints of the support of the equilibrium measure and no other zeros on Γ_φ. Then

$$d\mu_\varphi(z) = \frac{1}{i\pi} \sqrt{R(z)_+} \, dz.$$

   $\sqrt{R(z)_+}$ denotes the limiting value on the left of $\operatorname{supp} \mu_\varphi$ *with respect to its orientation.*

It is important to show that if V be one-cut regular, we can indeed find a nice contour with the above properties. Indeed, we show that there exists an admissible contour $\Gamma \in \mathcal{T}(\alpha, \alpha')$ with the following properties.

1. Γ solves the max-min energy problem in external field $\varphi = \Re V$.
2. Γ is the homeomorphic image of $\mathbb{R}$ under an infinitely differentiable injective function γ, $\gamma : \mathbb{R} \longrightarrow \mathbb{C}, \gamma(\mathbb{R}) = \Gamma$. $|\gamma(x)| \to +\infty$ as $x \to \pm\infty$ and there exist $K > 0$ such that

$$\begin{aligned} \arg \gamma(x) &= \frac{2\pi\alpha}{\kappa} & \forall x \in (-\infty, -K], \\ \arg \gamma(x) &= \frac{2\pi\alpha'}{\kappa} & \forall x \in [K, +\infty). \end{aligned}$$

3. For any $z \in \Gamma \backslash \operatorname{supp} \mu_\varphi$, $\varphi_{\text{eff}}(z) = \varphi(z) + U^{\mu_\varphi^\Gamma} - C_\varphi$ vanishes on the support $\gamma([0,1])$ of μ_φ^Γ and is strictly positive outside.

This implies that $\liminf_{\substack{|z| \to \infty \\ z \in \Gamma}} \frac{\Re V(z)}{|z|^\kappa} = \frac{1}{\kappa}$ and

$$\begin{aligned} \mathcal{Z}_{V,\Gamma}(N, \beta) &= \int_{\mathbb{R}^N} \prod_{i \neq j} (\gamma(x_i) - \gamma(x_j))^{\frac{\beta}{2}} \prod_{1 \leq i \leq N} \gamma'(x_i) e^{-NV(\gamma(x_i))} dx_i \\ &\simeq \int_{[-\epsilon, 1+\epsilon]^N} \prod_{i \neq j} (\gamma(x_i) - \gamma(x_j))^{\frac{\beta}{2}} \prod_{1 \leq i \leq N} \gamma'(x_i) e^{-NV(\gamma(x_i))} dx_i \end{aligned}$$

where we could finally localize the integration over a neighborhood of $[0,1]$ because the effective potential is positive outside so the integral over the complementary set is neglectable by large deviation estimates over the support.

Definition 5.21. *Let Γ as above. Let $\zeta_1, \zeta_2 \in \Gamma$ be the lower and upper endpoints of* $\operatorname{supp} \mu_\varphi$ *according to the orientation $\frac{1}{i\pi} \sqrt{R(z)} \, dz > 0$. The* complex effective potential *is given, if $g_\mu(z) = \int \ln(z - w)^{-1} d\mu(w)$, by*

$$\Phi_{\text{eff}} : \mathbb{C} \setminus \Gamma \longrightarrow \mathbb{C}, \Phi_{\text{eff}}(z) \stackrel{\text{def}}{=} \int_{\zeta_1}^{z} \sqrt{R(w)} \, dw = V(z) + g_\mu(z) - C$$

where the integration contour from $\zeta_1 \in \mathbb{C}$ to $z \in \mathbb{C}$ doesn't intersect with Γ except at ζ_1 itself.

Proposition 5.22. *Let $\pm$ refers to the left/ right boundary values according to the orientation of $\gamma([-\epsilon, 1+\epsilon])$. Then*

$$\Im\left(\Phi^+_{\text{eff}}(z) + \Phi^-_{\text{eff}}(z)\right) = 0 \qquad \forall z \in \gamma([-\epsilon, 1+\epsilon])$$

Furthermore $\Re\left(\Phi^+_{\text{eff}}(z) + \Phi^-_{\text{eff}}(z)\right) > 0 \qquad \forall z \in \gamma([-\epsilon, 1+\epsilon] \setminus [0,1])$.

Expanding the complex partition function I

Proposition 5.23. *Let V be one-cut regular and Γ, γ as before. Then there exists a positive constant C independent of N and N_0 finite so that for $N \geq N_0$:*

$$|\mathcal{Z}_{V,\Gamma}(N,\beta)| \geq C\mathcal{Z}^{\text{Re}}_{\varphi,\Gamma}(N,\beta)$$

- $$\frac{\mathcal{Z}_{V,\Gamma}(N,\beta)}{\mathcal{Z}^{\text{Re}}_{V,\Gamma}(N,\beta)} \simeq \frac{\mathcal{Z}_{V,\gamma([-\epsilon,1+\epsilon])}(N,\beta)}{\mathcal{Z}^{\text{Re}}_{V,\gamma([-\epsilon,1+\epsilon])}(N,\beta)} =: R_N$$

- Let $X_N = \frac{\beta N^2}{2}\int ad(L_N - \sigma)^{\otimes 2} + (1-\frac{\beta}{2})N\int pd(L_N-\sigma)$ with $\sigma = 8\sqrt{x(1-x)}dx/\pi$

  $$R_N = e^{i\frac{\beta N^2}{2}\int ad\sigma^{\otimes 2} + (1-\frac{\beta}{2})iN\int pd\sigma}\int_{[-\epsilon,1+\epsilon]^N} e^{iX_N} d\mathbb{P}_N^{[-\epsilon,1+\epsilon]}$$

  where $a(x,y) = \arg[(\gamma(x)-\gamma(y))/(x-y)]$, $p(x) = \arg[\gamma'(x)]$ and

  $$d\mathbb{P}_N^{[-\epsilon,1+\epsilon]} = \frac{\prod_{i<j}|\gamma(x_i)-\gamma(x_j)|^\beta}{\mathcal{Z}^{\text{Re}}_{V,\gamma([-\epsilon,1+\epsilon])}(N,\beta)} e^{-N\beta\sum_{k=1}^N \text{Re}(V\circ\gamma(x_k))}\prod_{k=1}^N|\gamma'(x_k)|dx_k$$

Lemma 5.24. *Assume V is one cut-regular. Then, under $\mathbb{P}_N^{[-\epsilon,1+\epsilon]}$, X_N converges towards a second chaos variable so that $\int_{[-\epsilon,1+\epsilon]^N} e^{iX_N} d\mathbb{P}_N^{[-\epsilon,1+\epsilon]}$ converges towards a non zero complex number $e^{F^{(0)}}$.*

Hints:

- a, p are smooth, compactly supported and hence can be written in terms of well-converging integrals of $F_N(\lambda) := \int e^{i\lambda x} dN(L_N-\sigma)(x)$. Hence it is enough to prove that $(F_N(\lambda))_{\lambda\in\mathbb{R}}$ converges towards Gaussian vectors.

- $\mathbb{P}_N^{[-\epsilon,1+\epsilon]}$ is a standard β-model with a potential depending quadratically on the empirical measure L_N. It can be shown (see Borot-G-Kozlowski) by using Dyson-Schwinger equations that if V is one-cut regular, $(F_N(\lambda))_{\lambda\in\mathbb{R}}$ converges towards Gaussian vectors.

Putting together the estimates above we get :

$$\mathcal{Z}_{V,\Gamma}(N,\beta) \simeq \mathcal{Z}^{\mathrm{Re}}_{V,\Gamma}(N,\beta) e^{i\frac{\beta N^2}{2}\int a d\sigma^{\otimes 2} + (1-\frac{\beta}{2}) iN \int p d\sigma} e^{F^{(0)}}(1+o(1))$$

We can deduce from that

$$\ln \mathcal{Z}_{V,\Gamma}(N,\beta) = \frac{\beta}{2} N \ln N + \frac{3+\frac{\beta}{2}+\frac{2}{\beta}}{12} \ln N + \sum_{k=-2}^{0} \frac{F_k(\beta,V)}{N^k} + o(1),$$

with

$$\begin{aligned} F_{-2}(\beta,V) &= -\frac{\beta}{2} I^V_{\Gamma_\varphi}[\mu_\varphi], \\ F_{-1}(\beta,V) &= \left(\frac{\beta}{2}-1\right)\left[\int_{\Gamma_\varphi} \ln \frac{d\mu_\varphi(z)}{dz}\, d\mu_\varphi(z) + \ln\frac{\beta}{2}\right] + \frac{\beta}{2}\ln\frac{2\pi}{e} - \ln\Gamma\left(\frac{\beta}{2}\right). \end{aligned}$$

where

$$I^V_\Gamma[\mu] \stackrel{\mathrm{def}}{=} \int_\Gamma \left(\frac{1}{2} g^+_\mu(z) + \frac{1}{2} g^-_\mu(z) + 2V(z)\right) d\mu(z)\,.$$

To show that the expansion holds at higher ranks, we see that

$$\ln \mathcal{Z}_{V,\Gamma}(N,\beta) = \frac{\beta}{2} N \ln N + \frac{3+\frac{\beta}{2}+\frac{2}{\beta}}{12} \ln N + \sum_{k=-2}^{K} \frac{F_k(\beta,V)}{N^k} + \mathcal{O}(N^{-K-1})\,.$$

- Find an interpolation of potentials V_t so that $\mathcal{Z}_{V_0,\Gamma}(N,\beta)$ is known

$$\ln \frac{\mathcal{Z}_{V,\Gamma}(N,\beta)}{\mathcal{Z}_{V_0,\Gamma}(N,\beta)} = -N^2 \int_0^1 \int L_N(\partial_t V_t) d\mathbb{P}^N_t \; dt$$

- Obtain Dyson-Schwinger equations for the cumulants of $L_N(P)$ by integration by parts
- invert the Master operator
- Control remainders by using previous bounds on partitions functions (so need to interpolate in the family of one-cut regular polynomials) and solve asymptotically the equations to derive the large N expansion of $\int L_N(\partial_t V_t) d\mathbb{P}^N_t$.

6 More about Dyson-Schwinger equations

- **Universality of Continuous β-ensembles**

  By introducing the idea of approximate transport map ideas developped in [6, 54], one can prove universality of (continuous) β-ensembles.

We will therefore consider again the β-ensembles $P_N^{\beta,V}$ introduced in Section 3 and will restrict ourselves to the setting of that section, namely the case where the equilibrium measure has a connected support and its density vanishes like a square root at the boundary. Thus, the global fluctuations are known and we now focus on the local fluctuations, such as the fluctuations of the largest eigenvalue or of the spacing between two eigenvalues. Our goal is to show that there is universality in the sense that the local fluctuations are the same than when the potential is quadratic. In fact, this is enough since local fluctuations could be studied in the case $V = x^2$. When $\beta = 1, 2, 4$, this was done by Riemann Hilbert techniques [88, 89, 76]. By using tridiagonal representation of the joint law of the eigenvalues of β ensembles with $V(x) = x^2$, derived by Dumitriu and Edelman [43], local fluctuations could be studied for general β [92, 80]. We are going to see here how to show that the same local fluctuations are true if we take another potential V, provided it is smooth enough, and so that the equilibrium measure has a connected support. Universality in the β-ensembles was first addressed in [21] (in the bulk, $\beta > 0$, $V \in C^4$), then in [22] (at the edge, $\beta \geq 1$, $V \in C^4$) and [70] (at the edge, $\beta > 0$, V convex polynomial) and finally in [83] (in the bulk, $\beta > 0$, V analytic, multi-cut case included). The approach we propose here, which was developed in [6], is based on the construction of approximate transport maps. More precisely, let us consider $P_N^{\beta,V}$ the previous β-ensemble distribution on $\mathbb{R}^N$. The goal is to construct a map $T^N : \mathbb{R}^N \to \mathbb{R}^N$ such that $T^N = T_0^{\otimes N} + \frac{1}{N} T_1^N$ where T_0 is smooth and increasing as well as T_1^N and so that

$$\lim_{N\to\infty} \|P_N^{\beta,V} - T^N \# P_N^{\beta,x^2}\|_{TV} = 0\,. \tag{79}$$

Here, we denote by $T\#\mu$ the push-forward of the measure μ by T given for any test function f by

$$\int f(x) dT\#\mu(x) = \int f(T(y)) d\mu(y)\,.$$

The name of approximate transport map emphasizes that the above equality only holds at the large N limit, and not for all N as a usual transport map would do. Eventhough existence of transport maps is well known for any given N, smoothness and dependency on the dimension of such maps is unknown in general. For this reason, we shall instead construct *explicit* approximate transport maps, for which we can investigate both regularity and dependency on the dimension. The main point is that the existence of approximate transport maps as in (79) implies universality. Roughly speaking, if T_1^N is bounded, we see that if λ_i^V are the ordered eigenvalues under $P_N^{\beta,V}$,

$$N^{2/3}(\lambda_N^V - T_0(2)) \simeq_{dist} N^{2/3}(T^N(\lambda_N^{x^2}) - T_0(2)) \simeq T_0'(2) N^{2/3}(\lambda_N^{x^2} - 2) + O(\frac{1}{N^{1/3}})\,.$$

This shows that the largest eigenvalue λ_N^V fluctuates around $T_0(2)$ as in the quadratic case. Moreover, if T_1^N is Hölder $\alpha \in (0,1)$,

$$N(\lambda_{i+1}^V - \lambda_i^V) = NT_0'(\lambda_i^{x^2})(\lambda_{i+1}^{x^2} - \lambda_i^{x^2}) + O(|\lambda_{i+1}^{x^2} - \lambda_i^{x^2}|^\alpha)$$

where the last term is negligible. Similarly, correlation functions can be considered. Hence, we also get universality in the bulk.

To construct the approximate transport map, we solve the Monge Ampere equations approximately. This is very similar to solving the Dyson-Schwinger equations.

- **Local fluctuations of Discrete Beta ensembles at the edge** Huang and I [62] showed that the largest particle of Discrete Beta ensembles fluctuate as the largest particle of continuous Beta ensembles for every Beta larger than one. This follows from rigidity and transport to continuous models in the region of the large particles whose distance is much bigger than the grid typical size.
- **Local laws for Beta ensembles** By reformulating cleverly the Dyson-Schwinger equations, Bourgade, Mody and Pain [24] could derive optimal local laws for (continuous) Beta ensembles.
- **Dynamical loop equations** Gorin and Huang [56] showed that loop equations can also be derived for two dimensional particle systems, including Dyson Brownian motion, nonintersecting Bernoulli and Poisson random walks, ?-corner processes, uniform and Jack-type Gelfand-Tsetlin patterns, models associated with Macdonald and Koornwider polynomials, as well as distributions of lozenge tilings. They showed that these imply central limit theorems and related convergence towards Gaussian fields of the fluctuations of the linear statistics.
- **Asymptotics of Jack Polynomials** Huang and I [63] used the dynamical loop equations to prove large deviations principle for Theta analogues of Bernoulli random walks, generalizing large deviations for Dyson's Brownian motions.
- **Fluctuations of Beta ensembles for critical potentials** Bekerman, Leblé, Serfaty [**?**] proved a central limit theorem for linear statistics for continuous Beta ensembles when the test functions belong to the orthogonal of the kernel of the master operator. The latter remains quite mysterious.
- **Fluctuations of β-Kratchouk ensembles** by Dimitrov and Knizel [40] introduce and study the β-Ktrachouk ensembles by developing multi-level loop equations.
- **Ramanujan Property and Edge Universality of Random regular graphs** Recently, McKenzie, Huang and Yau [66] proved that the first non-trivial eigenvalue of d-regular graph fluctuate like a Tracy-Widom law up

to $d = 3$. A key new tool they developed is the notion of microscopic loop equations.

- **Asymptotic expansion of smooth functions in polynomials in deterministic matrices and iid GUE** derived by Parraud [78], which were key in recent advances on asymptotic freeness [4, 7].

References

[1] S. Albeverio, L. A. Pastur, and M. Shcherbina, *On the $1/n$ expansion for some unitary invariant ensembles of random matrices*, Comm. Math. Physics **224** (2001), 271–305.

[2] J. Ambjø rn and Yu. M. Makeenko, *Properties of loop equations for the Hermitian matrix model and for two-dimensional quantum gravity*, Modern Phys. Lett. A **5** (1990), no. 22, 1753–1763. MR 1075933

[3] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, Cambridge, 2010. MR 2760897

[4] Afonso S. Bandeira, March T. Boedihardjo, and Ramon van Handel, *Matrix concentration inequalities and free probability*, Invent. Math. **234** (2023), no. 1, 419–487 (English).

[5] Zhigang Bao, László Erd¨ os, and Kevin Schnelli, *Local law of addition of random matrices on optimal scale*, Comm. Math. Phys. **349** (2017), no. 3, 947–990. MR 3602820

[6] F. Bekerman, A. Figalli, and A. Guionnet, *Transport maps for β-matrix models and universality*, Comm. Math. Phys. **338** (2015), no. 2, 589–619. MR 3351052

[7] Serban Belinschi and Mireille Capitaine, *Strong convergence of tensor products of independent G.U.E. matrices*, Preprint, arXiv:2205.07695 [math.OA] (2022), 2022.

[8] G. Ben Arous, A. Dembo, and A. Guionnet, *Aging of spherical spin glasses*, Probab. Theory Related Fields **120** (2001), no. 1, 1–67. MR 1856194

[9] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Related Fields **108** (1997), no. 4, 517–542. MR 1465640

[10] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Rel. **108** (1997), 517–542.

[11] Pavel M. Bleher and Alexander R. Its, *Asymptotics of the partition function of a random matrix model*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 6, 1943–2000. MR 2187941

[12] Charles Bordenave and Alice Guionnet, *Localization and delocalization of eigenvectors for heavy-tailed random matrices*, Probab. Theory Related Fields **157** (2013), no. 3-4, 885–953. MR 3129806

[13] ———, *Delocalization at small energy for heavy-tailed random matrices*, Comm. Math. Phys. **354** (2017), no. 1, 115–159. MR 3656514

[14] Alexei Borodin, Vadim Gorin, and Alice Guionnet, *Gaussian asymptotics of discrete β-ensembles*, Publications mathématiques de l'IHÉS (2016), 1–78.

[15] G. Borot and A. Guionnet, *Asymptotic expansion of β matrix models in the one-cut regime*, Comm. Math. Phys. **317** (2013), no. 2, 447–483. MR 3010191

[16] Gaëtan Borot, Bertrand Eynard, and Nicolas Orantin, *Abstract loop equations, topological recursion and new applications*, Commun. Number Theory Phys. **9** (2015), no. 1, 51–187. MR 3339853

[17] Gaetan Borot, Vadim Gorin, and Alice Guionnet, *Macroscopic asymptotics of discrete β-ensembles and tilings*, Work in progress, 2025.

[18] Gaëtan Borot and Alice Guionnet, *Asymptotic expansion of β matrix models in the multi-cut regime*, Forum Math. Sigma **12** (2024), 93 (English), Id/No e13.

[19] Gaëtan Borot, Alice Guionnet, and Karol K. Kozlowski, *Large-N asymptotic expansion for mean field models with Coulomb gas interaction*, Int. Math. Res. Not. IMRN (2015), no. 20, 10451–10524. MR 3455872

[20] ———, *Asymptotic expansion of a partition function related to the sinh-model*, Mathematical Physics Studies, Springer, [Cham], 2016. MR 3585351

[21] P. Bourgade, L. Erdös, and H.-T. Yau, *Edge universality of beta ensembles*, Comm. Math. Phys. **332** (2014), no. 1, 261–353. MR 3253704

[22] ———, *Universality of general β-ensembles*, Duke Math. J. **163** (2014), no. 6, 1127–1190. MR 3192527

[23] Paul Bourgade, Laszlo Erdos, Horng-Tzer Yau, and Jun Yin, *Universality for a class of random band matrices*, Adv. Theor. Math. Phys. **21** (2017), no. 3, 739–800. MR 3695802

[24] Paul Bourgade, Krishnan Mody, and Michel Pain, *Optimal local law and central limit theorem for β-ensembles*, Commun. Math. Phys. **390** (2022), no. 3, 1017–1079 (English).

[25] Mireille Bousquet-Mélou and Arnaud Jehanne, *Polynomial equations with one catalytic variable, algebraic series and map enumeration*, J. Combin. Theory Ser. B **96** (2006), no. 5, 623–672. MR 2236503

[26] A. Boutet de Monvel, L. Pastur, and M. Shcherbina, *On the statistical mechanics approach in the random matrix theory: integrated density of states*, J. Statist. Phys. **79** (1995), no. 3-4, 585–611. MR 1327898

[27] Jonathan Breuer and Maurice Duits, *Central limit theorems for biorthogonal ensembles and asymptotics of recurrence coefficients*, J. Amer. Math. Soc. **30** (2017), no. 1, 27–66. MR 3556288

[28] William G. Brown, *Enumeration of triangulations of the disk*, Proc. London Math. Soc. (3) **14** (1964), 746–768. MR 0168485

[29] S. Chatterjee, *Rigorous solution of strongly coupled so(n) lattice gauge theory in the large n limit*, To appear in Comm. Math. Phys., arXiv: 1502.07719 (2015).

[30] L. O. Chekhov, B. Eynard, and O. Marchal, *Topological expansion of the β-ensemble model and quantum algebraic geometry in the sectorwise approach*, Theoret. and Math. Phys. **166** (2011), no. 2, 141–185, Russian version appears in Teoret. Mat. Fiz. **1**66 (2011), no. 2, 163–215. MR 3165804

[31] Leonid Chekhov and Bertrand Eynard, *Matrix eigenvalue model: Feynman graph technique for all genera*, J. High Energy Phys. (2006), no. 12, 026, 29. MR 2276715

[32] T. Claeys, T. Grava, and K. D. T.-R. McLaughlin, *Asymptotics for the partition function in two-cut random matrix models*, Comm. Math. Phys. **339** (2015), no. 2, 513–587. MR 3370612

[33] B. Collins, A. Guionnet, and E. Maurel-Segala, *Asymptotics of unitary and orthogonal matrix integrals*, Adv. Math. **222** (2009), no. 1, 172–215.

[34] Antoine Dahlqvist, *Free energies and fluctuations for the unitary Brownian motion*, Comm. Math. Phys. **348** (2016), no. 2, 395–444. MR 3554890

[35] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou, *Uniform asymptotics for orthogonal polynomials*, Doc. Math. **III** (1998), 491–501, Extra Volume ICM 1998.

[36] P. A. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics, vol. 3, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. MR 1677884

[37] Percy Deift and Dimitri Gioev, *Random matrix theory: invariant ensembles and universality*, Courant Lecture Notes in Mathematics, vol. 18, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2009. MR 2514781

[38] Percy Deift, Alexander Its, and Igor Krasovsky, *Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities*, Ann. of Math. (2) **174** (2011), no. 2, 1243–1299. MR 2831118

[39] D.Feral, *On large deviations for the spectral measure of discrete coulomb gas.*, Seminaire de Probabilites XLI Lecture Notes in Mathematics **Vol. 1934** (2008), 19–49.

[40] Evgeni Dimitrov and Alisa Knizel, *Global asymptotics for β-Krawtchouk corners processes via multi-level loop equations*, Preprint, arXiv:2403.17895 [math.PR] (2024), 2024.

[41] A. L. Dixon, *Clt for fluctuations of β-ensembles with general potential*, arxiv 1706.09663.

[42] B. A. Dubrovin, *Theta-functions and nonlinear equations*, Uspekhi Mat. Nauk **36** (1981), no. 2(218), 11–80, With an appendix by I. M. Krichever. MR 616797

[43] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), 5830–5847.

[44] G. Parisi E. Brézin, C. Itzykson and J. B. Zuber, *Planar diagrams*, Comm. Math. Phys. **59** (1978), 35–51.

[45] N. M. Ercolani and K. D. T.-R. McLaughlin, *Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration*, Int. Math. Res. Not. (2003), no. 14, 755–820. MR 1953782

[46] ———, *Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration*, Int. Math. Res. Not. (2003), no. 14, 755–820. MR 1953782

[47] László Erd˝ os, *Universality for random matrices and log-gases*, Current developments in mathematics 2012, Int. Press, Somerville, MA, 2013, pp. 59–132. MR 3204344

[48] László Erdős, Benjamin Schlein, and Horng-Tzer Yau, *Wegner estimate and level repulsion for Wigner random matrices*, Int. Math. Res. Not. IMRN (2010), no. 3, 436–479.

[49] László Erdős and Horng-Tzer Yau, *Gap universality of generalized Wigner and β-ensembles*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 8, 1927–2036. MR 3372074

[50] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452. MR 2346575

[51] Bertrand Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, J. High Energy Phys. (2004), no. 11, 031, 35 pp. (2005). MR 2118807

[52] ———, *Counting surfaces*, Progress in Mathematical Physics, vol. 70, Birkhäuser/Springer, [Cham], 2016, CRM Aisenstadt chair lectures. MR 3468847

[53] C. Fan, A. Guionnet, Y. Song, and A. Wang, *Convergence of eigenvalues to the support of the limiting measure in critical β matrix models*, Random Matrices Theory Appl. **4** (2015), no. 3, 1550013, 22. MR 3385707

[54] A. Figalli and A. Guionnet, *Transport maps for several-matrix models and universality*, Acta Math. **217** (2016), 81–176.

[55] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Yu. Novokshenov, *Painlevé transcendents*, Mathematical Surveys and Monographs, vol. 128, American Mathematical Society, Providence, RI, 2006, The Riemann-Hilbert approach.

[56] Vadim Gorin and Jiaoyang Huang, *Dynamical loop equation*, Ann. Probab. **52** (2024), no. 5, 1758–1863 (English).

[57] I. P. Goulden and D. M. and Jackson, *Combinatorial enumeration*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983, With a foreword by Gian-Carlo Rota, Wiley-Interscience Series in Discrete Mathematics. MR 702512

[58] A. Guionnet and E. Maurel Segala, *Combinatorial aspects of matrix models*, Alea **1** (2006), 241–279.

[59] ———, *Second order asymptotics for matrix models*, Ann. Probab. **35** (2007), no. 6, 2160–2212. MR MR2353386 (2009b:60073)

[60] A. Guionnet and J. Novak, *Asymptotics of unitary multimatrix models: The schwinger-dyson lattice and topological recursion*, arXiv:1401.2703 (2014).

[61] A. Guionnet and O. Zeitouni, *Concentration of the spectral measure for large matrices*, Electron. Commun. Prob. **5** (2000), 119–136 (electronic).

[62] Alice Guionnet and Jiaoyang Huang, *Rigidity and edge universality of discrete β-ensembles*, Commun. Pure Appl. Math. **72** (2019), no. 9, 1875–1982 (English).

[63] ———, *Asymptotics of Symmetric Polynomials: A Dynamical Point of view*, Preprint, arXiv:2409.04621 [math.PR] (2024), 2024.

[64] Alice Guionnet, Karol Kozlowski, and Alex Little, *Asymptotic expansion of the partition function for β-ensembles with complex potentials*, Preprint, arXiv:2411.10610 [math-ph] (2024), 2024.

[65] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457–485.

[66] Jiaoyang Huang, Theo McKenzie, and Horng-Tzer Yau, *Ramanujan Property and Edge Universality of Random Regular Graphs*, Preprint, arXiv:2412.20263 [math.PR] (2024), 2024.

[67] A. Its and I. Krasovsky, *Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump*, Integrable systems and random matrices, Contemp. Math., vol. 458, Amer. Math. Soc., Providence, RI, 2008, pp. 215–247. MR 2411909

[68] K. Johansson, *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), 151–204.

[69] Maxim Kontsevich, *Vassiliev's knot invariants*, Gelfand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 137–150. MR 1237836

[70] Manjunath Krishnapur, Brian Rider, and Bálint Virág, *Universality of the stochastic Airy operator*, Comm. Pure Appl. Math. **69** (2016), no. 1, 145–199. MR 3433632

[71] A. B. J. Kuijlaars and K. T.-R. McLaughlin, *Generic behavior of the density of states in random matrix theory and equilibrium problems in the presence of real analytic external fields*, Communications on Pure and Applied Mathematics **53** (2000), no. 6, 736–785.

[72] Arno Kuijlaars and Guilherme Lima Ferreira de Silva, *S-curves in polynomial external fields*, Journal of Approximation Theory **191** (2015-03-01) (eng).

[73] Gaultier Lambert, Michel Ledoux, and Christian Webb, *Quantitative normal approximation of linear statistics of β-ensembles*, Ann. Probab. **47** (2019), no. 5, 2619–2685 (English).

[74] Thomas Leblé and Sylvia Serfaty, *Fluctuations of two dimensional Coulomb gases*, Geom. Funct. Anal. **28** (2018), no. 2, 443–508 (English).

[75] Mylène Maï da and Édouard Maurel-Segala, *Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices*, Probab. Theory Related Fields **159** (2014), no. 1-2, 329–356. MR 3201924

[76] M. L. Mehta, *Random matrices*, third ed., Pure and Applied Mathematics (Amsterdam), vol. 142, Elsevier/Academic Press, Amsterdam, 2004. MR 2129906 (2006b:82001)

[77] N. I. Muskhelishvili, *Singular integral equations*, (1992).

[78] Félix Parraud, *Asymptotic expansion of smooth functions in polynomials in deterministic matrices and iid GUE matrices*, Commun. Math. Phys. **399** (2023), no. 1, 249–294 (English).

[79] Leonid Pastur and Mariya Shcherbina, *Eigenvalue distribution of large random matrices*, Mathematical Surveys and Monographs, vol. 171, American Mathematical Society, Providence, RI, 2011. MR 2808038

[80] J. Ramirez, B. Rider, and B. Virag, *Beta ensembles, stochastic airy spectrum, and a diffusion*, J. Amer. Math. Soc. **24** (2011), 919–944.

[81] Edward B. Saff and Vilmos Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316, Springer-Verlag, Berlin, 1997, Appendix B by Thomas Bloom. MR 1485778

[82] M. Shcherbina, *Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime*, J. Stat. Phys. **151** (2013), no. 6, 1004–1034. MR 3063494

[83] ———, *Change of variables as a method to study general β-models: bulk universality*, J. Math. Phys. **55** (2014), no. 4, 043504, 23. MR 3390602

[84] Mariya Shcherbina, *Asymptotic expansions for beta matrix models and their applications to the universality conjecture*, Random matrix theory, interacting particle systems, and integrable systems, Math. Sci. Res. Inst. Publ., vol. 65, Cambridge Univ. Press, New York, 2014, pp. 463–482. MR 3380697

[85] Terence Tao and Van Vu, *Random matrices: universality of local eigenvalue statistics up to the edge*, Comm. Math. Phys. **298** (2010), no. 2, 549–572.

[86] ———, *Random matrices: universality of local eigenvalue statistics up to the edge*, Comm. Math. Phys. **298** (2010), no. 2, 549–572. MR 2669449

[87] G. t'Hooft, *Magnetic monopoles in unified gauge theories*, Nuclear Phys. B **79** (1974), 276–284.

[88] C. A. Tracy and H. Widom, *Level spacing distributions and the Airy kernel*, Commun. Math. Phys. **159** (1994), 151–174.

[89] ———, *On orthogonal and symplectic matrix ensembles*, Commun. Math. Phys. **177** (1996), 727–754.

[90] F. G. Tricomi, *Integral equations*, Dover Publications, New York, NY, 1985, Reprint of the 1957 original.

[91] W. T. Tutte, *A census of planar maps*, Canad. J. Math. **15** (1963), 249–271. MR 0146823

[92] Benedek Valkó and Bálint Virág, *Continuum limits of random matrices and the Brownian carousel*, Invent. Math. **177** (2009), no. 3, 463–508. MR 2534097

[93] A. Zvonkin, *Matrix integrals and maps enumeration: an accessible introduction*, Math. Comput. Modeling **26** (1997), 281–304.