Equidistribution results in Kähler geometry and asymptotic study of filtrations

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Abstract. These lecture notes accompany the mini-course delivered at the Budapest Summer School "Invitation to Complex Geometry" held at the Rényi Institute in August 2025. The primary aim of the course is to present recent developments in the study of submultiplicative filtrations, approached through the lens of Geometric Quantization and Pluripotential Theory. We situate these developments within the broader context of equidistribution phenomena in complex geometry and functional analysis, highlighting the underlying parallels between these areas.

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1 Equidistribution results and Kähler geometry

The aim of the first lecture is to present the main result of the mini-course, which concerns the asymptotic behavior of submultiplicative filtrations. We begin in Section 1.1 with a brief overview of the theory of Toeplitz matrices, which is the focus of the first subsection. In Section 1.2, we state the main theorem of the mini-course concerning submultiplicative filtrations. Finally, in Sections 1.3 and 1.4, we explain the connection between the two statements.

1.1 A brief detour through Toeplitz Matrices

We fix a sequence of numbers $a_i \in \mathbb{C}$, $i \in \mathbb{Z}$, verifying $a_i = \overline{a}_{-i}$, and consider the following $k \times k$ Hermitian matrix

$$T_k[a] := \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-k+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-k+2} \\ a_2 & a_1 & a_0 & \cdots & a_{-k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 \end{bmatrix}. \tag{1.1}$$

These matrices were first considered by Toeplitz in [76], and then systematically studied by Szegö [73]. To study such matrices in the regime $k \to \infty$, it is necessary to consider the function

$$f(\theta) = \sum_{j=-\infty}^{+\infty} a_j \exp(\sqrt{-1} j \theta), \text{ where } \quad \theta \in [0, 2\pi[. \tag{1.2}$$

We assume that $f \in L^\infty(\mathbb{S}^1)$, and denote $T_k[a]$ by $T_k[f]$.

The influential Szegö's result from [73] relates the spectral theory of $T_k[f]$, as $k \to \infty$, with the properties of the function f, sometimes called the symbol of the Toeplitz matrices $T_k[f]$.

To describe this result, we denote by $\lambda_{\min}(T_k[f])$ and $\lambda_{\max}(T_k[f])$ the minimal and the maximal eigenvalues of $T_k[f]$. Firstly, Szegö established that, as $k \to \infty$,

$$\lambda_{\min}(T_k[f]) \to \operatorname{ess\,inf} f, \qquad \lambda_{\max}(T_k[f]) \to \operatorname{ess\,sup} f. \tag{1.3}$$

Then for $I := [\operatorname{ess\,inf} f, \operatorname{ess\,sup} f]$ and any continuous function $g : I \to \mathbb{R}$, as $k \to \infty$,

$$\frac{1}{k} \sum_{\lambda \in \operatorname{Spec}(T_k[f])} g(\lambda) \to \frac{1}{2\pi} \int_0^{2\pi} g(f(\theta)) d\theta. \tag{1.4}$$

One particularly famous instance of this theorem concerns the special case when $f > \epsilon$ for some $\epsilon > 0$, and $g(x) = \log(x)$, giving the First Szegő theorem[1] stating that as $k \to \infty$, we have

$$\frac{1}{k} \log(\det T_k[f]) \to \frac{1}{2\pi} \int_0^{2\pi} \log(f(\theta)) d\theta. \tag{1.5}$$

[1] We invite the reader to consult a particularly engaging historical account of the unusual origins of the Second Szegő Theorem in Nikolski's book [57, §1.4.3]. This theorem, which refines the asymptotic expansion in (1.5) by capturing the next-order term, was prompted by a question from Nobel Laureate (in Chemistry) Lars Onsager in connection with the Lenz-Ising model [74]. It appeared only 37 years after the First Szegő Theorem – by which time Szegő himself was nearly three times as old as when he proved the first statement.

Toeplitz matrices has had a profound impact across various fields of mathematics. These include functional analysis (orthogonal polynomials), signal processing (convolution operators), and mathematical physics (Lenz-Ising model, Berezin quantization), among many others.

Although the primary focus of this mini-course is complex and algebraic geometry, many of the key objects and techniques we will study already arise in the theory of Toeplitz matrices. The author believes that for an audience just beginning to explore complex geometry, the motivations offered by more palpable Toeplitz theory may be easier to grasp. This is the main reason for our initial detour through the world of Toeplitz matrices.

To explain it in detail, we will assume that the symbol f is nonnegative, and such that $K := \mathrm{supp} f \subset \mathbb{S}^1$ is a finite union of intervals in $\mathbb{S}^1$, not covering $\mathbb{S}^1$ completely (for example, it might be an indicator function over a sub-interval in $[0, 2\pi[$). In this setting, although the statement (1.5) remains formally correct, it no longer reflects the leading-order term in the asymptotic expansion – in fact, the right-hand side equals $-\infty$. This leads naturally to the question: what is the asymptotics of $\log(\det T_k[f])$, as $k \to \infty$? A related refinement of (1.3) is the question: how rapidly does $\lambda_{\min}(T_k[f])$ converges towards $0 = \operatorname{ess\,inf} f$?

Perhaps surprisingly, this problem appears to have been investigated only quite recently—despite the fact that the spectral theory of $T_k[f]$ for other vanishing regimes of f have been extensively studied in the past, often spurred by connections to physics (see, for instance, the account of Dyson's work in [26, p. 32]).

Drewitz–Liu–Marinescu established[2] in [33, Theorem 1.23], that the minimal eigenvalue decays at most exponentially in k; that is, there exists $d > 0$, such that

$$\lambda_{\min}(T_k[f]) \geq \exp(-dk). \tag{1.6}$$

It was later established by the author [42, Corollary 10.4] (following [33, Question 5.13]) that for non-pathological f (in the sense that the set $K \cap f^{-1}(0)$ has zero Lebesgue measure) there is $c_K > 0$, which depends only on the set K, so that, as $k \to \infty$, we have

$$\frac{\log(\lambda_{\min}(T_k[f]))}{k} \to -c_K. \tag{1.7}$$

Moreover, the analogue of (1.4) remains correct if instead of all eigenvalues, one considers solely those which tend to 0 exponentially fast, and renormalizes the sum accordingly. More specifically, [42, Corollary 10.4] states that for any continuous function $g : \mathbb{R} \to \mathbb{R}$, as $k \to \infty$, we have

$$\frac{1}{k} \sum_{\lambda \in \mathrm{Spec}(T_k[f])} g\Big(\frac{\log(\lambda)}{k}\Big) \to \int_{t \in \mathbb{R}} g(-t) d\mu_K(t), \tag{1.8}$$

where μ_K is a certain compactly supported probability measure on $\mathbb{R}$, determined by set K.

The description of the measure μ_K from (1.8) is considerably less direct than in (1.4), and–naturally–it relies on recent developments in higher-dimensional Kähler geometry. After all, what else would one use to study the unit circle?

[2] Authors of [33] worked in the setting of Toeplitz operators on complex manifolds, which – strictly speaking – do not cover the case of Toeplitz matrices, see Section 1.3. Nevertheless, their result continues to hold also for the so-called generalized Toeplitz operators, which includes Toeplitz operators on complex manifolds and Toeplitz matrices, see [42, Corollary 10.3].

To give a flavor of the objects involved, we briefly mention that if one embeds $\mathbb{S}^1$ into a complex projective space $\mathbb{P}^1_{\mathbb{C}}$ as a great circle, then a fairly standard envelope construction from potential theory associates to the subsets $\mathbb{S}^1 \subset \mathbb{P}^1_{\mathbb{C}}$ and $K \subset \mathbb{P}^1_{\mathbb{C}}$ certain singular Hermitian metrics $h_{\mathbb{S}^1}$ and h_K on $\mathscr{O}(1)$ over $\mathbb{P}^1_{\mathbb{C}}$. Without entering too much details, we mention first that the line bundle $\mathscr{O}(1)$ carries a special metric, called the Fubini-Study metric. The metrics $h_{\mathbb{S}^1}$ and h_K are then defined as the minimal metrics among all metrics with subharmonic potentials, which are no smaller that the Fubini-Study metric over the subsets $\mathbb{S}^1 \subset \mathbb{P}^1_{\mathbb{C}}$ and $K \subset \mathbb{P}^1_{\mathbb{C}}$ respectively. Although it is not immediately evident from the definition, such minimal metrics do exist, and furthermore, they admit bounded potentials, cf. [45, Theorem 9.17].

The key tool for defining μ_K is the so-called Mabuchi-Darvas distance introduced in [56] and [20] that we shall describe in detail in Section 3.2. Roughly, it measures the distance between different metrics with bounded psh potentials. For now, we simply note that for every $p \in [1, +\infty)$, there exists a non-negative quantity $d_p(h_{\mathbb{S}^1}, h_K)$ arising from complex geometry.

Now, as Radon measures of compact support are fully determined by their moments, the measure μ_K is fully prescribed by the following condition: for any $p \in [1, +\infty[$, we have

$$\int_{t\in\mathbb{R}} t^p d\mu_K(t) = d_p(h_{\mathbb{S}^1}, h_K)^p. \tag{1.9}$$

While the general definition of $d_p(h_{\mathbb{S}^1}, h_K)$ is quite subtle, and its explicit computation appears to be a challenging problem, the special case $p = 1$ is significantly more tractable and can be further reformulated in the language of classical potential theory. We take inspiration below from the exposition of Boucksom [10].

We interpret $x, y \in \mathbb{C}$ as two identical particles with the same charge that repel each other, seeking to reduce the interaction energy $-\log|x-y|$. The interaction of a configuration of N particles $x_1, \ldots, x_N \in \mathbb{C}$ is modeled by averaging the contributions of all pairs, i.e.

$$E_N(x_1, \ldots, x_N) = \frac{2}{N(N-1)} \sum_{1\leq i<j\leq N} -\log|x-y|. \tag{1.10}$$

If these particles are confined to a compact set $K \subset \mathbb{C}$, which can be thought of as a conductor on which the N particles move freely, they will spread out so as to minimize the energy E_N, and the equilibrium position will then be given by a configuration $P = (x_1, \ldots, x_N) \in X^N$ such that $E_N(P) = \inf_{X^N} E_N$. As defined, equilibrium configurations are generally far from unique (think of a special case when K is a cricle). Remarkably, however, uniqueness is asymptotically restored as N tends to infinity. More precisely, a "continuous distribution" of charges on X is described by a probability measure μ, whose energy is given by

$$E(\mu) = \int_{x\in K} \int_{y\in K} -\log|x-y| \cdot d\mu(x) \cdot d\mu(y). \tag{1.11}$$

The following dichotomy holds true: either $E(\mu) = +\infty$ for every probability measure μ with support in K, then K is said to be *polar*, or there is a unique probability measure $\mu_{\text{eq}}(K)$ with support on K, called the *equilibrium measure* of K, so that

$$E(\mu_{\text{eq}}(K)) = \inf_{\mu} E(\mu), \tag{1.12}$$

where the infimum is taken over all probability measures with $\text{supp}\mu \subset K$.

Then the asymptotics (1.8) and (1.9) yield: as $k \to \infty$, we have

$$\frac{1}{k^2} \log(\det T_k[f]) \to E(\mu_{\text{eq}}(\mathbb{S}^1)) - E(\mu_{\text{eq}}(K)). \tag{1.13}$$

The asymptotics (1.13) can alternatively be established from Berman-Boucksom [3, Theorem A], cf. also [42, §9]. Remark that from this interpretation one immediately sees that the left-hand side of (1.13) is non-positive which is very much compatible with (1.7).

While this course will not delve into the theory of Toeplitz matrices and potential theory, the main result will closely resemble the statements in (1.8) and (1.9). As we shall explain in Section 1.3, this resemblance is no coincidence.

1.2 Submultiplicative filtrations

In this section, we introduce the key objects of study in this mini-course and present its main result.

A *submultiplicative filtration* on a commutative ring A is a decreasing filtration $\mathcal{F}$, indexed by $\lambda \in \mathbb{R}$, such that the multiplication map on A factors as

$$\mathcal{F}^\lambda A \otimes \mathcal{F}^\mu A \to \mathcal{F}^{\lambda+\mu} A \quad \text{for all } \lambda, \mu \in \mathbb{R}. \tag{1.14}$$

Throughout this mini-course, we impose the following assumptions: all filtrations will be decreasing, the ring A will be graded, i.e. $A = \oplus_{k \in \mathbb{N}} A_k$, and the filtration $\mathcal{F}$ will respect this grading, i.e. $\mathcal{F}^\lambda A = \oplus_{k \in \mathbb{N}} \mathcal{F}^\lambda_k A$. The primary goal of this mini-course is to study submultiplicative filtrations from an asymptotic perspective.

Historically, the asymptotic study of filtrations was first considered by Samuel [69]. Subsequent developments by Rees [64] [65], [66] established fundamental connections between the asymptotics of filtrations and the theory of valuations through his celebrated valuation theorems.

In these lecture notes, we focus primarily on the *statistical properties* of filtrations. Our main objective is to develop a framework that can effectively answer questions of the following form: *What proportion of elements in A_k have weights lying in a given interval $[ak, bk]$, where $a, b \in \mathbb{R}$?*

The answer to the above question will be formulated in geometric terms. To achieve this, we will work under the natural assumption that the ring A itself has geometric origin.

We fix a complex projective manifold X, $\dim X = n$, and an ample line bundle L over it. We shall be concerned with submultiplicative filtrations on the ring $A := R(X, L)$, where $R(X, L)$ is the section ring defined as

$$R(X, L) := \oplus_{k=0}^{+\infty} H^0(X, L^{\otimes k}). \tag{1.15}$$

We say that a submultiplicative filtration $\mathcal{F}$ on $R(X, L)$ is *bounded* if there is $C > 0$, such that for any $k \in \mathbb{N}^*$, $\mathcal{F}^{Ck} H^0(X, L^{\otimes k}) = \{0\}$. We say that $\mathcal{F}$ is *finitely generated* if it has integral weights and the associated $\mathbb{C}[\tau]$-algebra $\mathrm{Rees}(\mathcal{F}) := \sum_{(\lambda,k) \in \mathbb{Z} \times \mathbb{N}} \tau^{-\lambda} \mathcal{F}^\lambda H^0(X, L^{\otimes k})$, also called the *Rees algebra*, is finitely generated. Finitely generated submultiplicative filtrations are clearly automatically bounded. As the section ring $R(X, L)$ is finitely generated, cf. [50, Example 2.1.30], the set of finitely generated submultiplicative filtrations is non-empty, and for an arbitrary submultiplicative filtration, there is $C > 0$, such that for any $k \in \mathbb{N}^*$, $\mathcal{F}^{-Ck} H^0(X, L^{\otimes k}) = H^0(X, L^{\otimes k})$.

The most natural example of a submultiplicative filtration is the filtration given by the order of vanishing along a fixed divisor. The condition (1.14) then admits a natural geometric interpretation: for any product of functions, the vanishing order along the divisor is bounded below by the sum of the vanishing orders of the factors. Of course, when the divisor is irreducible, this inequality even becomes an equality.

Other examples include filtrations associated with the weight of a $\mathbb{C}^*$-action on the pair (X, L), filtrations associated with valuations or graded ideals [66], or finitely generated filtrations induced by an arbitrary filtration on $H^0(X, L^{\otimes k})$, for $k \in \mathbb{N}$ big enough.

To lighten the notation, we set $n_k := \dim H^0(X, L^{\otimes k})$, $k \in \mathbb{N}$. Define the *jumping numbers* $e_{\mathcal{F}}(j, k)$, $j = 1, \ldots, n_k$, as follows

$$e_{\mathcal{F}}(j, k) := \sup \Big\{ t \in \mathbb{R} : \dim \mathcal{F}^t H^0(X, L^{\otimes k}) \geq j \Big\}. \tag{1.16}$$

To simplify further presentation, we shall assume that $\mathcal{F}^0 H^0(X, L^{\otimes k}) = H^0(X, L^{\otimes k})$, which translates in the language of jumping numbers as $e_{\mathcal{F}}(j, k) \geq 0$.

Phong–Sturm [60] and Ross–Witt Nyström [68] showed that any submultiplicative filtration gives rise to a geodesic ray in the space of Hermitian metrics on L. We will describe this construction in detail in Lecture 4. Concretely, this means that one can associate to each filtration a ray of metrics $h^L_{\mathcal{F},t}$, for $t \in [0, +\infty[$, emanating from any given smooth positive metric $h^L_{\mathcal{F},0}$ on L.

This ray of metrics constitutes the analytic realization of the filtration, while the associated jumping numbers represent its algebraic counterpart. The main result of the mini-course asserts a precise compatibility between these two perspectives.

Theorem 1.1. *For any bounded submultiplicative filtration $\mathcal{F}$ on $R(X, L)$, there is the probability measure $\mu_{\mathcal{F}}$ on $\mathbb{R}$, so that for any continuous function $g : I \to \mathbb{R}$, as $k \to \infty$, we have*

$$\frac{1}{n_k} \sum_{j=1}^{n_k} g\Big(\frac{e_{\mathcal{F}}(j, k)}{k} \Big) \to \int g(t) d\mu_{\mathcal{F}}(t). \tag{1.17}$$

Moreover, $\mu_{\mathcal{F}}$ is characterized by the following identity

$$\int_{t \in \mathbb{R}} t^p d\mu_{\mathcal{F}}(t) = d_p(h^L_{\mathcal{F},0}, h^L_{\mathcal{F},1})^p. \tag{1.18}$$

Remark 1.2. a) The existence of the limit in (1.17) is due to Chen [13] and Boucksom-Chen [11].

b) For filtrations associated with a $\mathbb{C}^*$-action on the pair (X, L), Theorem 1.1 was established by Witt Nyström [77, Theorems 1.1 and 1.4]. For finitely generated filtrations, this result is due to Hisamoto [46, Theorem 1.1], proving a conjecture [77, after Theorem 1.4]. In full, Theorem 1.1 was established by the author in [35].

c) For further developments, the reader may consult [41], for a relative version of Theorem 1.1, cf. also Reboulet [63], and [39] for the relation with Bergman kernels.

The attentive reader will notice the similarity between (1.8) and (1.17), (1.9) and (1.18). In the next two sections, we explore the reasons behind these similarities.

1.3 Toeplitz operators and submultiplicative filtrations

The main goal of this section is to describe a result which unifies (1.8) and (1.17), (1.9) and (1.18). Roughly, we shall proceed as follows. We show that to any filtration one can naturally associate a certain linear operator, and this operator turns out to be the so-called Toeplitz operator. From this, (1.17) can be obtained as a consequence of a statement analogous to (1.8) in the setting of a Toeplitz operator.

Remark that any (decreasing) filtration $\mathcal{F}$ on a finitely dimensional Hermitian vector space (V, H) induces the *weight operator* $A(\mathcal{F}, H) \in \text{End}(V)$, defined as

$$A(\mathcal{F}, H)e_i = w_{\mathcal{F}}(e_i) \cdot e_i, \quad \text{where} \quad w_{\mathcal{F}}(e) := \sup\{\lambda \in \mathbb{R} : e \in \mathcal{F}^\lambda V\}, e \in V, \tag{1.19}$$

and $e_1, \ldots, e_r$, $r := \dim V$, is an orthonormal basis of (V, H) adapted to the filtration $\mathcal{F}$ in the sense that e_1 has the maximal weight, e_2 has the maximal weight among vectors orthogonal to e_1, and so on.

Any Hermitian metric h^L induces a natural L^2-Hermitian metric on the space of holomorphic sections $H^0(X, L^{\otimes k})$. More specifically, for any $s, t \in H^0(X, L^{\otimes k})$, we define

$$\langle s, t\rangle_{L^2} = \int_X \langle s(x), t(x)\rangle_{h^{L^{\otimes k}}} dv_X(x), \tag{1.20}$$

where dv_X is the Riemannian volume form on X, associated with a fixed Kähler form ω and normalized so that $\int_X dv_X(x) = 1$, i.e. $dv_X = \omega^n / \int[\omega]^n$. For any $f \in L^\infty(X)$, we define the Toeplitz operator

$$T_k^X[f] \in \text{End}(H^0(X, L^{\otimes k})), k \in \mathbb{N}, \text{ as } \quad T_k^X[f] := B_k \circ M_{f,k}, \tag{1.21}$$

where $B_k : L^\infty(X, L^{\otimes k}) \to H^0(X, L^{\otimes k})$ is the orthogonal (Bergman) projection to $H^0(X, L^{\otimes k})$, and $M_{f,k} : H^0(X, L^{\otimes k}) \to L^\infty(X, L^{\otimes k})$ is the multiplication map by f, acting as $s \mapsto f \cdot s$.

It is not a coincidence that these operators are called Toeplitz operators and the matrices considered in Section 1.1 were called Toeplitz matrices. In fact, both of them can be interpreted as a part of the general theory that we shall explain in Section 1.4.

The main result of [39] shows that the weight operator associated with a submultiplicative filtration is, up to a negligible error, a Toeplitz operator. This is captured in the following statement.

Theorem 1.3. *For any bounded submultiplicative filtration $\mathcal{F}$, there is a function $\phi(h^L, \mathcal{F}) \in L^\infty(X)$, such that for any $\epsilon > 0$, $p \in [1, +\infty[$, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, we have*

$$\left\| A(\mathcal{F}_k, \textit{Hilb}_k(h^L)) - kT_k^X[\phi(h^L, \mathcal{F})] \right\|_p \leq \epsilon k, \tag{1.22}$$

where $\|\cdot\|_p$ is the p-Schatten norm, defined for an operator $A \in \text{End}(V)$, of a finitely-dimensional Hermitian vector space (V, H) as $\|A\|_p = (\frac{1}{\dim V}\text{Tr}[|A|^p])^{\frac{1}{p}}$, $|A| := (AA^*)^{\frac{1}{2}}$. Moreover, the function $\phi(h^L, \mathcal{F})$ corresponds to the "speed" of the geodesic ray $h^L_{\mathcal{F},t}$ from Theorem 1.1, i.e. $\phi(h^L, \mathcal{F}) = -(h^L_{\mathcal{F},0})^{-1}\frac{d}{dt}h^L_{\mathcal{F},t}|_{t=0}$.

Now, to finally describe the relation between all the above results, we first point out that for Toeplitz operators, an analogue of (1.4) also holds, as established by Boutet de Monvel-Guillemin in [49]. This result was subsequently extended and its proof significantly simplified through the work on Bergman kernels by Dai-Liu-Ma [19] and Ma-Marinescu [53], [52], see the tutorial for Lecture 2 for more details. Roughly, the statement says that[3] for any $f \in L^\infty(X)$, and continuous $g : \mathbb{R} \to \mathbb{R}$, as $k \to \infty$, we have

$$\frac{1}{n_k}\text{Tr}\Big[g\big(T_k^X[f]\big)\Big] \to \frac{n!}{\int_X c_1(L)^n} \cdot \int_X g(f(x))dv_X(x). \tag{1.23}$$

[3]The works [53], [52] demand that f is continuous. For the statement allowing $f \in L^\infty(X)$, see [39, §5].

Remark that immediately from the definitions, in the notations of (1.17), we obtain

$$\mathrm{Tr}\Big[g\Big(\frac{A(\mathcal{F}_k, \mathrm{Hilb}_k(h^L))}{k}\Big)\Big] = \sum_{j=1}^{n_k} g\Big(\frac{e_{\mathcal{F}}(j,k)}{k}\Big). \tag{1.24}$$

Now, if we combine Theorem 1.3 with (1.23), we deduce that, as $k \to \infty$, we have

$$\frac{1}{n_k}\mathrm{Tr}\Big[g\Big(\frac{A(\mathcal{F}_k, \mathrm{Hilb}_k(h^L))}{k}\Big)\Big] \to \int_X g(\phi(h^L, \mathcal{F}))dv_X(x), \tag{1.25}$$

where – we recall – n_k was defined before (1.16). One can deduce Theorem 1.1 from Theorem 1.3 using the following (non-trivial) relationship between the speed and the distance

$$\int_X |\phi(h^L, \mathcal{F})|^p \frac{\omega^n}{n!} = d_p(h^L_{\mathcal{F},0}, h^L_{\mathcal{F},1})^p, \tag{1.26}$$

see [21, Theorem 7.2], [25, Lemma 4.5], [14], [30], for a detailed account on this relationship.

We will not establish Theorem 1.3 in the present lecture notes, and instead, we focus on a direct proof of Theorem 1.1. This not only simplifies the presentation significantly, but is also justified by the fact that the proof of Theorem 1.3 ultimately relies on the proof of Theorem 1.1 that we shall present. We invite the readers interested in the more refined Theorem 1.3 to consult [39].

1.4 Bernstein-Markov measures and Toeplitz theory

In this section, we clarify the relationship between Toeplitz operators and Toeplitz matrices. To this end, we show that the general theory of Toeplitz operators can be extended to a broader setting that encompasses both frameworks.

We fix a continuous Hermitian metric h^L on L and a positive Borel measure μ supported on a compact subset $K \subset X$. We assume that μ is *non-pluripolar*, i.e. it does not charge pluripolar sets – recall that a subset is pluripolar if it is contained in the $\{-\infty\}$-locus of some plurisubharmonic (*psh*) function. We denote by $\mathrm{Hilb}_k(h^L, \mu)$ the positive semi-definite form on $H^0(X, L^{\otimes k})$ defined for arbitrary $s_1, s_2 \in H^0(X, L^{\otimes k})$ as follows

$$\langle s_1, s_2 \rangle_{\mathrm{Hilb}_k(h^L, \mu)} = \int_X \langle s_1(x), s_2(x) \rangle_{(h^L)^k} \cdot d\mu(x). \tag{1.27}$$

Remark that since μ is non-pluripolar, the above form is positive definite.

For a fixed $f \in \mathscr{C}^0(X)$ and $k \in \mathbb{N}^*$, we define $T_k(f) \in \mathrm{End}(H^0(X, L^{\otimes k}))$ as $T_k(f) := B_k \circ M_k(f)$, where $B_k : \mathscr{C}^0(X, L^{\otimes k}) \to H^0(X, L^{\otimes k})$ is the orthogonal (Bergman) projection to $H^0(X, L^{\otimes k})$ with respect to (1.27), and $M_k(f) : H^0(X, L^{\otimes k}) \to \mathscr{C}^0(X, L^{\otimes k})$ is the multiplication map by f. By considering the products $\langle T_k(f)s_1, s_2 \rangle_{\mathrm{Hilb}_k(h^L, \mu)}$ for arbitrary $s_1, s_2 \in H^0(X, L^{\otimes k})$, we see that $T_k(f)$ depends solely on the restriction of f to K. As by Tietze-Urysohn-Brouwer extension theorem, any function from $\mathscr{C}^0(K)$ admits an extension to $\mathscr{C}^0(X)$, we can thus extend the definition of $T_k(f)$ for any $f \in \mathscr{C}^0(K)$.

The main results of [40] show that the analogues of the statements (1.4) and (1.23) hold for arbitrary measures μ satisfying a certain condition called the Bernstein-Markov condition. The latter condition roughly means that the L^2-norm (1.27) should not be very far away from the L^∞-norm on L. Many measures do satisfy this condition. For example, any volume form on a complex manifold satisfies it, and the above definition of $T_k(f)$ then coincides with the definition from Section 1.3. The results of [40] then imply (1.23).

To clarify the relationship with Toeplitz matrices, we encourage the reader to work through the following exercise, cf. [42, §10] and [40, §4]. We embed $\mathbb{S}^1$ into $X := \mathbb{P}^1$ as a great circle (the image of the map $\theta \mapsto [1 : \exp(i\theta)] \in \mathbb{P}^1$, for $\theta \in [0, 2\pi[$, where $[1 : z] \in \mathbb{P}^1$, $z \in \mathbb{C}$, denotes the standard affine chart). Let μ denote the Lebesgue measure on $\mathbb{S}^1$, viewed as a measure on X. Now consider the line bundle $L = \mathscr{O}(1)$ on X. The measure μ will be non-pluripolar and Bernstein-Markov.

The space of global sections $H^0(X, L^{\otimes(k-1)})$ can be naturally identified with the space of polynomials in z of degree at most $k - 1$. Consider $f \in L^\infty(\mathbb{S}^1)$ as in (1.2). The reader will verify that the matrix of the Toeplitz operator associated with f in the basis $\{z^0, z^1, \ldots, z^{k-1}\}$ of $H^0(X, L^{\otimes(k-1)})$ coincides with the Toeplitz matrix (1.1). In particular, the results of [40] then generalize (1.4).

1.5 Tutorial: introduction to Bergman kernel

The main goal of the first tutorial session will be to introduce the basic facts surrounding the Bergman kernel.

Let X be a compact complex manifold and L be an ample line bundle. We assume $H^0(X, L)$ has *no base points*, i.e. for any $x \in X$, there $s \in H^0(X, L)$ verifying $s(x) \neq 0$. Then for each $x \in X$, the set of sections $s \in H^0(X, L)$ vanishing at x, forms a hyperplane $H_p \subset H^0(X, L)$, and so we can define a map

$$\iota_L : X \to \mathbb{P}(H^0(X, L)^*), \tag{1.28}$$

by sending $x \in X$ to $H_p \in \mathbb{P}(H^0(X, L)^*)$.

We can describe the map ι_L more explicitly as follows. For every point $x \in X$, we get the evaluation map $ev_x : H^0(X, L) \to L_x$. As L_x is one-dimensional and ev_x is surjective, it gives us an element of $H^0(X, L)^*$, defined up to a constant, which is precisely $\iota_L(x)$.This map gives us the trivializing section of the line bundle

$$\iota_L^* \mathscr{O}(-1) \otimes L. \tag{1.29}$$

Even more explicitly, choose a basis $s_0, \ldots, s_N$ for $H^0(X, L)$. It defines the dual basis $s_0^*, \ldots, s_N^*$ of $H^0(X, L)^*$. In terms of this dual basis, the evaluation map can be written as

$$ev_x = s_1(x) \cdot s_1^* + \cdots + s_N(x) \cdot s_N^*. \tag{1.30}$$

Hence, upon the identification of $H^0(X, L)^*$ and $H^0(X, L)$ using the above basis, the map ι corresponds to the map

$$\iota_L : X \to \mathbb{P}(H^0(X, L)^*), \qquad x \mapsto [s_0(x), s_1(x), \ldots, s_N(x)]. \tag{1.31}$$

We see from this representation that ι_L is holomorphic.

The seminal work of Kodaira implies that if L is endowed with a positive Hermitian metric h^L, then there is $k_0 \in \mathbb{N}$, so that the above maps $\iota_{L^{\otimes k}}$, are well-defined and give embeddings for $k \geq k_0$. Familiarity with this theorem is assumed; readers who are not acquainted with it may consult Griffith-Harris [43, §1.4]. The main goal of this section is to describe a theorem due to Tian, which somehow refines the theorem of Kodaira on the metric level. It shows that the above embeddings become quasi-isometries for an arbitrary Kähler metric on X in the class $c_1(L)$, properly rescalled.

The relation between Tian's result and Kodaira's theorem is somewhat analogous to the relation between the Whitney and Nash embedding theorems, with one key distinction: the former concerns complex manifolds, whereas the latter pertains to real manifolds. The Whitney embedding theorem asserts that any smooth manifold can be embedded into a real Euclidean space, while Nash;s embedding theorem goes further, stating that such an embedding can be made isometric in the presence of a Riemannian metric.

To explain Tian's result, we need to recall the definition of the Fubini-Study metric.

Exercise 0. Let $Z_0, \dots, Z_n$ be coordinates on $\mathbb{C}^{n+1}$ and denote by

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$$

the standard projection map. Let $U \subset \mathbb{P}^n$ be an open set and $Z : U \to \mathbb{C}^{n+1} \setminus \{0\}$ a lifting of U, i.e., a holomorphic map with $\pi \circ Z = \mathrm{id}$. Consider the differential form

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2,$$

where $\|Z\|^2 = |Z_1|^2 + \ldots + |Z_{n+1}|^2$ is the usual ℓ^2-norm. Show that the definition doesn't depend on the lifting and gives a well-defined Kähler metric on $\mathbb{P}^n\mathbb{C}$.

Proof. If $Z' : U \to \mathbb{C}^{n+1} \setminus \{0\}$ is another lifting, then

$$Z' = f \cdot Z$$

with f a nonzero holomorphic function, so that

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z'\|^2 = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} (\log \|Z\|^2 + \log f + \log \bar{f}).$$

Thus,

$$\omega = \omega + \frac{\sqrt{-1}}{2\pi} (\partial\bar{\partial} \log f - \partial\bar{\partial} \log \bar{f}) = \omega.$$

Therefore, ω is independent of the lifting chosen; since liftings always exist locally, ω is a globally defined differential form in $\mathbb{P}^n$. Clearly, ω is of type $(1, 1)$.

To see that ω is positive, first note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ and leaves the form ω invariant, so that ω is positive everywhere if it is positive at one point.

Now let $\{w_i = Z_i/Z_0\}$ be coordinates on the open set

$$U_0 = \{Z_0 \neq 0\} \subset \mathbb{P}^n$$

and use the lifting

$$Z = (1, w_1, \dots, w_n)$$

on U_0. We then have

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(1 + \sum w_i \bar{w}_i) = \frac{\sqrt{-1}}{2\pi} \partial \left(\frac{\sum w_i d\bar{w}_i}{1 + \sum w_i \bar{w}_i} \right)$$

$$= \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum dw_i \wedge d\bar{w}_i}{1+\sum w_i\bar{w}_i} - \frac{(\sum \bar{w}_i dw_i)\wedge(\sum w_i d\bar{w}_i)}{(1+\sum w_i\bar{w}_i)^2}\right]. \quad (1.32)$$

At the point $[1, 0, \ldots, 0]$,

$$\omega = \frac{\sqrt{-1}}{2\pi}\sum dw_i \wedge d\bar{w}_i > 0.$$

Thus, ω defines a Hermitian metric on $\mathbb{P}^n$, called the *Fubini-Study metric*. □

Remark that a different choice of a Hermitian structure on $\mathbb{C}^{n+1}$ would yield a different Fubini-Study metric on $\mathbb{P}^n$.

We will now define the Bergman kernel, the relevance of which will become clear later on. We assume that $k \in \mathbb{N}$ is big enough, so that $H^0(X, L^{\otimes k})$ is base-point-free (it is always possible to choose such k, see Lecture 2 for a sheaf-theoretic proof of this fact).

We fix a positive Hermitian metric h^L on L and consider the L^2-scalar product on $H^0(X, L^{\otimes k})$, defined as in (1.20), where $\omega = c_1(L, h^L)$ is the first Chern class, the latter is defined as $c_1(L, h^L) = \frac{\sqrt{-1}}{2\pi}R^L$, where R^L is the curvature of the Chern connection on (L, h^L) (the only connection preserving the Hermitian and holomorphic structures). We choose an orthonormal basis $\{s_1, s_2, \ldots, s_{n_k}\}$ of $H^0(X, L^{\otimes k})$, where $n_k := \dim H^0(X, L^{\otimes k})$. The *Bergman kernel* of the Hermitian metric h^L is the function defined as follows

$$B_k : X \to \mathbb{R}, \quad x \mapsto \sum_{i=1}^{n_k} \left|s_i(x)\right|^2_{(h^L)^k}. \quad (1.33)$$

Exercise 1: B_k is independent of the choice of the orthonormal basis.
Exercise 2: show that for any $x \in X$ we have

$$B_k(x) = \sup\Big\{ |s(x)|^2_{h^{L^{\otimes k}}} : \|s\|_{L^2} = 1\Big\}. \quad (1.34)$$

Proof: As $H^0(X, L^{\otimes k})$ is base-point-free, the subspace

$$E_x = \Big\{ t \in H^0(X, L^{\otimes k}) : t(x) = 0\Big\} \quad (1.35)$$

has codimension 1. Let s be in the orthonormal complement of E_x with $\|s\|_{L^2} = 1$. Then it follows from the definition that $B_k(x) = |s(x)|^2_{h^{L^{\otimes k}}}$, since every section orthonormal to s vanishes at x. □

Exercise 3: Suppose the Kodaira embedding

$$\iota_k : X \longrightarrow \mathbb{P}(H^0(X, L^{\otimes k})^*),$$

is defined on all of X. Then

$$\frac{1}{k}\iota_k^*\omega_{FS} - \omega = \frac{\sqrt{-1}}{2\pi}\frac{\partial\bar{\partial}\log B_k}{k},$$

where ω_{FS} is the Fubini-Study metric defined by the L^2-metric induced by h^L.
Proof: Directly from the description of the Fubini-Study metric and the Kodaira embedding 1.31, we see that over the set $U \subset X$, verifying $s_1(x) \neq 0$ for $x \in U$, we have

$$\iota_k^*\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Big(1 + \Big|\frac{s_1(x)}{s_2(x)}\Big|^2 + \cdots + \Big|\frac{s_1(x)}{s_{n_k}(x)}\Big|^2\Big).$$

The result now follows directly from this, the definition of the Bergman kernel and the Lelong-Poincare equation. □

The following fundamental result was established by Tian [75], Zelditch [78], Bouche [9]. We shall discuss a proof of a version of it in the following lecture.

Theorem 1.4. *There are smooth functions* $a_i : X \to \mathbb{R}$, $i \in \mathbb{N}$, *such that* $a_0 = 1$ *and for any* $p, q \in \mathbb{N}$, *there is* $C > 0$ *such that*

$$\Big| B_k(x) - k^n \cdot \sum_{i=0}^{p} a_i(x) k^{-i} \Big|_{\mathscr{C}^q} \leq C\, k^{n-p-1}. \tag{1.36}$$

As an immediate corollary of Theorem 1.4 and last exercise, we deduce the following result.

Corollary 1.5. For any positive Hermitian metric h^L on L, as $k \to \infty$, we have the following convergence $\frac{1}{k}\iota_k^* \omega_{FS} \to \omega$ in the space of smooth differential forms on X.

We also note that Theorem 1.4 immediately implies the following asymptotic version of Riemann-Roch-Hirzerbruch formula.

Corollary 1.6. As $k \to \infty$, we have

$$\dim H^0(X, L^{\otimes k}) \sim \frac{\int_X c_1(L)^n}{n!} \cdot k^n. \tag{1.37}$$

Proof. Use the identity $\dim H^0(X, L^{\otimes k}) = \int B_k(x) \cdot \frac{\omega^n}{n!}$. □

2 Semiclassical extension theorem

In their seminal paper [58], Ohsawa and Takegoshi established a sufficient condition under which a holomorphic section of a vector bundle over a submanifold can be extended to a holomorphic section over the ambient manifold, with a controlled bound on the L^2-norm of the extension in terms of the original L^2-norm. The main objective of the second lecture is to present a refinement of this result in the semiclassical setting. Here, the semiclassical setting refers to the case where the vector bundle in question is a high tensor power of a fixed ample line bundle, and our interest lies in understanding the asymptotic behavior as the tensor power tends to infinity.

In its simplest form, semiclassical extensions have already been studied in the paper of Tian [75], where he introduced *peak sections*. In fact, peak sections can be seen as images of the *optimal extension operator*, see (2.5), applied for a submanifold given by a point. In this perspective, the main result of this lecture can be seen as a generalization of the result of Tian from points to general submanifolds of higher dimension.

2.1 Statement of the result

In this section, we review the results from [37], [38] on asymptotics of the L^2-optimal holomorphic extensions of holomorphic sections along submanifolds associated with high tensor powers of a positive line bundle.

Let us fix two complex manifolds X, Y of dimensions n and m respectively. For the sake of simplicity, we assume that X and Y are compact, although our results work in a more general setting of manifolds and embeddings of bounded geometry. We fix also a complex embedding $\iota : Y \to X$, a positive line bundle (L, h^L) over X. In particular, we assume that for the curvature R^L of the Chern connection on (L, h^L), the closed real $(1,1)$-differential form

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L, \tag{2.1}$$

is positive. We denote by g^{TX} the Riemannian metric on X so that its Kähler form coincides with ω. We denote by g^{TY} the induced metric on Y and by dv_X, dv_Y the associated Riemannian volume forms on X and Y.

Let TX, TY be the holomorphic tangent bundles of X and Y. We identify the (holomorphic) normal bundle $N = TX/TY$ of Y in X as orthogonal complement of TY in TX, so that we have the (smooth) orthogonal decomposition $TX|_Y \to TY \oplus N$.

For any smooth sections f, f' of L^k, $k \in \mathbb{N}$, over X, we define the L^2-scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_h$, induced by h^L as in (1.20).

We have the restriction operator

$$\mathrm{Res}_k : H^0(X, L^k) \to H^0(Y, \iota^* L^k), \qquad f \mapsto f|_Y. \tag{2.2}$$

A standard argument based on short exact sequences and Serre vanishing theorem implies that there is $p_0 \in \mathbb{N}$, such that for any $k \geq k_0$, the operator Res_k is surjective. Indeed, consider the short exact sequence of sheaves

$$0 \to \mathscr{O}_X(L^k) \otimes \mathscr{J}_Y \to \mathscr{O}_X(L^k) \to \mathscr{O}_X(L^k) \otimes \mathscr{O}_X/\mathscr{J}_Y \to 0, \tag{2.3}$$

where $\mathscr{J}_Y$ is the ideal sheaf of holomorphic germs on X, which vanish along Y. The associated long exact sequence in cohomology gives us

$$\cdots \to H^0(X, L^k) \to H^0(X, L^k \otimes \mathscr{O}_X/\mathscr{J}_Y) \to H^1(X, L^k \otimes \mathscr{J}_Y) \to \cdots . \tag{2.4}$$

By Serre vanishing theorem, for p big enough, the cohomology $H^1(X, L^k \otimes \mathscr{J}_Y)$ vanishes by the ampleness of L. This finishes the proof of the surjectivity of the restriction morphism, Res_k, as the first map of the above long exact sequence corresponds to Res_k under the natural isomorphism $H^0(X, L^k \otimes \mathscr{O}_X/\mathscr{J}_Y) \simeq H^0(Y, \iota^* L^k)$, and by (2.4) the vanishing of the first cohomology group means exactly that the map is surjective.

By the surjectivity of Res_k, we can define the *optimal extension operator*

$$\mathrm{E}_k : H^0(Y, \iota^* L^k) \to H^0(X, L^k), \tag{2.5}$$

by putting $\mathrm{E}_k g = f$, where $\mathrm{Res}_k(f) = g$, and f has the minimal L^2-norm among those $f' \in H^0(X, L^k)$ satisfying $\mathrm{Res}_k(f') = g$. Clearly, the minimizing f exists and it is unique. Moreover, the operator E_k is linear since the minimality of the L^2-norm among different extensions is characterized by a linear condition, requiring the image to be orthogonal to the space of holomorphic sections vanishing along Y. The main goal of [37] is to find an explicit asymptotic expansion of the operator E_k, as $k \to \infty$.

To describe the first term of the asymptotic expansion, we introduce some trivializations. For $y \in Y$, $Z_N \in N_y$, let $\mathbb{R} \ni t \mapsto \exp_y^X(tZ_N) \in X$ be the geodesic in X in direction Z_N. For $r_\perp > 0$ small enough, this map induces a diffeomorphism of $r_\perp$-neighborhood of the zero section in N with a tubular neighborhood U of Y in X. We use this identification, called *geodesic normal coordinates*, implicitly. We denote by $\pi : U \to Y$ the natural projection $(y, Z_N) \mapsto y$. Over U, we identify L, F to $\pi^*(L|_Y)$ by the parallel transport with respect to the Chern connections along the geodesic $[0,1] \ni t \mapsto (y, tZ_N) \in X$, $|Z_N| < r_\perp$.

We fix a smooth function $\rho : \mathbb{R}_+ \to [0,1]$, satisfying

$$\rho(x) = \begin{cases} 1, & \text{for } x < \frac{1}{4}, \\ 0, & \text{for } x > \frac{1}{2}. \end{cases} \tag{2.6}$$

We define the *trivial extension operator* $\mathrm{E}_k^0 : H^0(Y, \iota^* L^k) \to L^2(X, L^k)$ as follows. For $g \in H^0(Y, \iota^* L^k)$, we let $(\mathrm{E}_k^0 g)(x) = 0$ for $x \notin U$, and in U, we define $\mathrm{E}_k^0 g$ using the geodesic normal coordinates as follows

$$(\mathrm{E}_k^0 g)(y, Z_N) = g(y) \cdot \rho\Big(\frac{|Z_N|}{r_\perp}\Big) \cdot \exp\Big(- k\frac{\pi}{2}|Z_N|^2\Big). \tag{2.7}$$

The Gaussian integral calculation gives us for any $f \in H^0(Y, \iota^* L^k)$, as $k \to \infty$, the following asymptotics

$$\big\|\mathrm{E}_k^0 f\big\|_{L^2(X)} \sim \frac{1}{k^{\frac{n-m}{2}}} \cdot \big\|f\big\|_{L^2(Y)}. \tag{2.8}$$

In particular, as $k \to \infty$, we see that

$$\big\|\mathrm{E}_k^0\big\| \sim \frac{1}{k^{\frac{n-m}{2}}}, \tag{2.9}$$

where $\| \cdot \|$ is the operator norm, calculated with respect to the L^2-norms.

Now, the section $\mathrm{E}_k^0 g$ satisfies $(\mathrm{E}_k^0 g)|_Y = g$, but it is not holomorphic over X (unless g is null). Nevertheless, as our main result of [37] says, $\mathrm{E}_k^0 g$ approximates very well the holomorphic section $\mathrm{E}_k g$. More precisely, we have the following result.

Theorem 2.1. *There are $C > 0$, $k_1 \in \mathbb{N}^*$, such that for any $k \geq k_1$, we have*

$$\big\|\mathrm{E}_k - \mathrm{E}_k^0\big\| \leq \frac{C}{k^{\frac{n-m+1}{2}}}. \tag{2.10}$$

Remark 2.2. a) By (2.9), Theorem 2.1 tells us that the principal asymptotic term of the optimal extension operator is given by the trivial extension operator.

b) Theorem 2.1 refines previous works of Zhang [79, Theorem 2.2], Bost [8, Theorem A.1] and Randriambololona [62, Théorème 3.1.10], stating that for any $\epsilon > 0$, there is $k_1 \in \mathbb{N}^*$, such that $\big\|\mathrm{E}_k\big\| \leq \exp(\epsilon k)$ for $k \geq k_1$.

Clearly, Theorem 2.1 and (2.9) imply that for any $\epsilon > 0$, there is $k_1 \in \mathbb{N}$, such that for any $k \geq k_1$, we have

$$\|\mathrm{E}_k\| \leq \frac{1 + \epsilon}{k^{\frac{n-m}{2}}}. \tag{2.11}$$

This statement can be rephrased in the usual language of extension theorem as follows. For any $\epsilon > 0$ there is $k_1 \in \mathbb{N}$, such that for any $k \geq k_1$, $f \in H^0(Y, \iota^* L^k)$, there is $\tilde{f} \in H^0(X, L^k)$, verifying $\mathrm{Res}_k(\tilde{f}) = f$, and such that

$$\|\tilde{f}\|_{L^2(X)} \leq \frac{1 + \epsilon}{k^{\frac{n-m}{2}}} \cdot \|f\|_{L^2(Y)}. \tag{2.12}$$

Remark, however, that Theorem 2.1 and (2.8) say even more. For any $\tilde{f} \in H^0(X, L^k)$, verifying $\mathrm{Res}_k(\tilde{f}) = f$, we have

$$\|\tilde{f}\|_{L^2(X)} \geq \|\mathrm{E}_k(\mathrm{Res}_k \tilde{f})\|_{L^2(X)} \geq \frac{1-\epsilon}{k^{\frac{n-m}{2}}} \cdot \|f\|_{L^2(Y)}. \tag{2.13}$$

The above two statements will play a crucial role later on, and to illustrate this better, we shall develop a special notation. We denote the L^2-norm on $H^0(X, L^{\otimes k})$ by $\mathrm{Hilb}_k^X(h^L)$, and the L^∞-norm by $\mathrm{Ban}_k^X(h^L)$. To ease the notation, we denote the corresponding norms on $H^0(Y, \iota^* L^{\otimes k})$ by $\mathrm{Hilb}_k^Y(h^L)$ and $\mathrm{Ban}_k^Y(h^L)$ respectively. Here Ban stands for "Banach" and Hilb is for "Hilbert".

Recall that a norm $N_V = \| \cdot \|_V$ on a finitely dimensional vector space V naturally induces the norm $\| \cdot \|_Q$ on any quotient $Q, \pi : V \to Q$ of V through the following identity

$$\|f\|_Q := \inf \big\{ \|g\|_V; \quad g \in V, \pi(g) = f \big\}, \qquad f \in Q. \tag{2.14}$$

By a slight abuse of notations, we sometimes denote the quotient norm by $[N_V]$, i.e. without the reference to the quotient space. Of course, the norm $[N_V]$ is Hermitian whenever N_V is Hermitian.

In particular, by the surjectivity of the map (2.2), any norm on $H^0(X, L^k)$ induces a norm on $H^0(Y, \iota^* L^k)$. The bounds (2.12) and (2.13) then translate into this language in the following manner.

Corollary 2.3. For any $\epsilon > 0$ and a positive Hermitian metric h^L on L, there is $k \in \mathbb{N}^*$, so that for any $k \geq k_0$, we have

$$(1-\epsilon) \cdot k^{\frac{n-m}{2}} \cdot \mathrm{Hilb}_k^Y(h^L) \leq [\mathrm{Hilb}_k^X(h^L)] \leq (1+\epsilon) \cdot k^{\frac{n-m}{2}} \cdot \mathrm{Hilb}_k^Y(h^L). \tag{2.15}$$

A slightly more refined analysis would show the following statement.

Proposition 2.4. For any $\epsilon > 0$ and a positive Hermitian metric h^L on L, there is $k \in \mathbb{N}^*$, so that for any $k \geq k_0$, we have

$$\mathrm{Ban}_k^Y(h^L) \leq [\mathrm{Ban}_k^X(h^L)] \leq (1+\epsilon) \cdot \mathrm{Ban}_k^Y(h^L). \tag{2.16}$$

In this mini-course, however, the following statement due to Zhang [79, Theorem 2.2], Bost [8, Theorem A.1] and Randriambololona [62, Théorème 3.1.10] will always be sufficient.

Proposition 2.5. For any $\epsilon > 0$ and a continuous Hermitian metric h^L on L with psh potential, there is $k \in \mathbb{N}^*$, so that for any $k \geq k_0$, we have

$$\mathrm{Ban}_k^Y(h^L) \leq [\mathrm{Ban}_k^X(h^L)] \leq \exp(\epsilon k) \cdot \mathrm{Ban}_k^Y(h^L). \tag{2.17}$$

Proof. The lower bound is immediate: it basically says that the sup-norm of a restriction of a section is not bigger than the sup-norm of the section. To establish the upper bound, recall that according to Demailly's regularization theorem, see [27], [28], on ample line bundle, any continuous metric with psh potential is regularizable from below, meaning that there is an increasing sequence of smooth positive metrics h_i^L, $i \in \mathbb{N}$, converging uniformly to h^L, cf. [44, Theorem 8.1]. We choose $i \in \mathbb{N}$ so that $h^L \leq \exp(\epsilon/2) h_i^L$. Then we clearly have $[\mathrm{Ban}_k^X(h^L)] \leq \exp(\epsilon k/2)[\mathrm{Ban}_k^X(h_i^L)]$. The result then follows immediately from Proposition 2.4, applied for h_i^L. □

2.2 Schwartz kernel of the extension theorem

Theorem 2.1 was established in [37, Theorem 1.1] as an almost direct consequence of more precise results about the asymptotics of the *Schwartz kernel* $\mathrm{E}_k(x, y) \in L^k_x \otimes L^{p*}_y$, $x \in X$, $y \in Y$, of E_k. To describe these results, recall first that the *Schwartz kernel* $\mathrm{E}_k(x, y)$ is defined so that for any $g \in L^2(Y, \iota^* L^k)$, $x \in X$, we have

$$(\mathrm{E}_k g)(x) = \int_Y \mathrm{E}_k(x, y) \cdot g(y) dv_Y(y), \tag{2.18}$$

where we extended the domain of E_k from $H^0(Y, \iota^* L^k)$ to $L^2(Y, \iota^* L^k)$ by precomposing it with the orthogonal projection onto $H^0(Y, \iota^* L^k)$.

Then the first result needed for the proof of Theorem 2.1 shows that $\mathrm{E}_k(x, y)$ has exponential decay with respect to the distance between the parameters.

Theorem 2.6 ([37, Theorem 1.5]). *There are $c > 0$, $k_1 \in \mathbb{N}^*$, such that for any $k \geq k_1$, $x \in X$, $y \in Y$, the following estimate holds*

$$\big|\mathrm{E}_k(x, y)\big| \leq C k^m \exp\big(- c\sqrt{k} \cdot \mathrm{dist}(x, y)\big). \tag{2.19}$$

From Theorem 2.6, we see that to understand fully the asymptotics of $\mathrm{E}_k(x, y)$, it suffices to do so for x, y in a neighborhood of a fixed point $(y_0, y_0) \in Y \times Y$ in $X \times Y$. In [37, Theorem 1.6], we show that after a reparametrization, given by a homothety with factor $\sqrt{k}$ in the so-called Fermi coordinates around (y_0, y_0), the Schwartz kernel $\mathrm{E}_k(x, y)$ admits a complete asymptotic expansion in integer powers of $\sqrt{k}$, as $k \to \infty$. The first two terms of this expansion can be easily calculated explicitly, and the first term corresponds to the Schwartz kernel of the optimal extension operator of the so-called Fock-Bargmann space.

More precisely, we fix a point $y_0 \in Y$ and an orthonormal frame $(e_1, \ldots, e_{2m})$ (resp. $(e_{2m+1}, \ldots, e_{2n})$) in $(T_{y_0} Y, g^{TY}_{y_0})$ (resp. in $(N_{y_0}, g^N_{y_0})$), such that for $i = 1, \ldots, n$, $J e_{2i-1} = e_{2i}$, where J is the complex structure of X. For $Z \in \mathbb{R}^{2n}$, we denote by z_i, $i = 1, \ldots, n$, the induced complex coordinates $z_i = Z_{2i-1} + \sqrt{-1} Z_{2i}$. We frequently use the decomposition $Z = (Z_Y, Z_N)$, where $Z_Y = (Z_1, \ldots, Z_{2m})$ and $Z_N = (Z_{2m+1}, \ldots, Z_{2n})$ and implicitly identify Z (resp. Z_Y, Z_N) to an element in $T_y X$ (resp. $T_y Y$, N_y) by

$$Z = \sum_{i=1}^{2n} Z_i e_i, \quad Z_Y = \sum_{i=1}^{2m} Z_i e_i, \quad Z_N = \sum_{i=2m+1}^{2n} Z_i e_i. \tag{2.20}$$

We fix $r_Y, r_\perp > 0$ small enough. We define the coordinate system $\psi_{y_0} : B_0^{\mathbb{R}^{2m}}(r_Y) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \to X$, for $Z = (Z_Y, Z_N)$, $Z_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $Z_Y = (Z_1, \ldots, Z_{2m})$, $Z_N = (Z_{2m+1}, \ldots, Z_{2n})$, $|Z_Y| < r_Y$, $|Z_N| < r_\perp$, by

$$\psi_{y_0}(Z_Y, Z_N) := \exp^X_{\exp^Y_{y_0}(Z_Y)}(Z_N(Z_Y)), \tag{2.21}$$

where $Z_N(Z_Y) \in N_{\exp^Y_{y_0}(Z_Y)}$ is the parallel transport of Z_N along the geodesic $\exp^Y_{y_0}(t Z_Y)$, $t = [0, 1]$, with respect to the connection ∇^N on N given by the projection of the Levi-Civita connection on N, and $B_0^{\mathbb{R}^n}(\epsilon)$, $\epsilon > 0$ means the euclidean ball of radius ϵ around $0 \in \mathbb{R}^n$. The coordinates ψ_{y_0}

are called the *Fermi coordinates* at y_0. Fermi coordinates, in particular, provide a trivialization of the normal bundle, N, in a neighborhood of y_0.

We will now introduce a trivialization of the line bundle L using Fermi coordinates. We fix an orthonormal frame $l \in L_{y_0}$, and define the orthonormal frame $\tilde{l}$ by taking the parallel transport of l with respect to the Chern connection ∇^L of (L, h^L), done first along the path $\psi_{y_0}(tZ_Y, 0)$, $t \in [0,1]$, and then along the path $\psi_{y_0}(Z_Y, tZ_N)$, $t \in [0,1]$, $Z_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z_Y| < r_Y$, $|Z_N| < r_\perp$. This frame and the induced frame of the dual line bundle allows us to view $\mathrm{E}_k(x,y)$ as a complex-valued function of $x \in X$, $y \in Y$ in a $\min(r_\perp, r_Y)$-neighborhood of y_0.

Using the identifications similar to the ones before (2.20), we view the space $\mathbb{C}^{n-m}$ as a holomorphic normal bundle of $\mathbb{C}^m$ in $\mathbb{C}^n$ with basis $\frac{\partial}{\partial z_i}$, $i = m+1, \ldots, n$. We define the function $\mathscr{E}_{n,m} : \mathbb{R}^{2n} \times \mathbb{R}^{2m} \to \mathbb{C}$ for $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$ as follows

$$\mathscr{E}_{n,m}(Z, Z'_Y) = \exp\Big(-\frac{\pi}{2} \sum_{i=1}^{m} \big(|z_i|^2 + |z'_i|^2 - 2 z_i \overline{z}'_i \big) - \frac{\pi}{2} \sum_{i=m+1}^{n} |z_i|^2 \Big), \tag{2.22}$$

where $Z = (Z_1, \ldots, Z_{2n})$, $Z'_Y = (Z'_1, \ldots, Z'_{2m})$ and $z_i := Z_{2i-1} + \sqrt{-1} Z_{2i}$ and $z'_i := Z'_{2i-1} + \sqrt{-1} Z'_{2i}$. The rationale behind the function (2.22) is that it corresponds precisely to the Schwartz kernel, written in Fermi coordinates, of the optimal extension operator from an m-dimensional linear subspace of the n-dimensional Fock-Bargmann space to the entire Fock-Bargmann space, see [37, §3.2] for a justification. The result [37, Theorem 1.6] says that the general embedding of a submanifold in a manifold is not too far from this model one.

Theorem 2.7. *There are $\epsilon, c, C, Q > 0$, $k_1 \in \mathbb{N}^*$, such that for any $y_0 \in Y$, $k \geq k_1$, $Z = (Z_Y, Z_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z|, |Z'_Y| \leq \epsilon$, we have*

$$\begin{aligned} \Big| \frac{1}{k^m} \mathrm{E}_k\big(\psi_{y_0}(Z), \psi_{y_0}(Z'_Y)\big) &- \mathscr{E}_{n,m}(\sqrt{k} Z, \sqrt{k} Z'_Y) \Big| \\ &\leq \frac{C}{k^{\frac{1}{2}}} \cdot \Big(1 + \sqrt{k}|Z| + \sqrt{k}|Z'_Y| \Big)^Q \exp\Big(-c\sqrt{k}\big(|Z_Y - Z'_Y| + |Z_N| \big) \Big). \end{aligned} \tag{2.23}$$

Proof of Theorem 2.1. Let us now explain how (2.19) and (2.36) imply Theorem 2.1. As we explain in [37, (5.124) and (5.125)], directly from the off-diagonal asymptotic expansion of the Bergman kernel due to Dai-Liu-Ma [19], we deduce that the Schwartz kernel of the trivial extension operator, $\mathrm{E}^0_k(x,y)$, $x \in X$, $y \in Y$ has an asymptotic expansion as in (2.36) (in particular, with the higher order term). From this, (2.19), (2.36) and the exponential decay of the Bergman kernel proved by Ma-Marinescu [54], we deduce that the Schwartz kernel $K^0_k(x,y)$, $x \in X$, $y \in Y$, of $K_k := \mathrm{E}_k - \mathrm{E}^0_k$ satisfies the following bound, see [37, (5.127)]: there are $c, C > 0$, $k_1 \in \mathbb{N}^*$, such that for any $k \geq k_1$, $x \in X$, $y \in Y$, the following estimate holds

$$\Big| K_k(x,y) \Big| \leq C k^{m - \frac{1}{2}} \exp\big(-c\sqrt{k} \mathrm{dist}(x,y) \big). \tag{2.24}$$

From (2.24), we conclude that for any $k \in \mathbb{N}$, there are $c, C > 0$, $k_1 \in \mathbb{N}^*$, such that for any $k \geq k_1$, $y_0, y_1 \in Y$, the Schwartz kernel $G_k(y_0, y_1)$ of the operator $G_k := K^*_k \circ K_k$ satisfies the following estimate

$$\int_Y \big| G_k(y_0, y) \big| dv_Y(y) \leq \frac{C}{k^{n-m+1}}, \qquad \int_Y \big| G_k(y, y_0) \big| dv_Y(y) \leq \frac{C}{k^{n-m+1}}. \tag{2.25}$$

Directly from (2.25) and Young's inequality for integral operators, cf. [71, Theorem 0.3.1] applied for $p, q = 2$, $r = 1$ in the notations of [71], we obtain that $\|G_k\| \leq \frac{C}{k^{n-m+1}}$, which implies that

$$\|K_k\| \leq \Big(\frac{C}{k^{n-m+1}} \Big)^{\frac{1}{2}}. \tag{2.26}$$

But the latter statement is a restatement of Theorem 2.1 by the very definition of K_k. □

The proof of (2.19) and (2.36) lies beyond the scope of these lecture notes. We invite the reader to consult the articles [37] and [38] for two (different) proofs of these results and [36] for an overview of them.

Proof of Proposition 2.4. As in the proof of Proposition 2.5, the lower bound is immediate: it basically says that the sup-norm of a restriction of a section is not bigger than the sup-norm of the section. For the upper bound, it suffices to show that for any $f \in H^0(Y, \iota^* L^k)$,

$$\big\| \mathrm{E}_p f \big\|_{L^\infty(X)} \leq \Big(1 + \frac{C}{\sqrt{k}} \Big) \cdot \big\| f \big\|_{L^\infty(Y)}. \tag{2.27}$$

From (2.24) and the usual Gaussian integral calculation, there are $C > 0$, $k_1 \in \mathbb{N}^*$, such that for any $k \geq k_1$, $f \in H^0(Y, \iota^* L^k)$, we have $\|K_k f\|_{L^\infty(X)} \leq \frac{C}{\sqrt{k}} \cdot \|f\|_{L^\infty(Y)}$. Remark also that by construction, we have $\|\mathrm{E}^0_k f\|_{L^\infty(X)} = \|f\|_{L^\infty(Y)}$. These estimates clearly imply (2.27). □

2.3 Explicit kernels in Fock-Bargmann space

In this section, we consider the model situation of the complex vector space, for which an explicit formula for the Schwartz kernels of Bergman projectors and the extension operator can be given.

Endow $X := \mathbb{C}^n$ with the standard metric and consider a trivialized complex line bundle L_0 on $\mathbb{C}^n$. We endow L_0 with the Hermitian metric h^{L_0}, given by

$$\|1\|_{h^{L_0}}(Z) = \exp\Big(-\frac{\pi}{2} |Z|^2 \Big), \tag{2.28}$$

where Z is the natural real coordinate on $\mathbb{C}^n$, and 1 is the trivializing section of L_0. An easy verification shows that (2.28) implies that (2.1) holds in our setting. Recall that [52, §4.1.6] shows that the Kodaira Laplacian $\mathscr{L}$ on $\mathscr{C}^\infty(X, L_0)$, multiplied by 2, and viewed as an operator on $\mathscr{C}^\infty(X)$ using the orthonormal trivialization, given by $1 \cdot \exp(\frac{\pi}{2}|Z|^2)$, is given by

$$\mathscr{L} = \sum_{i=1}^n b_i b_i^+, \tag{2.29}$$

where b_i, b_i^+ are *creation* and *annihilation* operators, defined as

$$b_i = -2 \frac{\partial}{\partial z_i} + \pi \overline{z}_i, \qquad b_i^+ = 2 \frac{\partial}{\partial \overline{z}_i} + \pi z_i. \tag{2.30}$$

A classical calculation, cf. [52, Theorem 4.1.20], shows that the orthonormal basis with respect to the induced L^2-norm of $\ker \mathscr{L}$ is given in the orthonormal trivialization above by

$$\Big(\frac{\pi^{|\beta|}}{\beta!} \Big)^{1/2} z^\beta \exp\Big(-\frac{\pi}{2} |Z|^2 \Big), \qquad \beta \in \mathbb{N}^n. \tag{2.31}$$

In particular, [52, (4.1.84)], the Bergman kernel $\mathscr{P}_n$ of $\mathbb{C}^n$ is given for $Z, Z' \in \mathbb{C}^n$ by

$$\mathscr{P}_n(Z, Z') = \exp\Big(-\frac{\pi}{2} \sum_{i=1}^{n} \big(|z_i|^2 + |z_i'|^2 - 2 z_i \overline{z}_i' \big) \Big). \tag{2.32}$$

Let us calculate the L^2-extension operator $\mathscr{E}_{n,m}$, extending each element from $(\ker \mathscr{L})|_Y$ to an element from $\ker \mathscr{L}$ with the minimal L^2-norm. From (2.31), we easily see that for $Z_Y \in \mathbb{C}^m$, $Z_N \in \mathbb{C}^{n-m}$ and $g \in (\ker \mathscr{L})|_Y$, we have

$$(\mathscr{E}_{n,m} g)(Z_Y, Z_N) = g(Z_Y) \exp\Big(-\frac{\pi}{2} |Z_N|^2 \Big). \tag{2.33}$$

We extend $\mathscr{E}_{n,m}$ to the whole L^2-space by $g \mapsto (\mathscr{E}_{n,m} \circ \mathscr{P}_m) g$. From (2.32) and (2.33), we see that the kernel of $\mathscr{E}_{n,m}$ corresponds precisely to the quantity, defined in (2.22).

2.4 Bergman kernel: study away from the diagonal

In this section, we undertake a more detailed examination of the Bergman kernel and its relationship with the extension operator. More precisely, we study the off-diagonal expansion of the Bergman kernel due to Dai-Liu-Ma [19] and show that it can be seen as a special case of the asymptotic expansion of the extension operator stated in Theorem 2.7. It is important to stress, however, that the result of Dai-Liu-Ma [19] plays a foundational role in both known proofs of Theorem 2.7, namely those in [37] and [38], and should therefore not be viewed as a consequence of it.

We continue to use the notations introduced in Section 1.5. We fix a positive Hermitian metric h^L on L and consider the L^2-scalar product on $H^0(X, L^{\otimes k})$, defined as in (1.20), where $\omega = c_1(L, h^L)$ is the first Chern class, the latter is defined as $c_1(L, h^L) = \frac{\sqrt{-1}}{2\pi} R^L$, where R^L is the curvature of the Chern connection on (L, h^L). We choose an orthonormal basis $\{s_1, s_1, \ldots, s_{n_k}\}$ of $H^0(X, L^{\otimes k})$, where $n_k := \dim H^0(X, L^{\otimes k})$. The *Bergman kernel*, $B_k(x, y) \in L^k_x \otimes (L^k_y)^*$ for any $x, y \in X$ is defined as

$$B_k(x, y) := \sum_{i=1}^{n_k} s_i(x) \cdot s_i(y)^*. \tag{2.34}$$

Remark that by (1.33), we have the following relation $B_k(x, x) = B_k(x)$.

The word kernel in "Bergman kernel" refers to the fact that this section can be viewed as the Schwartz kernel of the orthogonal projection $B_k : L^2(X, L^{\otimes k}) \to H^0(X, L^{\otimes k})$, i.e. we have

$$(B_k f)(x) = \int_{y \in X} B_k(x, y) \cdot f(y) \cdot dv_X(y), \tag{2.35}$$

for any $f \in L^2(X, L^{\otimes k})$.

The result of Dai-Liu-Ma [19] says that the Bergman kernel of a general complex manifold is not too far from the model one of the Fock-Bargmann space, denoted by $\mathscr{P}_n$ in (2.32). To state their result precisely, we introduce a trivialization of the line bundle L using geodesic coordinates around $x_0 \in X$. For $Z \in \mathbb{R}^{2n}$, identified with $T_{x_0}X$ by means of a fixed orthonormal frame, we let $\phi_{x_0}(Z) := \exp^X_{x_0}(Z)$. We fix an orthonormal frame $l \in L_{x_0}$, and define the orthonormal frame $\tilde{l}$ by taking the parallel transport of l with respect to the Chern connection ∇^L of (L, h^L), done along the path $\phi_{x_0}(t \cdot Z)$, $t \in [0,1]$, $t \in [0,1]$, $Z \in \mathbb{R}^{2n}$, $|Z| < r_X$, where r_X is small enough. This frame and the induced frame of the dual line bundle allows us to view $B_k(x, y)$ as a complex-valued function of $x \in X$, $y \in Y$ in a r_X-neighborhood of x_0.

Theorem 2.8 (Dai-Liu-Ma [19]). *There are $\epsilon, c, C, Q > 0$, $k_1 \in \mathbb{N}^*$, such that for any $x_0 \in X$, $k \geq k_1$, $Z, Z' \in \mathbb{R}^{2n}$, $|Z|, |Z'| \leq \epsilon$, we have*

$$\Big| \frac{1}{k^n} B_k\big(\phi_{x_0}(Z), \phi_{x_0}(Z')\big) - \mathscr{P}_n(\sqrt{k}Z, \sqrt{k}Z') \Big| \leq \frac{C}{\sqrt{k}} \cdot \Big(1 + \sqrt{k}|Z| + \sqrt{k}|Z'|\Big)^Q \exp\Big(- c\sqrt{k}\big(|Z - Z'|\big)\Big). \tag{2.36}$$

The above result is often used in conjunction with the following rough estimate on the Bergman kernel away from the diagonal.

Theorem 2.9 (Christ [17], Ma-Marinescu [54]). *There are $c > 0$, $k_0 \in \mathbb{N}^*$, such that for any $k \geq k_0$, $x, x' \in X$, the following estimate holds*

$$\big|B_k(x, x')\big| \leq Ck^n \exp\big(- c\sqrt{k} \cdot \operatorname{dist}(x, x')\big). \tag{2.37}$$

Let us point out one particularly significant corollary of Theorem 2.8 and Theorem 2.9. We assume that k is large enough so that $n_k > 0$, where $n_k := \dim H^0(X, L^{\otimes k})$. We define a sequence of measures on $X \times X$ as follows

$$\mu_k^{\rm Berg} := \frac{1}{n_k} |B_k(x, y)|^2_{(h^L)^k} \cdot dv_X(x) \cdot dv_X(y). \tag{2.38}$$

We shall explain later on, cf. (2.51), that $\mu_k^{\rm Berg}$ are probability measures for any $k \in \mathbb{N}$.

Corollary 2.10. The measures $\mu_k^{\rm Berg}$ converge weakly, as $k \to \infty$, to the measure $\Delta_* d\overline{v}_X$, where $\Delta : X \to X \times X$ is the diagonal embedding and $\overline{v}_X = c \cdot v_X$ where $c > 0$ is so that $\int_X d\overline{v}_X = 1$.

Proof. Immediately from Theorem 2.9, we see that for any compact subsets $K_1, K_2 \subset X$ verifying $K_1 \cap K_2 = \emptyset$, we have

$$\lim_{k \to \infty} \int_{K_1 \times K_2} \mu_k^{\rm Berg} = 0. \tag{2.39}$$

Let us moreover explain that for any $f \in \mathscr{C}^0(X)$, we have

$$\lim_{k \to \infty} \int_{x \in X} \int_{x' \in X} f(x) \cdot d\mu_k^{\rm Berg}(x, y) = \int_{x \in X} f(x) \cdot d\overline{v}_X(x). \tag{2.40}$$

Indeed, as we shall see later in (2.51), we have

$$\int_{x' \in X} |B_k(x, y)|^2_{(h^L)^k} \cdot dv_X(y) = B_k(x, x). \tag{2.41}$$

Hence, we see that

$$\int_{x \in X} \int_{x' \in X} f(x) \cdot d\mu_k^{\rm Berg}(x, y) = \frac{1}{n_k} \int_{x \in X} f(x) \cdot B_k(x, x) \cdot dv_X(x). \tag{2.42}$$

The proof of (2.40) is then finished in the same way as in Exercise 3 from Section 2.5.

Let us now show that Corollary 2.10 is a formal consequence of the above two statements. Let us fix $f \in \mathscr{C}^0(X \times X)$. We would like to verify that $\int f(x, y) \cdot d\mu_k^{\rm Berg}(x, y) \to \int f(x, x) \cdot d\overline{v}_X)$, as $k \to \infty$. We define $g \in \mathscr{C}^0(X \times X)$ as $g(x, y) := f(x, x)$. Then directly from (2.40), we obtain $\int g(x, y) \cdot d\mu_k^{\text{Berg}}(x, y) \to \int f(x, x) \cdot d\overline{v}_X$, as $k \to \infty$. By considering the difference $f - g$, we see that it suffices to show that $\int h(x, y) \cdot d\mu_k^{\text{Berg}}(x, y) \to 0$, as $k \to \infty$, for continuous h vanishing on the diagonal.

According to (2.39), the above holds for h lying in the space of the functions $\mathcal{V}$ spanned by $a(x) \cdot b(y)$ where $a, b \in \mathscr{C}^0(X)$ have non-intersecting support. Hence it also holds for the functions from the uniform closure $\overline{\mathcal{V}}$ of $\mathcal{V}$. The proof of Corollary 2.10 will be complete once we establish that $\overline{\mathcal{V}}$ coincides with the space of functions vanishing on the diagonal.

To establish this, note first that it is immediate that every function from $\overline{\mathcal{V}}$ vanishes on the diagonal. Also any function vanishing along the diagonal can be uniformly approximated by functions vanishing in a neighborhood of the diagonal. It is hence enough to show that a continuous function h vanishing in a neighborhood of the diagonal lies in $\overline{\mathcal{V}}$. To see this, consider a partition of unity ρ_i, $i \in I$, subordinate to a sufficiently small mesh. Then from the uniform continuity of h, one sees that the functions $\sum_{i,j \in I} h(x_{i,j}) \rho_i(x) \rho_j(y)$, where $x_{i,j} \in \operatorname{supp}(\rho_i) \times \operatorname{supp}(\rho_j)$ are chosen in an arbitrary way, approximate uniformly the function h if the size of the mesh is small enough, and – again if the mesh is small enough – these approximations lie in $\mathcal{V}$ by our assumption on the vanishing of h in a neighborhood of the diagonal. □

2.5 Tutorial: spectral theory of Toeplitz operators

In this tutorial we shall explain that Theorem 2.7 refines Theorem 2.8 and Theorem 2.6 refines Theorem 2.9. Then we discuss the applications of Theorem 2.8 towards the study of Toeplitz operators.

Exercise 1. We shall fix a point $x \in X$ and consider the trivial embedding $\{x\} \hookrightarrow X$. We denote by $\mathrm{E}_k^x : L_x^k \to H^0(X, L^{\otimes k})$ the optimal extension operator associated with this embedding. The main goal of the exercise is to prove the following formula relating the Bergman kernel and the extension operator

$$B_k(x, y) = B_k(y, y) \cdot \mathrm{E}_k^y(x, y). \tag{2.43}$$

Proof. We assume $k_0 \in \mathbb{N}$ is big enough so that $n_k > 0$ for any $k \geq k_0$ and the base loci of L^k are empty for any $k \geq k_0$. Hence, for any $x \in X$ and $k \geq k_0$, there is $s \in H^0(X, L^{\otimes k})$ so that $s(x) \neq 0$.

Now, for any $x \in X$, we denote by $s_{k,x} \in H^0(X, L^{\otimes k})$ the *peak section* at x with respect to the scalar product $\mathrm{Hilb}_k(h^L)$. Recall that this means that $s_{k,x}$ is of unit norm with respect to $\mathrm{Hilb}_k(h^L)$ and orthogonal to the subspace $H^0(X, L^{\otimes k} \otimes \mathcal{J}_x)$ of holomorphic sections of $L^{\otimes k}$ vanishing at x.

Directly from the definition of the Bergman kernel and the fact that it doesn't depend on the choice of an orthonormal basis, we deduce that the Bergman kernels $B_k(x, y)$, $x \in X, y \in X$, and the Schwartz kernel $\mathrm{E}_k^y(x, y)$ verify

$$B_k(x, y) = s_{k,y}(x) \cdot s_{k,y}(y)^*, \qquad \mathrm{E}_k^y(x, y) = \frac{s_{k,y}(x) \cdot s_{k,y}(y)^*}{|s_{k,y}(y)|^2_{(h^L)^k}}. \tag{2.44}$$

The result then follows immediately from this. □

Let us now establish the following weak version of Theorem 1.4.

Exercise 2: Derive that there are $C > 0$, $k_0 \in \mathbb{N}$, so that for any $k \geq k_0$, $x \in X$, we have $|B_k(x) - k^n| \leq C k^{n - \frac{1}{2}}$, as $k \to \infty$.

Proof: Remark first that by (2.44), we have

$$\int_{x\in X} \left|\mathrm{E}_k^y(x,y)\right|^2_{(h^L)^k} dV_X(x) = \frac{1}{B_k(y,y)}. \tag{2.45}$$

By using notations from Theorem 2.7, we see that for any $x \in X$, there are $\epsilon, c, C, Q > 0$, $k_0 \in \mathbb{N}$, so that for any $k \geq k_0$, $Z \in \mathbb{R}^{2n}$, $|Z| \leq \epsilon$, we have

$$\left|\mathrm{E}_k^y(\phi_y(Z), y) - \exp\Big(-\frac{\pi}{2}\sum_{i=1}^n |z_i|^2\Big)\right| \leq \frac{C}{\sqrt{k}} \cdot \Big(1 + \sqrt{k}|Z|\Big)^Q \exp(-c\sqrt{k}\cdot|Z|). \tag{2.46}$$

The result then follows from this, Theorem 2.6 and the Gaussian integral calculation. □

The above exercise implies that Theorem 2.7 refines Theorem 2.8 and Theorem 2.6 refines Theorem 2.9. We leave the details to the interested reader and proceed instead with the proof of (1.23).

Exercise 3: For any continuous $f : X \to \mathbb{R}$, the associated Toeplitz operator $T_k[f] \in \mathrm{End}(H^0(X, L^{\otimes k}))$ verifies

$$\lim_{k\to\infty} \frac{1}{n_k}\mathrm{Tr}[T_k(f)] = \int_{x\in X} f(x) \cdot d\overline{v}_X(x) \tag{2.47}$$

Proof: Remark first that the operator $T_k(f)$ admits a Schwartz kernel $T_k(f)(x,y) \in L_x^k \otimes (L_y^k)^*$, defined as

$$T_k(f)(x,y) := \int_{z\in X} B_k(x,z) f(z) B_k(z,y) dv_X(z). \tag{2.48}$$

It means that for any $s \in H^0(X, L^{\otimes k})$, we have

$$(T_k(f)s)(x) = \int_{y\in X} T_k(f)(x,y) \cdot s(y) dv_X(x). \tag{2.49}$$

Immediately from the fact that trace can be calculated through the integral of the Schwartz kernel, we deduce

$$\mathrm{Tr}[T_k(f)] = \int_{X\times X} f(x) \cdot |B_k(x,y)|^2_{(h^L)^k} \cdot d\mu(x) \cdot d\mu(y). \tag{2.50}$$

Remark, however, that immediately from (2.44), we have

$$\int_{y\in X} |B_k(x,y)|^2_{(h^L)^k} \cdot d\mu(y) = B_k(x,x). \tag{2.51}$$

The result now follows from Exercise 2. □

Exercise 4: The minimal and the maximal eigenvalues $\lambda_{\min}(T_k[f])$, $\lambda_{\max}(T_k[f])$ of $T_k[f]$ satisfy $\lambda_{\min}(T_k[f]) \geq \inf_{x\in X} f(x)$ and $\lambda_{\max}(T_k[f]) \leq \sup_{x\in X} f(x)$.

Proof: Replacing f with $-f$, the problem reduces to analyzing $\lambda_{\max}(T_k(f))$. The bound then follows immediately from the min-max characterization of the eigenvalues and the bound $\langle T_k(f)s, s\rangle_{\mathrm{Hilb}_k(h^L)} \leq \sup_{x\in K} f(x) \cdot \langle s, s\rangle_{\mathrm{Hilb}_k(h^L)}$ for any $s \in H^0(X, L^{\otimes k})$, which follows immediately from the definition of the L^2-norm. □

For the following exercise, recall that the p-Schatten norm $\|\cdot\|_p$ is defined for an operator $A \in \mathrm{End}(V)$, of a finitely-dimensional Hermitian vector space (V, H) as $\|A\|_p = (\frac{1}{\dim V}\mathrm{Tr}[|A|^p])^{\frac{1}{p}}$, $|A| := (AA^*)^{\frac{1}{2}}$. Classical properties of Schatten norms yield that for any $T, S \in \mathrm{End}(V)$, we have

$$\|T\|_p \leq \|T\|_2^{\frac{1}{p}} \cdot \|T\|^{\frac{p-1}{p}}, \qquad \|S \circ T\|_p \leq \|S\| \cdot \|T\|_p. \tag{2.52}$$

Exercise 5: For any $f, g \in \mathscr{C}^0(X)$, we have

$$\lim_{k\to\infty} \|T_k(f) \circ T_k(g) - T_k(f \cdot g)\|_2 = 0. \tag{2.53}$$

Proof: It is immediate to see using the reproducing property $B_k \circ B_k = B_k$ that we can write

$$T_k(f) \circ T_k(g) - T_k(f \cdot g) = B_k \circ S_k \circ B_k, \tag{2.54}$$

where $S_k \in \mathrm{End}(H^0(X, L^{\otimes k}))$, is defined as

$$(S_k s)(x) := \int_{y\in X} S_k(x, y) \cdot s(y) \cdot dv_X(y), \quad \text{for any } s \in H^0(X, L^{\otimes k}), \tag{2.55}$$

for $S_k(x, y) \in L_x^{\otimes k} \otimes (L_y^{\otimes k})^*$ given by

$$S_k(x, y) = (f(x)g(x) - f(x)g(y))B_k(x, y). \tag{2.56}$$

From (2.52), we obtain

$$\|B_k \circ S_k \circ B_k\|_2 \leq \|S_k\|_2. \tag{2.57}$$

We now rely on the fact that the 2-Schatten norm is the rescaled Hilbert-Schmidt norm, and the latter can be calculated using the L^2-norm of Schwartz kernel of the operator, which gives us

$$\|S_k\|_2^2 = \frac{1}{n_k}\int_{X\times X} |S_k(x, y)|^2_{(h^L)^k} \cdot dv_X(x) \cdot dv_X(y). \tag{2.58}$$

But from Corollary 2.10 and (2.56), we deduce that

$$\lim_{k\to\infty} \frac{1}{n_k}\int_{X\times X} |S_k(x, y)|^2_{(h^L)^k} \cdot dv_X(x) \cdot dv_X(y) = 0, \tag{2.59}$$

which along with (2.54), (2.56), (2.57) and (2.58) finishes the proof of (2.53). □

Exercise 6. Show that for any $h \in \mathscr{C}^0([\min f, \max f])$ and $f \in \mathscr{C}^0(X)$, we have

$$\lim_{k\to\infty} \|h(T_k[f]) - T_k[h(f)]\|_2 = 0. \tag{2.60}$$

Proof. Since T_k has a spectrum in a compact interval $I := [\min f, \max f]$, the operator $h(T_k(f))$ is well-defined. By Weierstrass approximation theorem, applied over I, we see that it suffices to establish the result for h given by monomials. However, remark that from Exercises 4, 5 and (2.52), we see by induction that for any $l \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \|T_k[f]^l - T_k[f^l]\|_2 = 0, \tag{2.61}$$

which finishes the proof. □

Exercise 7: Derive (1.23).

Proof: It follows immediately from Exercises 3, 6 and (2.52). □

3 Metric structures on the space of norms and Kähler potentials

The main objective of this section is to present one of the key technical results of the minicourse, which establishes a comparison between two types of metrics: those defined on spaces of Hermitian structures on finite-dimensional vector spaces, and those defined on the space of Kähler potentials. In Section 3.1, we introduce and study various metrics on spaces of Hermitian structures. Section 3.2 is devoted to metrics on the space of Kähler potentials. In Section 3.3, we state the main comparison result linking these two settings. Finally, Section 3.4 provides some elements of the proof.

3.1 Metrics on the space of Hermitian structures

In this section, we introduce some metric structures on the space of Hermitian structures of a finite-dimensional vector space.

Let V be a complex vector space, $\dim V = n$. We denote by $\mathcal{H}_V$ the space of Hermitian norms H on V, viewed as an open subset of the Hermitian operators $\mathrm{Herm}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be the ordered spectrum of $h \in \mathrm{Herm}(V)$ with respect to a norm $H \in \mathcal{H}_V$. For $p \in [1, +\infty[$, we define

$$\|h\|_p^H := \sqrt[p]{\frac{\sum_{i=1}^{\dim V} |\lambda_i|^p}{\dim V}}. \tag{3.1}$$

By Ky Fan inequality, one can establish that $\| \cdot \|_p^H$, $p \in [1, +\infty[$, is a Finsler norm for any H, i.e. it satisfies the triangle inequality, cf. [25, Theorem 2.7].

We then define the length metric $d_p(H_0, H_1)$, $H_0, H_1 \in \mathcal{H}_V$, as

$$d_p(H_0, H_1) = \inf_{\gamma} l_p(\gamma), \tag{3.2}$$

where the infimum is taken over all piecewise smooth path $\gamma : [0,1] \to \mathcal{H}_V$, joining H_0, H_1, and the length $l_p(\gamma)$ is defined as

$$l_p(\gamma) := \int_0^1 \|\gamma'(t)\|_p^{\gamma(t)} dt. \tag{3.3}$$

One can verify, cf. [25, Theorem 2.7], that this metric admits the following explicit description. Let $T \in \mathrm{Herm}(V)$, be the *transfer map* between Hermitian norms $H_0, H_1 \in \mathcal{H}_V$, i.e. the Hermitian products $\langle \cdot, \cdot \rangle_{H_0}, \langle \cdot, \cdot \rangle_{H_1}$ induced by H_0 and H_1, are related as $\langle \cdot, \cdot \rangle_{H_1} = \langle T \cdot, \cdot \rangle_{H_0}$, then

$$d_p(H_0, H_1) = \sqrt[p]{\frac{\mathrm{Tr}[|\log T|^p]}{\dim V}}. \tag{3.4}$$

Moreover, the Hermitian norms H_t, $t \in [0, 1]$, corresponding to the scalar products

$$\langle \cdot, \cdot \rangle_{H_t} := \langle T^t \cdot, \cdot \rangle_{H_0}, \tag{3.5}$$

are geodesics in $(\mathcal{H}_V, d_p)$, $p \in [1, +\infty]$. Later on, we call them the *distinguished geodesics*. For $p \in]1, +\infty[$, it is possible to verify that $(\mathcal{H}_V, d_p)$ is a uniquely geodesic space, cf. [7, Theorem 6.1.6], and hence these are the only geodesic segments between H_0 and H_1.

It is possible to prove that $(\mathcal{H}_V, d_2)$, is isometric to the space $SL(V)/SU(V)$, endowed with the distance coming from the standard $SL(V)$-invariant metric, cf. [25, Theorem 1.1]. The later space is known to be of non-positive sectional curvature, see [48, Theorem XI.8.6], and contractible (by Cartan decomposition). In particular, by Cartan-Hadamard theorem, it is uniquely geodesic.

3.2 Mabuchi-Darvas geometry on the space of Kähler metrics

In this section, we introduce the metric structures on the space of Kähler potentials of a compact Kähler manifold.

Let us fix a Kähler form ω on X and consider the space $\mathcal{H}_\omega$ of Kähler potentials, consisting of $u \in \mathscr{C}^\infty(X, \mathbb{R})$, such that $\omega_u := \omega + \sqrt{-1}\partial\overline{\partial}u$ is strictly positive. We denote by $\mathrm{PSH}(X, \omega)$ the set of ω-psh potentials; these are upper semi-continuous functions $u \in L^1(X, \mathbb{R} \cup \{-\infty\})$, such that ω_u is positive as a $(1,1)$-current.

One can introduce on the space of Kähler potentials $\mathcal{H}_\omega$ a collection of L^p-type Finsler metrics, $p \in [1, +\infty[$, defined as follows. If $u \in \mathcal{H}_\omega$ and $\xi \in T_u\mathcal{H}_\omega \simeq \mathscr{C}^\infty(X, \mathbb{R})$, then the L^p-length of ξ is given by the following expression

$$\|\xi\|_p^u := \sqrt[p]{\frac{1}{\int \omega^n} \int_X |\xi(x)|^p \cdot \omega_u^n(x)}. \tag{3.6}$$

For $p = 2$, this was introduced by Mabuchi [56], and for $p \in [1, +\infty[$ by Darvas [20]. For brevity, we omit ω from our further notations.

We then define the length metric $d_p(u_0, u_1)$, $u_0, u_1 \in \mathcal{H}_\omega$, as

$$d_p(u_0, u_1) = \inf_\gamma l_p(\gamma), \tag{3.7}$$

where the infimum is taken over all piecewise smooth path $\gamma : [0,1] \to \mathcal{H}_\omega$, joining u_0, u_1, and the length $l_p(\gamma)$ is defined as

$$l_p(\gamma) := \int_0^1 \|\gamma'(t)\|_p^{\gamma(t)} dt. \tag{3.8}$$

Darvas in [20] studied the completion $(\mathcal{E}^p, d_p)$ of $(\mathcal{H}, d_p)$ and proved that these completions are geodesic metric spaces (in other words, any two points in $\mathcal{H}_\omega$ can be joined by a geodesic).

Certain geodesic segments of $(\mathcal{E}^p, d_p)$ can be constructed as upper envelopes of quasi-psh functions. More precisely, we identify paths $u_t \in \mathcal{E}^p$, $t \in [0,1]$, with rotationally-invariant functions $\hat{u}$ over $X \times \mathbb{D}(e^{-1}, 1)$ through the following formula

$$\hat{u}(x, \tau) = u_t(x), \quad \text{where} \quad x \in X \text{ and } t = -\log|\tau|. \tag{3.9}$$

We say that a curve $[0,1] \ni t \to v_t \in \mathcal{E}^p$ is a *weak subgeodesic* connecting $u_0, u_1 \in \mathcal{E}^p$ if $d_p(v_t, u_i) \to 0$, as $t \to 0$ for $i = 0$ and $t \to 1$ for $i = 1$, and $\hat{u}$ is $\pi^*\omega$-psh on $X \times \mathbb{D}(e^{-1}, 1)$. As shown in [20, Theorem 2], the following envelope

$$u_t := \sup\Big\{v_t \,:\, t \to v_t \text{ is a weak subgeodesic connecting } v_0 \leq u_0 \text{ and } v_1 \leq u_1\Big\}, \tag{3.10}$$

is a d_p-geodesic connecting u_0, u_1. It will be later called the *distinguished geodesic segment*.

One can establish, cf. Guedj-Zeriahi [45, Exercise 10.2], that

$$\bigcap_{p \in [1, +\infty[} \mathcal{E}^p = \mathrm{PSH}(X, \omega) \cap L^\infty(X). \tag{3.11}$$

When $u_0, u_1 \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$, Berndtsson in [5, §2.2] proved that u_t, $t \in [0,1]$, defined by (3.10), verifies $u_t \in L^\infty(X)$ and it can be described as the only path connecting u_0 to u_1, so that $\hat{u}$ is the solution of the following Monge-Ampère equation

$$(\pi^*\omega + \sqrt{-1}\partial\overline{\partial}\hat{u})^{n+1} = 0, \tag{3.12}$$

where the wedge power is interpreted in Bedford-Taylor sense [1]. For smooth geodesic segments in $(\mathcal{H}, d_2)$, Semmes [70] and Donaldson [32] have made similar observations before. The uniqueness of the solution of (3.12) is assured by [45, Lemma 5.25]. Remark, in particular, that for any $u_0, u_1 \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$, the distinguished weak geodesic connecting them is the same if we view u_0, u_1 as elements in any of $\mathcal{E}^p$, $p \in [1, +\infty[$.

Theorem 3.1 (Darvas-Lu [24, Theorem 2])**.** *For any $p \in]1, +\infty[$, $(\mathcal{E}^p_\omega, d_p)$ is uniquely geodesic, i.e. any two points are joined by a unique geodesic.*

Theorem 3.1 allows us to verify identity (3.12) using methods from metric geometry: it suffices to check that a given path of metrics is a geodesic with respect to some distance d_p, for $p \in]1, +\infty[$. We shall see this method in action in Section 4.

The distance on $\mathcal{E}^1_\omega$ can be alternatively described in terms of the *Monge-Ampère energy* functional E. Recall that E is explicitly given for $u, v \in \mathcal{H}_\omega$ by

$$E(u) - E(v) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (u-v) w_u^j \wedge w_v^{n-j}. \tag{3.13}$$

By [45, Proposition 10.14], E is monotonic, i.e. for any $u \leq v$, we have $E(u) \leq E(v)$. From this, it is reasonable to extend the domain of the definition of E to $\mathrm{PSH}(X, \omega)$ as

$$E(u) := \inf \Big\{ E(v) : v \in \mathcal{H}_\omega, u \leq v \Big\}. \tag{3.14}$$

Darvas proved in [20] that $\mathcal{E}^1_\omega$ coincides with the set of $u \in \mathrm{PSH}(X, \omega)$, verifying $E(u) > -\infty$. Moreover, for any $u, v \in \mathcal{E}^1_\omega$, verifying $u \leq v$, according to [20, Corollary 4.14], we have

$$d_1(u, v) = E(v) - E(u). \tag{3.15}$$

In particular, $(\mathcal{E}^1_\omega, d_1)$ is not a uniquely geodesic space – a fact originally observed by Darvas [22, comment after Theorem 4.17].

Let us now discuss the relation between the speed of Mabuchi geodesic and the distance between its endpoints. We fix $u_0, u_1 \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ and consider u_t, $t \in [0, 1]$ as in (3.10). From Berndtsson [5, §2.2], the limits $\lim_{t \to 0} u_t = u_0$, $\lim_{t \to 1} u_t = u_1$ hold in the uniform sense. Also, the psh-condition from the definition of the weak geodesics implies that for a fixed $x \in X$, the function $u_t(x)$ is convex in $t \in [0, 1]$, see [29, Theorem I.5.13]. Hence, one-sided derivatives $\dot{u}_t^-$, $\dot{u}_t^+$ of u_t are well-defined for $t \in]0, 1[$ and they increase in t. We denote $\dot{u}_0 := \lim_{t \to 0} \dot{u}_t^- = \lim_{t \to 0} \dot{u}_t^+$. From [5, §2.2], we know that $\dot{u}_0$ is bounded and by Darvas [22, Theorem 1], we, moreover, have

$$\sup |\dot{u}_0| \leq \sup |u_1 - u_0|. \tag{3.16}$$

Then according to Berndtsson [6], Darvas-Lu-Rubinstein [25, Lemma 4.5], for any $u_0 \in \mathcal{H}_\omega$, $u_1 \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$, we have

$$d_p(u_0, u_t) = t \cdot \sqrt[p]{\int_X |\dot{u}_0|^p \cdot MA(u_0)}. \tag{3.17}$$

See also Di Nezza-Lu [30, Theorem 3.2] for a related statement in this direction.

3.3 Comparison of the two metric structures

When the De Rham cohomology class $[\omega]$ of ω satisfies $[\omega] \in 2\pi H^2(X, \mathbb{Z})$, there is a Hermitian line bundle (L, h_0^L), such that $\omega = 2\pi c_1(L, h_0^L)$. Hence, upon fixing h_0^L (which is uniquely defined up to a multiplication by a locally constant function), the set $\mathcal{H}_\omega$ (resp. $\mathrm{PSH}(X, \omega)$) can be identified with the set of smooth positive (resp. psh) metrics on L through the correspondence

$$u \mapsto h^L := e^{-u} \cdot h_0^L. \tag{3.18}$$

Remark that we then have $\omega_u = 2\pi c_1(L, h^L)$. This identification will be implicit later on, and all the constructions (of distances, geodesics, etc.) for elements from $\mathcal{H}_\omega$ and $\mathrm{PSH}(X, \omega) \cap L^\infty(X)$ will be implicitly extended to the corresponding sets of metrics on the line bundle L.

The following result says that the metrics d_p on the space of Hermitian norms on $H^0(X, L^{\otimes k})$ are quantisations of Mabuchi-Darvas metrics d_p on the space of bounded psh metrics on L.

Theorem 3.2 (Chen-Sun [15], Berndtsson [6], Darvas-Lu-Rubinstein [25, Theorem 1.2])**.** *For any bounded psh metrics h_0^L, h_1^L on an ample line bundle L and any $p \in [1, +\infty[$, we have*

$$\lim_{k \to \infty} \frac{1}{k} d_p\Big(\mathrm{Hilb}_k(h_0^L), \mathrm{Hilb}_k(h_1^L)\Big) = d_p(h_0^L, h_1^L). \tag{3.19}$$

Remark 3.3. These results go in line with the general philosophy that the geometry of the space of psh metrics on L can be approximated by the geometry of the space of norms on $H^0(X, L^{\otimes k})$, as $k \to \infty$, see Donaldson [32] and Phong-Sturm [59].

3.4 Curvature of the L^2-metrics and a proof of Theorem 3.2

The main objective of this section is to prove Theorem 3.2. We will do so under additional assumptions: the endpoints h_0^L and h_1^L are smooth metrics with strictly positive curvature, and the Mabuchi geodesic h_t^L, for $t \in [0, 1]$, is a smooth path of smooth metrics with strictly positive curvature.

It was shown by Lempert and Vivas [51], as well as Darvas and Lempert [23], that for general endpoints, such a smooth Mabuchi geodesic does not exist.

The proof can be extended to the general case by considering smooth approximations of Mabuchi geodesics, known as ε-geodesics [14]. However, this leads to more technical arguments, which we prefer to avoid here for the sake of clarity. We refer to [39] for details.

We follow closely [39], which is itself inspired by a method due to Berndtsson [6]. The idea is to compare the geodesic $H_{k,t}$, defined for $k \in \mathbb{N}$ and $t \in [0, 1]$, between the L^2-metrics associated with $\mathrm{Hilb}_k(h_0^L)$ and $\mathrm{Hilb}_k(h_1^L)$, with the L^2-metric $\mathrm{Hilb}_k(h_t^L)$ corresponding to the Mabuchi geodesic h_t^L between h_0^L and h_1^L. This comparison is based on the curvature properties of $\mathrm{Hilb}_k(h_t^L)$.

To explain the relation between the comparison of the norms and their curvatures, similarly to (3.9), we identify a smooth path $H_t \in \mathcal{H}_V$, $t \in [0, 1]$, with the rotationally-invariant Hermitian metric $\hat{H}$ on the (trivial) vector bundle $V \times \mathbb{D}(e^{-1}, 1)$ over $\mathbb{D}(e^{-1}, 1)$, through the formula

$$\hat{H}(\tau) = H_t, \quad \text{where} \quad t = -\log|\tau|, \tag{3.20}$$

that we suggest to compare with (3.9). We also introduce the speed of the path, defined as follows $\dot{H}_t := H_t^{-1} \frac{d}{dt} H_t \in \mathrm{End}(V)$.

Below, we use the Loewner's order on $\mathrm{End}(V)$. More specifically, it means that for two Hermitian operators $A, B \in \mathrm{End}(V)$, we say $A \geq B$ if $A - B$ is positive semi-definite.

Recall that we say that a Hermitian vector bundle (E, h^E) over a Riemann surface S has a positive curvature, cf. [29, §VII.6], if the curvature R^E of the Chern connection of (E, h^E), for any $s \in S$, can be written as $R^E_s = dz \wedge d\overline{z} \cdot A(s)$, for a positively definite $A(s) \in \mathrm{End}(E_s)$ and a local holomorphic coordinate z on S, centered at s.

Theorem 3.4 ([67, Theorem 4.2], [18, Theorem 4.1, §15]). *Assume that a smooth path $H^0_t \in \mathcal{H}_V$, $t \in [0, 1]$, is such that the associated Hermitian metric $\hat{H}_0$ on $V \times \mathbb{D}(e^{-1}, 1)$ has a positive (resp. negative) curvature. Then for the geodesic H_t between H^0_t and H^1_t, we have $H^0_t \geq H_t$ (resp. $H^0_t \leq H_t$). In particular, the following inequality is satisfied $\dot{H}^0_0 \geq \dot{H}_0$ (resp. $\dot{H}^0_0 \leq \dot{H}_0$). The path H^0_t is then called a superinterpolating (resp. subinterpolating) family.*

Now, according to Theorem 3.4, it suffices to compute effectively the curvature of the associated Hermitian structure associated with $\mathrm{Hilb}_k(h^L_t)$ as in (3.20).

To do so, Berndtsson in his proof proof [6] relies on his positivity result from [4], which asserts that the curvature of the metric associated with $\mathrm{Hilb}_k(h^L_t)$ becomes Nakano positive when k is sufficiently large. This yields a lower bound $\mathrm{Hilb}_k(h^L_t) \geq H_{k,t}$. However, as it provides no control on the upper bound of the curvature, not so much can be said about the inverse inequality. As a result, it becomes necessary to compare the derivatives of $\mathrm{Hilb}_k(h^L_t)$ and $H_{k,t}$ at both endpoints of the interval $[0, 1]$. To get Theorem 3.2, one has to use the fact that along a Mabuchi geodesic, the norm of its velocity, defined in (3.6), remains constant along the entire segment. While this property is straightforward for geodesics on (finitely-dimensional) manifolds, its analogue in Mabuchi geometry requires some work, as detailed in [6] and [30]. Moreover, the result holds only when both endpoints of the geodesics are regular enough.

The approach from [39] that we outline here uses the curvature computations for L^2-metrics developed by Ma-Zhang [55]. In this way, we show that L^2-metrics can be used not only to construct superinterpolating families along the geodesic between two L^2-metrics, but also subinterpolating ones. This allows for a significantly more refined analysis of the transfer operator, leading to stronger results. Moreover, we ultimately do not rely on the constancy of the speed norm along the entire geodesic segment. What matters for our purposes is simply that the distance can be computed as the norm of the speed evaluated at the initial point of the geodesic. This latter property is significantly more robust with respect to the regularity of the geodesic endpoints as discussed, for example, in [25, Lemma 4.5].

We shall now explain the result of Ma-Zhang from [55], which we present below in a special case that we shall need later on. We fix a smooth family of (strictly) positive Hermitian metrics h^L_τ, $\tau \in \mathbb{D}(e^{-1}, 1)$ on L and a smooth family of Kähler forms χ_τ, $\tau \in \mathbb{D}(e^{-1}, 1)$, on X. We denote by $\omega := c_1(L \times \mathbb{D}(e^{-1}, 1), h^L_\tau)$ the curvature of h^L_τ, viewed as a metric on the line bundle $L \times \mathbb{D}(e^{-1}, 1)$ over $X \times \mathbb{D}(e^{-1}, 1)$.

For $\tau \in \mathbb{D}(e^{-1}, 1)$, we define $\omega_H(\tau) \in \mathscr{C}^\infty(X)$ as

$$\omega_H(\tau)(x) := \frac{1}{n+1} \frac{\omega^{n+1}}{\omega^n \wedge \sqrt{-1} dz \wedge d\overline{z}}(x, \tau). \tag{3.21}$$

The denominator above is nonzero, as ω is positive along the fibers.

We denote by R_k the curvature of the Chern connection on the trivial vector bundle $H^0(X, L^{\otimes k}) \times \mathbb{D}(e^{-1}, 1)$ associated with the fiberwise L^2-metric $\mathrm{Hilb}_k(h^L_\tau)$, $\tau \in \mathbb{D}(e^{-1}, 1)$, induced by h^L_τ. We define $D_k(\tau) \in \mathrm{End}(H^0(X, L^{\otimes k}))$ so that $\frac{\sqrt{-1}}{2\pi} R_{k,\tau} := \sqrt{-1} dz \wedge d\overline{z} \cdot D_k(\tau)$.

Theorem 3.5 (Ma-Zhang [55, Theorem 0.4])**.** *There are $C > 0$, $k_0 \in \mathbb{N}$, such that in the notations from (1.21), for any $k \geq k_0$, $\tau \in \mathbb{D}(e^{-1}, 1)$, we have*

$$\Big\| D_k(\tau) - kT_k(\omega_H(\tau)) \Big\| \leq C, \tag{3.22}$$

where $\|\cdot\|$ is the operator norm subordinate with Hilb$_k(h^L_\tau, \chi_\tau)$. In particular, $\{\frac{1}{k} D_k(\tau)\}_{k=1}^{+\infty}$ forms a Toeplitz operator with symbol $\omega_H(\tau)$.

We denote by $T_k(h^L_0, h^L_1) \in \mathrm{End}(H^0(X, L^{\otimes k}))$ the transfer map between $\mathrm{Hilb}_k(h^L_0)$ and $\mathrm{Hilb}_k(h^L_1)$, i.e. it is a map so that $\langle T_k(h^L_0, h^L_1)\cdot, \cdot\rangle_{\mathrm{Hilb}_k(h^L_0)} = \langle \cdot, \cdot\rangle_{\mathrm{Hilb}_k(h^L_1)}$. We denote by $\phi(h^L_0, h^L_1)$ the speed at time 0 of the Mabuchi geodesic connecting h^L_0 and h^L_1, defined as $(h^L_0)^{-1} \cdot \frac{d}{dt}|_{t=0} h^L_t$.

Theorem 3.6. *Under the described above assumptions, there are $C > 0$, $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, we have*

$$\big\| \log T_k(h^L_0, h^L_1) - k \cdot T_k(\phi(h^L_0, h^L_1)) \big\| \leq C, \tag{3.23}$$

where $T_k(\phi(h^L_0, h^L_1))$ is the Toeplitz operator associated with the symbol $\phi(h^L_0, h^L_1)$, defined as in Section 1.3.

Proof. We consider the rotationally-invariant Hermitian metric $\hat{H}^0_k$ on the (trivial) vector bundle $H^0(X, L^{\otimes k}) \times \mathbb{D}(e^{-1}, 1)$ over $\mathbb{D}(e^{-1}, 1)$, constructed from $\mathrm{Hilb}_k(h^L_t)$, $t \in [0, 1]$, as in (3.20). Directly from the fact that smooth Mabuchi geodesics solve the homogeneous Monge-Ampère equation (3.12), by Theorem 3.5, we deduce that there are $C_0 > 0$, $k_0 \in \mathbb{N}$, such that the curvature R_k of $\hat{H}^0_k$ satisfies $\|R_k\| \leq C_0$ for any $k \geq k_0$. We denote

$$g : \mathbb{D}(e^{-1}, 1) \to \mathbb{R}, \qquad \tau \mapsto g(\tau) := (2 \log |\tau|^2 - 1)^2 - 1. \tag{3.24}$$

Remark that g is strictly subharmonic and verifies $g(e^{-1+i\theta}) = g(e^{i\theta}) = 0$, for any $\theta \in [0, 2\pi]$. Directly from the bound $\|R_k\| \leq C_0$, and strict subharmonicity of g, there is $C_1 > 0$, such that the curvature of Hermitian metrics $\hat{H}^1_k = \hat{H}^0_k \cdot \exp(-C_1 g)$ (resp. $\hat{H}^2_k = \hat{H}^0_k \cdot \exp(C_1 g)$) is positive (resp. negative). We denote by $H^1_{k,t}$, $H^2_{k,t}$, $t \in [0, 1]$, the paths of metrics on $H^0(X, L^{\otimes k})$ induced through (3.20) by $\hat{H}^1_k$ and $\hat{H}^2_k$ respectively. Our boundary condition on g implies that $H^1_{k,0} = \mathrm{Hilb}_k(h^L_0) = H^2_{k,0}$ and $H^1_{k,1} = \mathrm{Hilb}_k(h^L_1) = H^2_{k,1}$. From this and the above curvature calculation, we deduce by Theorem 3.4 that for any $t \in [0, 1]$, we have

$$H^1_{k,t} \geq H_{k,t} \geq H^2_{k,t}, \tag{3.25}$$

where $H_{k,t}$ is the geodesic between $\mathrm{Hilb}_k(h^L_0)$ and $\mathrm{Hilb}_k(h^L_1)$. By taking derivatives at $t = 0$ from the above inequality, we deduce that

$$\dot{H}^1_{k,0} \geq T_k(h^L_0, h^L_1) \geq \dot{H}^2_{k,0}. \tag{3.26}$$

Directly from the definition of $H^1_{k,t}$, $H^2_{k,t}$, we deduce that

$$\begin{aligned} \dot{H}^1_{k,0} &= (\mathrm{Hilb}_k(h^L_0))^{-1} \frac{d}{dt} \mathrm{Hilb}_k(h^L_t)|_{t=0} + C_1 g'(1) \mathrm{Id}, \\ \dot{H}^2_{k,0} &= (\mathrm{Hilb}_k(h^L_0))^{-1} \frac{d}{dt} \mathrm{Hilb}_k(h^L_t)|_{t=0} - C_1 g'(1) \mathrm{Id}. \end{aligned} \tag{3.27}$$

From the definition of the L^2-norm, it is direct (see the Exercise session to this Section) to see that

$$(\mathrm{Hilb}_k(h_0^L))^{-1}\frac{d}{dt}\mathrm{Hilb}_k(h_t^L)|_{t=0} = kT_{\phi(h_0^L,h_1^L),k} + T_{\dot{v}_{X,0},k}, \tag{3.28}$$

where $\dot{v}_{X,0} := v_{X,0}^{-1}\frac{d}{dt}|_{t=0}dv_{X,t}$. From (3.26), (3.27) and (3.28), the result follows directly. □

Proof of Theorem 3.2. Directly from (3.4), we see

$$d_p\Big(\mathrm{Hilb}_k(h_0^L), \mathrm{Hilb}_k(h_1^L)\Big) = \sqrt[p]{\frac{\mathrm{Tr}[|\log T_k(h_0^L,h_1^L)|^p]}{\dim H^0(X,L^{\otimes k})}} \tag{3.29}$$

Remark that there is $C > 0$, so that $\|\log T_k(h_0^L,h_1^L)\| \leq Ck$, where $\|\cdot\|$ is the operator norm. From this and Theorem 3.6, we deduce that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ so that for any $k \geq k_0$, we have

$$\Big|\mathrm{Tr}[|\log T_k(h_0^L,h_1^L)|^p] - k^p \cdot \mathrm{Tr}[|T_k(\phi(h_0^L,h_1^L))|^p]\Big| \leq \epsilon \cdot k^p \cdot \dim H^0(X,L^{\otimes k}). \tag{3.30}$$

However, by the Exercise session from Lecture 2, for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ so that for any $k \geq k_0$, we have

$$\Big|\mathrm{Tr}[|T_k(\phi(h_0^L,h_1^L))|^p] - \mathrm{Tr}[T_k(|\phi(h_0^L,h_1^L)|^p)]\Big| \leq \epsilon \cdot \dim H^0(X,L^{\otimes k}). \tag{3.31}$$

From (1.23), however, we have

$$\lim_{k\to\infty}\frac{\mathrm{Tr}[T_k(|\phi(h_0^L,h_1^L)|^p)]}{\dim H^0(X,L^{\otimes k})} = \frac{\int_X |\phi(h_0^L,h_1^L)|^p \cdot dv_X}{\int_X dv_X}. \tag{3.32}$$

Which implies the result by (3.17). □

4 Graded normed algebras and submultiplicative filtrations

The primary goal of this section is to complete the proof of Theorem 1.1. We present the proof in Section 4.1, subject to several intermediate results that will be developed later in the lecture. We begin by defining the geodesic rays that appear in the statement of Theorem 1.1, and provide some examples of their calculation in Section 4.2. Section 4.3 introduces one of the main technical results of the section, concerning the asymptotic behavior of graded norms. In Section 4.4, we explain how this result is applied in the proof of Theorem 1.1. The key ideas underlying the technical statements from Section 4.3 are discussed in Sections 4.5 and 4.6.

4.1 The general strategy of the proof of Theorem 1.1

The primary objective of this section is to establish Theorem 1.1, subject to several intermediate results that will be addressed later in the lecture.

Our proof relies on comparing two constructions of geodesic rays: one in the space of Hermitian structures on finite-dimensional vector spaces, and the other in the space of Kähler potentials. We begin by describing the construction in the space of Hermitian structures, as the corresponding construction in the space of Kähler potentials builds upon it.

We fix a *Hermitian* norm $H_0 := \| \cdot \|_H$ on V and a filtration $\mathcal{F}$ on V. Consider an orthonormal basis $s_1, \ldots, s_r$, $r := \dim V$, of V, adapted to the filtration $\mathcal{F}$, i.e. verifying $s_i \in \mathcal{F}^{e_{\mathcal{F}}(i)}V$, where $e_{\mathcal{F}}(i)$ are the jumping numbers of the filtration $\mathcal{F}$, defined as

$$e_{\mathcal{F}}(i) := \sup \Big\{ t \in \mathbb{R} : \dim \mathcal{F}^t V \geq i \Big\}. \tag{4.1}$$

We define the ray of Hermitian norms $H_t^{\mathcal{F}} := \| \cdot \|_t^{\mathcal{F}}$, $t \in [0, +\infty[$, on V by declaring the basis

$$(s_1^t, \ldots, s_r^t) := \big(e^{t e_{\mathcal{F}}(1)} s_1, \ldots, e^{t e_{\mathcal{F}}(r)} s_r \big), \tag{4.2}$$

to be orthonormal with respect to $H_t^{\mathcal{F}}$.

The reader should view the above construction as an interpolation between H_0 and the filtration $\mathcal{F}$. To explain this in more detail, we recall that filtrations $\mathcal{F}$ on V are in one-to-one correspondence with functions $\chi_{\mathcal{F}} : V \to [0, +\infty[$, defined as

$$\chi_{\mathcal{F}}(s) := \exp(-w_{\mathcal{F}}(s)). \tag{4.3}$$

where $w_{\mathcal{F}}(s)$ is the weight associated with the filtration, defined as $w_{\mathcal{F}}(s) := \sup\{ \lambda \in \mathbb{R} : s \in F^{\lambda} V \}$. An easy verification shows that $\chi_{\mathcal{F}}$ is a non-Archimedean norm on V with respect to the trivial absolute value on $\mathbb{C}$, i.e. it satisfies the following axioms

1. $\chi_{\mathcal{F}}(f) = 0$ if and only if $f = 0$,
2. $\chi_{\mathcal{F}}(\lambda f) = \chi_{\mathcal{F}}(f)$, for any $\lambda \in \mathbb{C}^*$, $k \in \mathbb{N}^*$, $f \in V$,
3. $\chi_{\mathcal{F}}(f + g) \leq \max\{ \chi_{\mathcal{F}}(f), \chi_{\mathcal{F}}(g) \}$, for any $k \in \mathbb{N}^*$, $f, g \in V$.

To convince oneself that it is reasonable to call $H_t^{\mathcal{F}}$ the interpolation between H_0 and the filtration $\mathcal{F}$, we suggest the reader to verify that for any $f \in V$, we have

$$\log \chi_{\mathcal{F}}(f) = \lim_{t \to +\infty} \frac{\log \| f \|_t^{\mathcal{F}}}{t}. \tag{4.4}$$

The construction of the geodesic ray on the space of Kähler potentials is based on the above construction. To explain this, we need to introduce a correspondence between the norms on $H^0(X, L^{\otimes k})$ and metrics on L.

We fix an ample line bundle L over a compact complex manifold X. For $k \in \mathbb{N}$ so that $L^{\otimes k}$ is very ample, Fubini-Study operator associates for any norm $N_k = \| \cdot \|_k$ on $H^0(X, L^{\otimes k})$, a continuous metric $FS(N_k)$ on L, constructed in the following way. Consider the Kodaira embedding

$$\mathrm{Kod}_k : X \hookrightarrow \mathbb{P}(H^0(X, L^{\otimes k})^*), \tag{4.5}$$

which embeds X in the space of hyperplanes in $H^0(X, L^{\otimes k})$. The evaluation maps provide the isomorphism $L^{\otimes(-k)} \to \mathrm{Kod}_k^* \mathscr{O}(-1)$, where $\mathscr{O}(-1)$ is the tautological bundle over $\mathbb{P}(H^0(X, L^{\otimes k})^*)$. We endow $H^0(X, L^{\otimes k})^*$ with the dual norm N_k^* and induce from it a metric $FS^{\mathbb{P}}(N_k)$ on $\mathscr{O}(1)$ over $\mathbb{P}(H^0(X, L^{\otimes k})^*)$. We define the metric $FS(N_k)$ on $L^{\otimes k}$ as the only metric verifying under the dual of the above isomorphism the identity

$$FS(N_k) = \mathrm{Kod}_k^*(FS^{\mathbb{P}}(N_k)). \tag{4.6}$$

A statement below can be seen as an alternative definition of $FS(N_k)$.

Lemma 4.1. For any $x \in X$, $l \in L_x^{\otimes k}$, the following identity takes place

$$|l|_{FS(N_k),x} = \inf_{\substack{s \in H^0(X, L^{\otimes k}) \\ s(x) = l}} \|s\|_k. \tag{4.7}$$

When N_k is a Hermitian norm, $FS(N_k)$ is the only metric on $L^{\otimes k}$, which for any $x \in X$, and for an orthonormal basis $s_1, \ldots, s_{n_k}$ of $(H^0(X, L^{\otimes k}), N_k)$ satisfies the following equation

$$\sum_{i=1}^{n_k} \big|s_i(x)\big|^2_{FS(H_k)} = 1. \tag{4.8}$$

Proof. The second part follows immediately from the first, and the first part is left to the interested reader. □

When the norm N_k is Hermitian, the definition of the Fubini-Study map is standard, and explicit evaluation shows that in this case $c_1(\mathscr{O}(1), FS^{\mathbb{P}}(N_k))$ coincides up to a positive constant with the Kähler form of the Fubini-Study metric on $\mathbb{P}(H^0(X, L^k)^*)$ induced by N_k. In particular, $c_1(\mathscr{O}(1), FS^{\mathbb{P}}(N_k))$ is a positive $(1,1)$-form. From Kobayashi [47], for general norms N_k, the $(1,1)$-current $c_1(\mathscr{O}(1), FS^{\mathbb{P}}(N_k))$ is positive, cf. [35, §2.1] for details. In particular, the curvature of the metric $FS(N_k)$ on $L^{\otimes k}$ is positive for any norm N_k on $H^0(X, L^{\otimes k})$.

In what follows, we fix a submultiplicative filtration $\mathcal{F}$ on $R(X, L)$ which we assume to be bounded. We fix a metric h^L on L with positive curvature. We denote by $H_{k,t}$ the geodesic ray emanating from $\text{Hilb}_k(h^L)$ and associated with the restriction of $\mathcal{F}$ to $H^0(X, L^{\otimes k})$.

We define the path of metrics $h^{L,\mathcal{F}}_{t,0}$, $t \in [0, +\infty[$ as

$$h^{L,\mathcal{F}}_{t,0} := \lim_{k \to \infty} \inf_{l \geq k} FS(H_{l,t})^{\frac{1}{l}}, \tag{4.9}$$

where for a bounded metric $h^{L,0}$ on L we denote by $h^{L,0}_*$ the lower semicontinuous regularization of $h^{L,0}$. Not so much can be said concerning the regularity of $h^{L,\mathcal{F}}_{t,0}$ (in general it is not even continuous as we shall describe in Section 4.2), but an easy verification shows that boundness of the filtration implies that there is $C > 0$, so that $h^L \cdot \exp(-Ct) \leq h^{L,\mathcal{F}}_{t,0} \leq h^L \cdot \exp(Ct)$. Also, the limit from (4.9) exists since the metrics under the limit sign form an increasing sequence.

To increase the regularity as well as other properties of the resulting ray of metric, we will apply the lower semi-continuous regularization to this path. Recall that for a given function f on a topological space, we denote by f^* (resp. f_*) the upper (resp. lower) semi-continuous regularization of f, defined as

$$f^*(x) = \limsup_{\substack{y \to x \\ y \neq x}} f(y), \quad (\text{resp. } f^*(x) = \liminf_{\substack{y \to x \\ y \neq x}} f(y)). \tag{4.10}$$

The same notations are used for metrics on line bundles. We then define

$$h^{L,\mathcal{F}}_t := (h^{L,\mathcal{F}}_{t,0})_*. \tag{4.11}$$

We shall establish the following result in Section 4.4.

Proposition 4.2. For any $t \in [0, +\infty[$, $h^{L,\mathcal{F}}_t$ has a bounded psh potential.

For now, we accept the preceding proposition and the following key result.

Theorem 4.3. *For any $t \in [0, +\infty[$, we have*

$$\lim_{k \to \infty} \frac{1}{k} d_p\big(\mathit{Hilb}_k(h_t^{L,\mathcal{F}}), H_{k,t}\big) = 0. \tag{4.12}$$

The first important corollary of the above two statements will concern the metric properties of $h_t^{L,\mathcal{F}}$. It was first established by Phong-Sturm in [60] for filtrations arising from test configurations and later by Ross-Witt Nyström [68] in the general case. The proof we present below is from [35].

Corollary 4.4. The ray of metrics $h_t^{L,\mathcal{F}}$, $t \in [0, +\infty[$ is a geodesic ray with respect to the distances d_p, for any $p \in [1, +\infty[$. As a consequence, it resolves the analogue of the equation (3.12).

Proof. First of all, by Proposition 4.2 and Theorem 3.2, for any $s, t \in [0, +\infty[$, the following limit exists, and can be expressed using Darvas-Mabuchi distances as follows

$$\lim_{k \to \infty} \frac{1}{k} d_p\big(\mathrm{Hilb}_k(h_t^{L,\mathcal{F}}), \mathrm{Hilb}_k(h_s^{L,\mathcal{F}})\big) = d_p(h_t^{L,\mathcal{F}}, h_s^{L,\mathcal{F}}). \tag{4.13}$$

By Theorem 4.3 and the fact that on the space of Hermitian structures, the function d_p forms a distance (and hence verifies the triangle inequality), we see that for any $s, t \in [0, +\infty[$,

$$\lim_{k \to \infty} \frac{1}{k} d_p\big(\mathrm{Hilb}_k(h_t^{L,\mathcal{F}}), \mathrm{Hilb}_k(h_s^{L,\mathcal{F}})\big) = \lim_{k \to \infty} \frac{1}{k} d_p\big(H_{k,t}, H_{k,s}\big). \tag{4.14}$$

Note, however, that by the construction of $H_{k,t}$, it forms a geodesic ray with respect to the distance d_p on the space of Hermitian structures. In particular, for any $k \in \mathbb{N}$, $t, s \in [0, +\infty[$, we have

$$d_p\big(H_{k,t}, H_{k,s}\big) = |t - s| \cdot d_p\big(H_{k,0}, H_{k,1}\big). \tag{4.15}$$

If we now combine all of the above statements, we establish

$$d_p(h_t^{L,\mathcal{F}}, h_s^{L,\mathcal{F}}) = |t - s| \cdot d_p(h_0^{L,\mathcal{F}}, h_1^{L,\mathcal{F}}), \tag{4.16}$$

which finishes the proof. □

Proof of Theorem 1.1. Immediately from the definition of d_p-distances and our assumption $\mathcal{F}^0 R(X, L) = \{0\}$, we conclude

$$d_p\big(H_{k,0}, H_{k,1}\big)^p = \frac{1}{n_k} \sum_{j=1}^{n_k} \Big(\frac{e_{\mathcal{F}}(j,k)}{k}\Big)^p. \tag{4.17}$$

Directly from (4.13) and (4.14), we deduce the following

$$\lim_{k \to \infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \Big(\frac{e_{\mathcal{F}}(j,k)}{k}\Big)^p = d_p(h_0^{L,\mathcal{F}}, h_1^{L,\mathcal{F}})^p. \tag{4.18}$$

In particular, we get the convergence from Theorem 1.1 for monomials g, and (4.18) establishes the characterization of the limiting measure (1.18). Since polynomials form a uniformly dense subset in the space of continuous functions on an arbitrary interval, we see that the limit from Theorem 1.1 exists for an arbitrary continuous function, which then finishes the proof. □

4.2 Examples of geodesic rays

The main goal of this section is to help the reader to familiarize with the construction of the geodesic ray on a number of examples.

Example 1. We consider the projective space $(X, L) := (\mathbb{P}^1, \mathscr{O}(1))$, and the filtration $\mathcal{F}$ associated the weight function $w_{\mathcal{F}_k}(s) := k \min\{\mathrm{ord}_0(s), 1\}$, where $\mathrm{ord}_0(s)$ is the order of vanishing of $s \in H^0(\mathbb{P}^1, \mathscr{O}(k))$ at the point $0 := [1, 0] \in \mathbb{P}^1$. A straightforward verification reveals that the filtration $\mathcal{F}$ is submultiplicative and bounded.

We identify $\mathbb{P}^1$ to $\mathbb{P}(V^*)$, where V is a vector space generated by two elements: x and y. Let us consider a metric H on V, which makes x and y an orthonormal basis, and denote by h^{FS} the induced Fubini-Study metric on $\mathscr{O}(1)$. For any $k \in \mathbb{N}^*$, $i, j \in \mathbb{N}$, $i + j = k$, under the isomorphism $\mathrm{Sym}^k(V) \to H^0(\mathbb{P}(V^*), \mathscr{O}(k))$, an easy calculation shows that we have

$$\left\| x^i \cdot y^j \right\|^2_{\mathrm{Hilb}_k(h^{FS})} = \frac{i!j!}{(k+1)!}. \tag{4.19}$$

We then denote by $H^{\mathcal{F}}_{t,k}$ the geodesic ray departing from $\mathrm{Hilb}_k(h^{FS})$ and associated with $\mathcal{F}_k$. For any $a, b \in \mathbb{C}$, not simultaneously equal to zero, we have

$$\frac{FS(\mathrm{Hilb}_k(h^{FS}))}{FS(H^{\mathcal{F}}_{t,k})}\Big([ax^* + by^*]\Big) = \frac{e^{tk}(|a|^2 + |b|^2)^k + (1 - e^{tk})|a|^{2k}}{(|a|^2 + |b|^2)^k}. \tag{4.20}$$

In particular, for any $t \in [0, +\infty[$, we conclude that

$$\lim_{k \to \infty} \left(\frac{FS(\mathrm{Hilb}_k(h^{FS}))}{FS(H^{\mathcal{F}}_{t,k})} \right)^{\frac{1}{k}} \Big([ax^* + by^*]\Big) = \begin{cases} e^t, & \text{if} \quad b \neq 0, \\ 1, & \text{otherwise.} \end{cases} \tag{4.21}$$

In the notations of (4.9), we then see that $h^{L,\mathcal{F}}_{t,0}$ is not continuous for $t > 0$, but its lower semicontinuous regularization corresponds to $h^{\mathcal{F}}_t = e^{-t} h^{FS}$.

Example 2. We consider the projective space $(X, L) := (\mathbb{P}^1, \mathscr{O}(2))$, and the filtration $\mathcal{F}$ associated the weight function $w_{\mathcal{F}_k}(s) := \min\{\mathrm{ord}_0(s), k\}$ in the notations of the previous example. Similar calculation to the ones behind (4.20) will reveal that for any $a, b \in \mathbb{C}$, not simultaneously equal to zero, we have

$$\begin{aligned} \frac{FS(\mathrm{Hilb}_{2k}(h^{FS}))}{FS(H^{\mathcal{F}}_{t,k})}\Big([ax^* + by^*]\Big) = \frac{1}{(|a|^2 + |b|^2)^{2k}} \Big(\sum_{i=0}^{k} e^{ti} |a|^{2i} |b|^{2(2k-i)} \frac{(2k)!}{i!(2k-i)!} \\ + e^{tk} \cdot \sum_{i=k+1}^{2k} |a|^{2i} |b|^{2(2k-i)} \frac{(2k)!}{i!(2k-i)!} \Big). \end{aligned} \tag{4.22}$$

Cramér's theorem from large deviations theory applied for the binomial distribution yields that for any $x < 1$, we have

$$\lim_{k \to \infty} \frac{1}{2k} \log \Big(\sum_{i=k+1}^{2k} x^i \frac{(2k)!}{i!(2k-i)!} \Big) = \frac{1}{2} \log(4x). \tag{4.23}$$

From (4.23) and binomial formula, it is immediate to recover that for any $t \in [0, +\infty[$, we have

$$\frac{(h^{FS})^2}{h_t^{\mathcal{F}}}\Big([ax^* + by^*]\Big) = \begin{cases} \big(\frac{e^t|a|^2+|b|^2}{|a|^2+|b|^2}\big)^2, & e^{t/2}|a| < |b|, \\ \big(\frac{2e^{t/2}|a||b|}{|a|^2+|b|^2}\big)^2, & e^{-t/2}|b| < |a| < |b|, \\ e^t, & |b| < |a|. \end{cases} \tag{4.24}$$

We see in particular that the above geodesic ray is $\mathscr{C}^{1,1}$, but not smooth. Our example here is of course related to the well-known phenomena that one can expect at most $\mathscr{C}^{1,1}$-regularity for the envelopes, cf. [2], [16], [72].

Example 3. Let (X, L) be a polarized projective manifold with a $\mathbb{C}^*$-action. It induces the filtration on $R(X, L)$, defined as

$$\mathcal{F}^\lambda H^0(X, L^{\otimes k}) = \Big\{ s \in H^0(X, L^{\otimes k}) | \lim_{\tau \to 0} \tau^{-\lceil \lambda \rceil} \cdot \tau_* s \text{ exists} \Big\}. \tag{4.25}$$

The reader will easily check that the resulting filtration is submultiplicative. It is also bounded, as it can be seen from the fact that the ring $R(X, L)$ is finitely generated, cf. [50, Example 1.2.22]. Now, let h_0^L be a smooth metric on L with positive curvature, which is invariant under the induced $\mathbb{S}^1$-action (such h_0^L exists by the compactness of $\mathbb{S}^1$ and the usual averaging procedure). We claim that in the notations of (4.9), we then have $h_{t,0}^{L,\mathcal{F}} = h_t^{L,\mathcal{F}} = \tau(t)_* h_0^L$, where $\tau(t) := \exp(-t)$, and $\tau_* : (X, L) \to (X, L)$ is the automorphism induced by the $\mathbb{C}^*$-action by $\tau \in \mathbb{C}^*$.

To see this, remark that by considering an equivariant orthonormal basis of $\mathrm{Hilb}_k(h_0^L)$ (which exists by our assumption of $\mathbb{S}^1$-invariance), we observe $H_{k,t} := \tau(t)_* \mathrm{Hilb}_k(h_0^L)$. However, immediately from the definition of the L^2-norm, we see that $\tau(t)_* \mathrm{Hilb}_k(h_0^L) = \mathrm{Hilb}_k(h_t^L)$. The result then follows from (4.9) and the fact that for any smooth metric h^L with positive curvature, we have

$$FS(\mathrm{Hilb}(h^L)) = h^L, \tag{4.26}$$

which follows immediately from (4.8) and Tian's theorem on Bergman kernel expansion.

4.3 Submultiplicative norms and their asymptotic study

The main goal of this section is to introduce the principal tool used in the proof of Theorem 4.3, which involves the asymptotic analysis of submultiplicative norms. Since this result has applications that extend beyond the context of filtrations, we present it in full generality before explaining how it can be specialized to the study of filtrations.

A graded norm $N = \sum N_k$, $N_k := \| \cdot \|_k$, over $R(X, L) := \oplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$. is called *submultiplicative* if for any $k, l \in \mathbb{N}^*$, $f \in H^0(X, L^{\otimes k})$, $g \in H^0(X, L^{\otimes l})$, we have

$$\|f \cdot g\|_{k+l} \le \|f\|_k \cdot \|g\|_l. \tag{4.27}$$

As a basic example, any bounded metric h^L on L induces the sequence of sup-norms $\mathrm{Ban}_k^X(h^L) := \| \cdot \|_{L_k^\infty(X, h^L)}$ over $H^0(X, L^{\otimes k})$, defined for $f \in H^0(X, L^{\otimes k})$ as follows

$$\|f\|_{L_k^\infty(X, h^L)} = \sup_{x \in X} |f(x)|_{h^L}. \tag{4.28}$$

The associated graded norm $\mathrm{Ban}^X(h^L) = \sum \mathrm{Ban}_k^X(h^L)$ is clearly submultiplicative (Ban here stands for "Banach"). The main goal of this section is to prove that under some mild assumptions on L and N, asymptotically, these are the only possible examples.

More precisely, we say that two graded norms $N = \sum N_k$, $N' = \sum N'_k$ over $R(X, L)$ are *equivalent* ($N \sim N'$) if the multiplicative gap between the graded pieces, N_k and N'_k, is subexponential. This means that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}^*$, such that for any $k \geq k_0$, we have

$$\exp(-\epsilon k) \cdot N_k \leq N'_k \leq \exp(\epsilon k) \cdot N_k. \tag{4.29}$$

We say N is *bounded* if $N \geq \mathrm{Ban}^X(h^L)$ for a certain smooth metric h^L on L.

For any $k \in \mathbb{N}^*$, such that $L^{\otimes k}$ is very ample, any norm N_k on $H^0(X, L^{\otimes k})$ induces the Fubini-Study metric $FS(N_k)$ on $L^{\otimes k}$ through the associated Kodaira embedding, see (4.6). The following basic construction plays a fundamental role in what follows.

Lemma 4.5. The sequence of Fubini-Study metrics $FS(N_k)$, $k \in \mathbb{N}^*$, is submultiplicative for any submultiplicative graded norm $N = \sum N_k$. In particular, by Fekete's lemma, the sequence of metrics $FS(N_k)^{\frac{1}{k}}$ on L converges, as $k \to \infty$, to a (possibly only bounded from above and even null) upper semi-continuous metric, which we denote by $FS(N)$. We, moreover, have

$$FS(N) = \inf FS(N_k)^{\frac{1}{k}}. \tag{4.30}$$

If N is bounded, then $FS(N)_*$ has a bounded psh potential. If $FS(N)$ is lower semi-continuous and everywhere non-null, the convergence is uniform.

Proof. The first part follows easily from Lemma 4.1. The second part follows from Lemma 4.1 and some classical results, cf. [29, Proposition I.4.24]. The third part is a consequence of the well-known subadditive analogue of Dini's theorem and a statement asserting that a pointwise limit of subadditive sequence of continuous functions is upper semi-continuous, cf. [34, Appendix A]. □

We can now state the first main theorem of this section.

Theorem 4.6. *Assume that a graded norm $N = \sum N_k$ over the section ring $R(X, L)$ of an ample line bundle L is submultiplicative and $FS(N)$ is continuous. Then*

$$N \sim \mathrm{Ban}^X(FS(N)). \tag{4.31}$$

The first example from Section 4.2 shows that in the context of filtrations, the norms with non-continuous Fubini-Study potentials arise naturally. To be able to treat these examples, we define a weaker equivalence relation on the set of graded norms.

Let $N_i = \| \cdot \|_i$, $i = 1, 2$, be two norms on a finite dimensional vector space V. We define the *logarithmic relative spectrum* of N_1 with respect to N_2 as a non-increasing sequence $\lambda_j := \lambda_j(N_1, N_2)$, $j = 1, \cdots, \dim V$, defined as follows

$$\lambda_j := \sup_{\substack{W \subset V \\ \dim W = j}} \inf_{w \in W \setminus \{0\}} \log \frac{\|w\|_2}{\|w\|_1}. \tag{4.32}$$

For $p \in [1, +\infty[$, we let

$$d_p(N_1, N_2) := \sqrt[p]{\frac{\sum_{i=1}^{\dim V} |\lambda_i|^p}{r}}. \tag{4.33}$$

We say that graded norms $N = \sum N_k$ and $N' = \sum N'_k$ are *p-equivalent* ($N \sim_p N'$) if

$$\frac{1}{k} d_p(N_k, N'_k) \to 0, \qquad \text{as } k \to \infty. \tag{4.34}$$

We show in Proposition 4.10 that $\sim_p$, $p \in [1, +\infty]$, is an equivalence relation and $\sim$ equals $\sim_{+\infty}$.

Theorem 4.7. *Assume that a graded norm $N = \sum N_k$ over the section ring $R(X, L)$ of an ample line bundle L is submultiplicative and bounded. Then for any $p \in [1, +\infty[$, we have*

$$N \sim_p \mathrm{Ban}^X(FS(N)). \tag{4.35}$$

4.4 Submultiplicative interpolations of filtrations

The main objective of this section is to establish Theorem 4.3, which will follow as a consequence of Theorem 4.7. A central difficulty arises from the fact that it is unclear whether the ray of norms $H_{k,t}$ appearing in Theorem 4.3 satisfies any form of submultiplicativity. To address this issue, we begin by replacing $H_{k,t}$ with an alternative interpolation construction that does enjoy submultiplicative properties. Most of the results will then follow from Theorem 4.7 and the fact that the two interpolation constructions are sufficiently close – a fact we will establish later.

We fix a finitely-dimensional normed vector space (V, N_V), $\| \cdot \|_V := N_V$ and a filtration $\mathcal{F}$ of V. We construct a ray of norms $N^t_{V,\mathcal{F}} := \| \cdot \|^t_{V,\mathcal{F}}$, $t \in [0, +\infty[$, emanating from N_V, as follows

$$\|f\|^t_{V,\mathcal{F}} := \inf \Big\{ \sum e^{-t\mu_i} \cdot \|f_i\|_V \; : \; f = \sum f_i, \; f_i \in \mathcal{F}^{\mu_i} V \Big\}. \tag{4.36}$$

The main advantage of the above construction of the ray, compared to the previous one in (4.2), lies in its behavior with respect to algebraic operations. To make this precise, we fix an algebra A equipped with a submultiplicative filtration $\mathcal{F}$.

Proposition 4.8. For any submultiplicative norm N on a ring A, the norm $N^t_{\mathcal{F}}$ is submultiplicative for any $t \in [0, +\infty[$.

Proof. Immediate verification left to the reader. □

We now compare this ray with the earlier construction in (4.2). Fix a *Hermitian* norm $N_H := \| \cdot \|_H$ on V. We denote by $N^{\perp,t}_{H,\mathcal{F}}$ the ray of Hermitian norms emanating from N_H and associated with the filtration $\mathcal{F}$ as in (4.2). The following result provides a comparison between the two constructions. To state it precisely, for two norms N_1 and N_2 on V, we denote by $d_{+\infty}(N_1, N_1)$ the minimal constant $C > 0$ so that $N_1 \cdot \exp(-C) \leq N_2 \leq N_1 \cdot \exp(C)$.

Lemma 4.9. For any (resp. Hermitian) norm N_V (resp. N_H) on V and any $t \in [0, +\infty[$, we have

$$d_{+\infty}(N^{\perp,t}_{H,\mathcal{F}}, N^t_{V,\mathcal{F}}) \leq d_{+\infty}(N_H, N_V) + \log \dim V. \tag{4.37}$$

Proof. Let us denote by $N^t_{H,\mathcal{F}}$ the ray of norms emanating from N_H by the construction from (4.36). Let us establish first that

$$\dim V \cdot N^t_{H,\mathcal{F}} \geq N^{\perp,t}_{H,\mathcal{F}}. \tag{4.38}$$

By the definition of $N^t_{H,\mathcal{F}}$, we conclude that for any $\lambda \in \mathbb{R}$, $f \in V$, we have

$$\|f\|^t_{H,\mathcal{F}} \geq e^{-t\lambda} \|Q_\lambda(f)\|_H, \tag{4.39}$$

where $Q_\lambda(f) := f - P_\lambda(f)$ and $P_\lambda(f)$ is the projection of f to $\cup_{\epsilon>0} \mathcal{F}^{\lambda+\epsilon} V$ with respect to the norm N_H. We take now the decomposition $f = \sum a_i s^t_1$, $a_i \in \mathbb{C}$ of $f \in V$ in basis $(s^t_1, \dots, s^t_r)$ from (4.2). Then by the definition of $N^{\perp,t}_{H,\mathcal{F}}$, we have

$$\|f\|^{\perp,t}_{H,\mathcal{F}} := \sqrt{\sum |a_i|^2}. \tag{4.40}$$

By taking sums of (4.39) over all jumping numbers, using (4.40) and the fact that for any $i = 1, \dots, r$, we have $\|Q_{e_{\mathcal{F}}(i)}(f)\|_H \geq e^{te_{\mathcal{F}}(i)} \cdot |a_i|$, we deduce (4.38).

Now, directly by the definition of $N^t_{H,\mathcal{F}}$, we obtain $\|f\|^t_{H,\mathcal{F}} \leq \sum |a_i|$. From this, (4.40) and mean value inequality, we establish

$$N^t_{H,\mathcal{F}} \leq \sqrt{\dim V} \cdot N^{\perp,t}_{H,\mathcal{F}}. \tag{4.41}$$

From (4.38) and (4.41), we conclude that $d_{+\infty}(N^{\perp,t}_{H,\mathcal{F}}, N^t_{H,\mathcal{F}}) \leq \log \dim V$. To finish the proof, it is only left to use the following trivial bound $d_{+\infty}(N^t_{V,\mathcal{F}}, N^t_{H,\mathcal{F}}) \leq d_{+\infty}(N_H, N_V)$. □

Proof of Proposition 4.2. First of all, we denote by $N_{k,t}$ the ray of norms departing from $\mathrm{Ban}^X_k(h^L_0)$ constructed as in (4.36) associated with the restriction of $\mathcal{F}$ to $H^0(X, L^{\otimes k})$.

Immediately from Tian's theorem on the Bergman kernel expansion, there is $C > 0$ so that for any $k \in \mathbb{N}^*$, we have

$$C^{-1} k^{-n} \cdot \mathrm{Ban}^X_k(h^L_0) \leq \mathrm{Hilb}_k(h^L_0) \leq \mathrm{Ban}^X_k(h^L_0). \tag{4.42}$$

From this and Lemma 4.9, we conclude that there is $C > 0$ such that for any $t \geq 0$, $k \in \mathbb{N}^*$,

$$C^{-1} k^{-2n} \cdot N_{k,t} \leq H_{k,t} \leq C k^{2n} \cdot N_{k,t}. \tag{4.43}$$

From this, (4.9), and the fact that $\lim_{k\to\infty} k^{\frac{1}{k}} = 1$, we conclude that

$$h^{L,\mathcal{F}}_{t,0} = \lim_{k\to\infty} \inf_{l \geq k} FS(N_{l,t})^{\frac{1}{l}}. \tag{4.44}$$

However, since for any $t > 0$, the norm $N_t := \sum N_{k,t}$ on $R(X, L)$ is submultiplicative, we conclude by Lemma 4.5 that

$$h^{L,\mathcal{F}}_{t,0} = FS(N_t). \tag{4.45}$$

Moreover, by the submultiplicativity, we have $FS(N_{l,t})^2 \geq FS(N_{2l,t})$ for any $l \in \mathbb{N}$, and so $FS(N_t)$ is a decreasing limit of metrics with psh potentials, which are uniformly bounded from below. By the standard results from complex analysis, cf. [29, Proposition I.4.24], we conclude that $h^{L,\mathcal{F}}_t$ has a bounded psh potential, which finishes the proof. □

Next, we shall use the following basic result.

Proposition 4.10. For any $p \in [1, +\infty[$, $\sim_p$ is an equivalence relation.

Proof. Directly from (4.32), for any norms N_1, N_2, N_3, $j = 1, \cdots, \dim V$, we have

$$\lambda_j(N_1, N_2) + \lambda_{\dim V}(N_2, N_3) \leq \lambda_j(N_1, N_3) \leq \lambda_j(N_1, N_2) + \lambda_1(N_2, N_3), \tag{4.46}$$

and whenever $N_1 \leq N_2 \leq N_3$, we have

$$\lambda_j(N_1, N_2) \leq \lambda_j(N_1, N_3). \tag{4.47}$$

When both N_i, $i = 1, 2$, are Hermitian norms associated with the scalar products $\langle \cdot, \cdot \rangle_i$, the logarithmic relative spectrum coincides (up to a multiplication by a factor 2) with the logarithm of the spectrum of the transfer map $A \in \mathrm{End}(V)$ between N_1 and N_2, which is the Hermitian operator, verifying $\langle A\cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_2$. In particular, we then have

$$d_p(N_1, N_2) = \sqrt[p]{\frac{\mathrm{Tr}[|\log A|^p]}{\dim V}}, \tag{4.48}$$

and so the normalization from (4.33) gives in this case the distance on the space of Hermitian norms coming from the natural Finsler structure.

By [12, Theorem 3.1], the functions d_p, $p \in [1, +\infty[$, defined in (4.33), are such that

$$d_p \text{ satisfies the triangle inequality over the space of Hermitian norms.} \tag{4.49}$$

Remark also that the John ellipsoid theorem, cf. [61, p. 27], says that for any normed vector space (V, N_V), there is a Hermitian norm N_V^H on V, verifying

$$N_V^H \leq N_V \leq \sqrt{\dim V} \cdot N_V^H. \tag{4.50}$$

From (4.46), (4.47), (4.49), (4.50) and the fact that $\dim H^0(X, L^{\otimes k})$ grow polynomially (hence, subexponentially) in $k \in \mathbb{N}$, we see that $\sim_p$ for $p \in [1, +\infty[$ is indeed an equivalence relation. □

Proof of Theorem 4.3. To simplify the presentation, we additionally assume that the filtration $\mathcal{F}$ is such that

$$h_{t,0}^{L,\mathcal{F}} = h_t^{L,\mathcal{F}}, \tag{4.51}$$

for any $t \in [0, +\infty[$. The above condition is equivalent to the fact that $h_{t,0}^{L,\mathcal{F}}$ is a continuous metric for any $t \in [0, +\infty[$. The proof provided in [35] remains valid even in the absence of this assumption, and we refer the interested reader to that work for further details.

Immediately from (4.43), we see that the norms $H_t := \sum H_{k,t}$ and $N_t := \sum N_{k,t}$ on $R(X, L)$ are $\sim_p$-equivalent for any $p \in [1, +\infty[$. However, since N_t is submultiplicative, we conclude by Theorem 4.7 that N_t is $\sim_p$-equivalent to $\mathrm{Ban}^X(FS(N_t))$. By (4.45) and our standing assumption, we see that $\mathrm{Ban}^X(FS(N_t)) = \mathrm{Ban}^X(h_t^{L,\mathcal{F}})$. From this, by Proposition 4.10, we conclude that it would suffice to establish that $\mathrm{Hilb}(h_t^{L,\mathcal{F}})$ is $\sim_p$-equivalent to $\mathrm{Ban}^X(h_t^{L,\mathcal{F}})$ for any $t \in [0, +\infty[$. But this follows from the general fact that for an arbitrary continuous metric h^L with psh potential, the norms $\mathrm{Ban}^X(h^L)$ and $\mathrm{Hilb}(h^L)$ are $\sim_p$-equivalent, see (4.42). □

4.5 Elements of the proof of Theorems 4.6, 4.7

The main goal of this section is to describe some main ideas from the proof of Theorems 4.6, 4.7. To simplify our presentation, we shall make the following two simplifications. First, we assume that $R(X, L)$ is generated in degree 1, i.e. as an algebra it is generated by $H^0(X, L)$. By a standard result from commutative algebra, this can always be achieved upon replacing L by its sufficiently big tensor power. Second we shall fix a norm N_1 on $H^0(X, L)$, and will only establish Theorem 4.6 for the norm $N = \sum N_k$, which is the maximal submultiplicative norm on $R(X, L)$, coinciding with N_1 on $H^0(X, L)$. The full proof can be derived from this special case via an approximation argument; however, to keep the presentation concise, we will not pursue it here, and refer to [35] for details.

In order to describe the maximal submultiplicative norm N_k explicitly, we need to recall some basic definitions from functional analysis. Recall that there is no canonical way of inducing a norm on the tensor product $V_1 \otimes V_2$ of two normed vector spaces (V_1, N_1), (V_2, N_2). Instead, several natural constructions are possible. The construction which is particularly relevant in what follows is the one of *projective tensor norm* $N_1 \otimes_\pi N_2 = \|\cdot\|_{\otimes_\pi}$ on $V_1 \otimes V_2$ which is defined for $f \in V_1 \otimes V_2$ as

$$\|f\|_{\otimes_\pi} = \inf\Big\{ \sum \|x_i\|_1 \cdot \|y_i\|_2 : \quad f = \sum x_i \otimes y_i \Big\}, \tag{4.52}$$

where the infimum is taken over different ways of partitioning f into a sum of decomposable terms.

For the next proposition, we denote by $\mathrm{Sym}_\pi^k N_1$ the norm on $\mathrm{Sym}^k H^0(X, L)$ induced by the projective tensor norm on $H^0(X, L)^{\otimes k}$ induced by N_1. We denote by $[\mathrm{Sym}_\pi^k N_1]$ the induced quotient norm on $H^0(X, L^{\otimes k})$ constructed as in (2.14) under the multiplication map

$$\mathrm{Mult}_k : \mathrm{Sym}^k H^0(X, L) \to H^0(X, L^{\otimes k}), \tag{4.53}$$

which is surjective by our assumption of generation in degree 1.

Proposition 4.11. We have $N_k = [\mathrm{Sym}_\pi^k N_1]$.

Proof. The submultiplicativity property immediately yields the inequality $N_k \leq [\mathrm{Sym}_\pi^k N_1]$. To establish the reverse inequality, it suffices to show that the norm $\|\cdot\|_k := [\mathrm{Sym}_\pi^k N_1]$ satisfies the submultiplicative estimate $\|f \cdot g\|_{k+l} \leq \|f\|_k \cdot \|g\|_l$ for any sections $f \in H^0(X, L^{\otimes k})$, $g \in H^0(X, L^{\otimes l})$. This inequality can be verified directly: once the definitions are unpacked, the norm $\|f \cdot g\|_{k+l}$ is defined as the infimum of a certain expression over all decompositions of $f \cdot g$ into monomials, whereas $\|f\|_k \cdot \|g\|_l$ corresponds to the infimum of the same expression taken only over decompositions that arise as products of decompositions of f and g. The inequality then follows immediately. □

Remark now that for an arbitrary vector space V, there is an isomorphism between $\mathrm{Sym}^k V$ and $H^0(\mathbb{P}(V^*), \mathscr{O}(k))$. We shall now use the notations inspired by (4.6). We denote by $FS^{\mathbb{P}(V^*)}(N_V)$ the norm on $\mathscr{O}(1)$ over $\mathbb{P}(V^*)$ induced by N_V. We denote by $\mathrm{Ban}_k^{\mathbb{P}}(N_V)$ the L^∞-norm on $H^0(\mathbb{P}(V^*), \mathscr{O}(k))$ induced by $FS^{\mathbb{P}(V^*)}(N_V)$. The central point in our proof lies in the following statement, which will be established in Section 4.6.

Theorem 4.12. *For any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, so that for any $k \geq k_0$, we have*

$$\mathrm{Ban}_k^{\mathbb{P}}(N_V) \leq \mathrm{Sym}_\pi^k N_1 \leq \exp(\epsilon k) \cdot \mathrm{Ban}_k^{\mathbb{P}}(N_V). \tag{4.54}$$

Proof of Theorems 4.6. As explained above, we shall only establish Theorem 4.6 for the norm N described in the beginning of this section. We shall first establish that $FS(N_k) = FS(N_1)^k$ for any $k \in \mathbb{N}^*$. For this, we shall use the characterization of the Fubini-Study metric from Lemma 4.1. Let us first show the inequality $FS(N_k) \leq FS(N_1)^k$. For this, let $s_1 \in H^0(X, L)$, $s_1(x) \leq 0$, be such that $|s_1(x)|_{FS(N_1),x} = \|s\|_1$. By considering $s := s_1^k$ in (4.7), we then obtain $|s_1^k(x)|_{FS(N_k),x} \leq \|s\|_1^k$, which finishes the proof of $FS(N_k) \leq FS(N_1)^k$.

Let us now show the opposite bound $FS(N_k) \geq FS(N_1)^k$. By the above reasoning, it is sufficient to establish that in the formula (4.7) for $FS(N_k)$, it is sufficient to restrict only to decomposable sections, i.e. those which can be written as a product of sections from $H^0(X, L)$.

For this, remark that by definition, for any $s \in H^0(X, L^{\otimes k})$, $\epsilon > 0$, there is a decomposition $s_k = \sum_\alpha s_{\alpha,1} \cdot \ldots \cdot s_{\alpha,k}$, where $s_{\alpha,1}, \ldots, s_{\alpha,k} \in H^0(X, L)$ are such that

$$\|s\|_k \geq \sum_\alpha \|s_{\alpha,1}\|_1 \cdot \ldots \cdot \|s_{\alpha,k}\|_1 - \epsilon. \tag{4.55}$$

Also, by the triangle inequality, we obviously have

$$|s_k(x)|_{FS(N_k),x} \leq \sum_\alpha |s_{\alpha,1}(x)|_{FS(N_k)^{\frac{1}{k}},x} \cdot \ldots \cdot |s_{\alpha,k}|_{FS(N_k)^{\frac{1}{k}},x}. \tag{4.56}$$

Remark now that if $s_1, s_2 \in H^0(X, L^{\otimes k})$ are such that $\frac{\|s_1\|_k}{|s_1(x)|_{(^hL)^k}} \geq \frac{\|s_2\|_k}{|s_2(x)|_{(^hL)^k}}$, then in the formula (4.7) for $FS(N_k)$, the quantity corresponding to s from the complex line induced by s_2 shall be smaller than the complex line induced by s_1. This implies that in the formula (4.7) for $FS(N_k)$, it is sufficient to restrict only to decomposable sections by (4.55), (4.56) and the following elementary inequality: for any $a, b, c, d > 0$, we have $\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d})$.

From above, to finish the proof of Theorems 4.6, it is then sufficient to establish the following statement: for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, so that for any $k \geq k_0$, we have

$$\mathrm{Ban}_k^X(FS(N_1)) \leq N_k \leq \exp(\epsilon k) \cdot \mathrm{Ban}_k^X(FS(N_1)). \tag{4.57}$$

Let us first show that the lower bound is immediate. Indeed, for any norm N_k' on $H^0(X, L^{\otimes k})$, immediately from (4.7), we see that for any $s \in H^0(X, L^{\otimes k})$, $x \in X$, we have

$$\|s\|_{N_k'} \geq |s(x)|_{FS(N_k')}, \tag{4.58}$$

which translates after taking supremum into the inequality

$$\mathrm{Ban}_k^X(FS(N_k')) \leq N_k'. \tag{4.59}$$

Taken into account that $FS(N_k) = FS(N_1)^k$, the above inequality implies the lower bound of (4.57).

Let us establish now the upper bound by relying on Theorem 4.12 and the semiclassical extension theorem – more specifically – Proposition 2.5. Immediately from Theorem 4.12, we see that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, so that for any $k \geq k_0$, we have

$$N_k \leq \exp(\epsilon k) \cdot [\mathrm{Ban}_k^{\mathbb{P}}(N_1)], \tag{4.60}$$

where the brackets above are for the quotient norm associated with the multiplication operator Mult_k from (4.53). We shall establish that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, so that for any $k \geq k_0$, we have

$$[\mathrm{Ban}_k^{\mathbb{P}}(N_1)] \leq \exp(\epsilon k) \cdot \mathrm{Ban}_k^X(FS(N_1)), \tag{4.61}$$

which would clearly finish the proof.

Let us consider the Kodaira embedding Kod_1 from (4.5). We denote by Res_k : $H^0(\mathbb{P}(H^0(X,L)^*), \mathscr{O}(k)) \to H^0(X, L^{\otimes k})$ the associated restriction operator, and by Res_k, $k \in \mathbb{N}^*$, the restriction operators on the associated graded pieces. The multiplication operator Mult_k from (4.53) factorizes through symmetrization and restriction as

$$\begin{array}{ccc} H^0(X,L)^{\otimes k} & \xrightarrow{\mathrm{Sym}} & \mathrm{Sym}^k(H^0(X,L)) \\ & & \| \\ & \searrow^{\mathrm{Mult}_k} & H^0(\mathbb{P}(H^0(X,L)^*), \mathscr{O}(k)) \\ & & \downarrow \mathrm{Res}_k \\ & & H^0(X, L^{\otimes k}). \end{array} \tag{4.62}$$

The bound (4.61) is then a direct application of (4.62) and Proposition 2.5. □

4.6 Projective tensor norms and holomorphic extension theorem

The main goal of this section is to establish Theorem 4.12. We first establish this result by the classical Fourier analysis in the special case $V = \mathbb{C}^r$, $r \in \mathbb{N}^*$, and $N_V := \| \cdot \|_V := l_1$, and then prove it in its full generality by relying on some tools from functional analysis and – quite surprisingly – the tools from complex geometry; more specifically, the semiclassical extension theorem in the form of Proposition 2.5.

To establish Theorem 4.12 for $V = \mathbb{C}^r$, $r \in \mathbb{N}^*$, and $N_V := \| \cdot \|_V := l_1$, consider a vector space $V_{r,k}$ of homogeneous complex polynomials of degree k in r variables. We represent an element $P \in V_{r,k}$ as

$$P(x_1, \cdots, x_r) = \sum_{|\alpha|=k} a_\alpha x^\alpha. \tag{4.63}$$

Since $\dim V_{r,k} = \binom{r+k}{r} < +\infty$, any two norms on $V_{r,k}$ are equivalent. In particular, there is a constant $B_{r,k} > 0$, such that for any $P \in V_{r,k}$, we have

$$\sum_{|\alpha|=k} |a_\alpha| \leq B_{r,k} \cdot \|P\|, \tag{4.64}$$

where the sup-norm $\|P\|$ is defined as follows

$$\|P\| := \sup_{\substack{x_i \in \mathbb{C} \\ |x_i| \leq 1}} \big| P(x_1, \cdots, x_r) \big|. \tag{4.65}$$

We assume that the constants $B_{r,k}$ for $r, k \in \mathbb{N}^*$, are the minimal constants verifying the inequality (4.64). Let us establish the following statement.

Proposition 4.13. For any fixed $r \in \mathbb{N}^*$, the sequence $B_{r,k}$, $k \in \mathbb{N}$, grows at most polynomially in k.

Proof. Remark first that by the maximum principle, we have

$$\|P\| = \sup_{\substack{x_i \in \mathbb{C} \\ |x_i|=1}} \big|P(x_1, \cdots, x_r)\big|. \tag{4.66}$$

Parseval's identity shows us

$$\int_{\substack{x_i \in \mathbb{C} \\ |x_i|=1}} \big|P(x_1, \cdots, x_r)\big|^2 d\nu = \sum_{|\alpha|=k} |a_\alpha|^2, \tag{4.67}$$

where $d\nu$ is the Lebesgue measure on $\{(x_1, \ldots, x_n) \in \mathbb{C}^n \| x_i| = 1, \quad i = 1, \ldots, n\}$. The result then follows by the Generalized mean inequality and the fact that $\dim V_{r,k} = \binom{r+k}{r}$ grows polynomially in k for a fixed r. □

Proof of Theorem 4.12 in the special case when $V = \mathbb{C}^r$ and $N_V := \|\cdot\|_V := l_1$. From (4.59), it is sufficient to show that $\mathrm{Sym}_\pi(N_V)$, considered up to a subexponential constant, is bounded from above by $\mathrm{Ban}_k^{\mathbb{P}}(N_V)$.

Let us denote by $x_1, \ldots, x_r$ the coordinate vectors in $\mathbb{C}^r$. We use the notation (4.63) for $P \in \mathrm{Sym}^k(V)$, $k \in \mathbb{N}^*$. Since the dual of the l_1-norm is given by the l_∞-norm on $\mathbb{C}^r$, we have

$$\|P\|_{\mathrm{Ban}_k^{\mathbb{P}}(N_V)} = \|P\|. \tag{4.68}$$

On another hand, immediately from the definitions, we have

$$\|P\|_{\mathrm{Sym}_\pi(N_V)} \leq \sum_{|\alpha|=k} |a_\alpha|. \tag{4.69}$$

We conclude by Proposition 4.13 and (4.68), (4.69) that $\mathrm{Sym}_\pi(N_V)$, considered up to a subexponential constant, is bounded from above by $\mathrm{Ban}_k^{\mathbb{P}}(N_V)$. □

Let us now establish Theorem 4.12 in its full generality. Surprisingly, our main technical tool in the proof of this purely functional-analytic statement comes from complex geometry. We also use the following classical result.

Lemma 4.14 (cf. [31, Lemma 2.2]). For any finite dimensional complex normed vector space $(V, \|\cdot\|_V)$, and any $\epsilon > 0$, there is $l \in \mathbb{N}^*$ and a surjective map $\pi : \mathbb{C}^l \to V$, such that $\|\cdot\|_V$ is related to the quotient norm associated with the l_1-norm on $\mathbb{C}^l$ as follows

$$\exp(-\epsilon) \cdot [l_1] \leq \|\cdot\|_V \leq [l_1]. \tag{4.70}$$

Proof of Theorem 4.12. Since Theorem 4.12 holds for l_1-norms by the above considerations, we deduce by Lemma 4.14 that it is enough to show that the validity of Theorem 4.12 is stable under taking quotients, i.e. if Theorem 4.12 holds for a normed vector space (U, N_U), then it holds for any normed quotient (V, N_V), $\pi : U \to V$. As we shall see below, this is a consequence of the semiclassical version of Ohsawa-Takegoshi extension theorem. We consider the embedding

$$\mathrm{Im}_\pi : \mathbb{P}(V^*) \to \mathbb{P}(U^*). \tag{4.71}$$

Clearly, under this embedding, the associated restriction operator, which we denote by $\mathrm{Res}_{\pi,k}$, and the projection map to the symmetric tensors induced by π, which we denote by $\mathrm{Sym}^k\pi$, can be put into the following commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}(U^*), \mathscr{O}(k)) & \xrightarrow{\mathrm{Res}_{\pi,k}} & H^0(\mathbb{P}(V^*), \mathscr{O}(k)) \\ \Big\| & & \Big\| \\ \mathrm{Sym}^k(U) & \xrightarrow{\mathrm{Sym}^k\pi} & \mathrm{Sym}^k(V). \end{array} \tag{4.72}$$

Since (V, N_V) is a quotient of (U, N_U), we also have

$$FS(N_V) = FS(N_U)|_{\mathbb{P}(V^*)}. \tag{4.73}$$

From Proposition 2.5, (4.72) and (4.73), we conclude that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}^*$, such that for any $k \geq k_0$, $f \in \mathrm{Sym}^k(V)$, there is $g \in \mathrm{Sym}^k(U)$, such that $\mathrm{Sym}^k\pi(g) = f$, and

$$\|f\|_{\mathrm{Ban}_k^{\mathbb{P}}(N_V)} \geq \exp(-\epsilon k) \cdot \|g\|_{\mathrm{Ban}_k^{\mathbb{P}}(N_U)} \tag{4.74}$$

Now, since Theorem 4.12 holds for (U, N_U), we deduce that there is $k_1 \in \mathbb{N}^*$, such that for any $k \geq k_1$, $g \in \mathrm{Sym}^k(U)$, we have

$$\|g\|_{\mathrm{Ban}_k^{\mathbb{P}}(N_U)} \geq \exp(-\epsilon k) \cdot \|g\|_{\mathrm{Sym}_\pi(N_U)}. \tag{4.75}$$

Since (V, N_V) is a quotient of (U, N_U), for any $x \in U$, we have

$$\|x\|_U \geq \|\pi(x)\|_V. \tag{4.76}$$

From this, the definition of the projective tensor norm, we deduce that for any $k \in \mathbb{N}^*$, $f \in \mathrm{Sym}^k(V)$ and $g \in \mathrm{Sym}^k(U)$, verifying $\mathrm{Sym}^k\pi(g) = f$, we have

$$\|g\|_{\mathrm{Sym}_\pi(N_U)} \geq \|f\|_{\mathrm{Sym}_\pi(N_V)}. \tag{4.77}$$

From (4.74), (4.75) and (4.77), we see that for any $k \geq \max\{k_0, k_1\}$, $f \in \mathrm{Sym}^k(V)$, we have

$$\|f\|_{\mathrm{Ban}_k^{\mathbb{P}}(N_V)} \geq \exp(-2\epsilon k) \cdot \|f\|_{\mathrm{Sym}_\pi(N_V)}. \tag{4.78}$$

As $\epsilon > 0$ is arbitrary, from (4.59) and (4.78), we conclude that $\mathrm{Ban}_k^{\mathbb{P}}(N_V)$ and $\mathrm{Sym}_\pi(N_V)$ are asymptotically equivalent. This finishes the proof. □

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