

EXERCISES ON BIRATIONAL GEOMETRY

For simplicity, we work over an algebraically closed field k .

1. SURFACES

Exercise 1. Assume that $\text{char}(k) = 0$ (or that it is big enough with respect to the equations below).

- (a) Find the minimal resolution of the du Val singularity $(x^n = yz) \subseteq \mathbb{A}^3$, with $n \geq 1$.
- (b) Consider the surface singularity $S = (x^3 + y^2z + z^4 = 0) \subseteq \mathbb{A}^3$. Does S have log canonical singularities? (remember that verifying log canonical singularities should be done on a *log* resolution!).

Exercise 2 (Negativity lemma (very useful!)). (a) Let $f: S \rightarrow T$ be a projective birational morphism of surfaces, with S smooth. Let $E = E_1 \cup \cdots \cup E_n$ be the decomposition of the exceptional divisor E into irreducible components. Show that the matrix $(E_i \cdot E_j)$ is negative definite.

Hint: There are several ways to see this, here is one. First, you may assume that S and T are projective. You have to show that if D is f -exceptional and non-zero, then $D^2 < 0$. You may assume that $D \geq 0$ to achieve this (why?). In this case, show that if H is sufficiently ample, then $h^0(X, \mathcal{O}_X(nD + H)) \rightarrow \infty$ as $n \rightarrow \infty$, contradicting the fact that D is f -exceptional (why?).

- (b) Let $f: X \rightarrow Y$ be a projective birational morphism of normal varieties, and assume that B is an f -nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. Show that $B \leq 0$.

In several exercises below, the negativity lemma will be useful!

Exercise 3. Let X be a smooth variety. Show that X has terminal singularities. Deduce that if S is a smooth projective surface with K_S big and nef but not ample, its canonical model must be singular. Find such an example.

Hint for the example: cyclic cover to improve the positivity of K_S can be useful!

Exercise 4. Show that terminal surface singularities are smooth. Show that this is not true in higher dimension, for example by considering $(x^2 + y^2 + z^2 + t^2 = 0) \subseteq \mathbb{A}^4$.

Exercise 5. Let S be a smooth projective surface with $\kappa(S) \geq 0$. Show that S admits no (-1) -curve if and only if K_S is nef. Similarly, let $f: S \rightarrow T$ be a projective birational morphism. Show that K_S is f -nef if and only if f does not contract any (-1) -curve (now without any assumption on $\kappa(S)$).

Exercise 6 ((Non-)uniqueness of minimal models). (a) Let $f: X \dashrightarrow Y$ be a birational map between two normal projective varieties with terminal singularities such that both K_X and K_Y are nef. Show that f does not contract any divisor.

- (b) Deduce that minimal models of smooth projective surfaces are unique.
- (c) What happens if we replace terminal singularities by canonical singularities in the first point?

Exercise 7 (Classifying surface log canonical singularities with Cartier canonical class). Do the following:

- (a) Let $f: S \rightarrow T$ be a minimal resolution of a normal singularity such that K_T is $\mathbb{Q}$ -Cartier, and write

$$K_S = f^*K_T + \sum_i a_i E_i.$$

Show that $a_i \leq 0$ for all i , and that either $a_i = 0$ for all i , or $a_i < 0$ for all i .

- (b) Use the above to show that if T is a log canonical singularity with K_T Cartier, then either T is a du Val singularity, or that there are precisely two possibilities for its dual graph:
- a cycle of $\mathbb{P}^1$'s
 - a single vertex, represented by an elliptic curve.
 - a single vertex, represented by a nodal curve of arithmetic genus 1 (i.e. a $\mathbb{P}^1$ with one node).

2. HIGHER DIMENSION

Exercise 8. Let X be a smooth Fano variety in characteristic zero. Show that for *any* $\mathbb{Q}$ -Cartier divisor D on X , one can run a D -MMP. Show also that if D is nef, then it is semiample.

Hint: Write a multiple of D as $K_X + \Delta$, with (X, Δ) klt.

Remark 1. The previous statement can be generalized to log Fano pairs, but the proof is slightly trickier.

Exercise 9. Let $f: X \dashrightarrow Y$ be a birational map between normal projective varieties, which is an isomorphism at each codimension one point of X and Y (e.g. a flip or a flop). Show that if H is an ample divisor and f_*H is $\mathbb{Q}$ -Cartier, then f_*H cannot be nef.

Exercise 10 (A singular minimal model in dimension three, unique in its birational class). For simplicity, assume that $\text{char}(k) = 0$. Let A be an abelian threefold, and consider X to be the quotient of A by its involution. Proceed as follows:

- (a) Show that X is singular. You may assume that X has terminal singularities (in fact, its singularities are equivalent to the $\mathbb{Z}/2\mathbb{Z}$ -quotient of $\mathbb{A}^3$ given by $(x, y, z) \mapsto (-x, -y, -z)$, which are also equivalent of the cone of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$).
- (b) Assume that there exists a birational map $X \dashrightarrow Y$ where K_Y is nef and Y has terminal singularities, and consider the composition $A \rightarrow X \dashrightarrow Y$. Show that this rational map is étale in codimension one (use Exercise 6).
- (c) Let Z be the integral closure of Y in A , so that we get a finite morphism $Z \rightarrow Y$ with Y normal. Show that Y is in fact smooth, and $Y \rightarrow Z$ is étale.
- (d) Note that $A \dashrightarrow Y$ is an isomorphism at each codimension one point. Show that $A \dashrightarrow Y$ is an isomorphism, as follows: if not, use the cone theorem on an appropriate klt pair (A, Δ) and the previous exercise to show that A contains a rational curve, hence getting a contradiction (there is no rational curve on an abelian variety).
- (e) Deduce that neither map $X \dashrightarrow Y$ or $Y \dashrightarrow X$ can contract a curve, and conclude that $X \cong Y$.

Exercise 11. Show that klt and lc $\mathbb{Q}$ -factorial singularities are preserved through all steps of an MMP. If there is no boundary, show also that terminal and canonical singularities are preserved through all steps of an MMP.

Exercise 12. (a) Let $f: Y \rightarrow X$ be a projective birational morphism between normal varieties. Show that if X is $\mathbb{Q}$ -factorial, then the exceptional locus of f is purely divisorial (unless f is an isomorphism).

Hint: Take a relatively ample effective divisor H on Y , and compare it with f^*f_*H .

- (b) Let (X, Δ) be a klt and $\mathbb{Q}$ -factorial singularity, and let $f: Y \rightarrow X$ be a log resolution. Find a divisor D on Y such that a $(K_Y + D)$ -MMP over X finishes at X . Show also that the last step of this MMP must be a divisorial contraction (if we remove the

$\mathbb{Q}$ -factorial hypothesis, then we instead get a birational morphism whose exceptional locus has codimension at least two).

- Remark 2.* (a) This can be very useful to study klt singularities! For example, Bernasconi and Kollár used this method and a Kawamata-Viewheg vanishing theorem on log del Pezzo surfaces to show that klt threefold singularities in characteristic $p > 5$ are rational.
- (b) The exceptional divisor in the last step of this MMP is usually referred to as a *Kollár component*. Extracting these component and using inversion of adjunction type arguments can be useful to extract information from $(n - 1)$ -log Fano varieties to n -dimension klt singularities.