Polynomial method – Exercises

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Combinatorial Nullstellensatz. Let $f \in \mathbb{F}[x_1, \dots, x_n]$ and $S_i \subseteq \mathbb{F}$ (for $1 \le i \le n$). Let $g_i(x_i) = \prod_{s \in S_i} (x_i - s)$. If $f(s_1, \dots, s_n) = 0$ for every $s_i \in S_i$ ($1 \le i \le n$), then there exist some $h_i \in \mathbb{F}[x_1, \dots, x_n]$ with $\deg h_i \le \deg f - \deg g_i$ such that $f = \sum h_i g_i$.

Nonvanishing Criterion. Let $f = c x_1^{d_1} \dots x_n^{d_n} + \dots \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial of degree $d = d_1 + \dots + d_n$ such that $c \ne 0$. Let $S_i \subseteq \mathbb{F}$ (for $1 \le i \le n$) with size $|S_i| > d_i$. Then there exist some $s_1 \in S_1, \dots, s_n \in S_n$ such that $f(s_1, \dots, s_n) \ne 0$.

1. Show that the Combinatorial Nullstellensatz implies the Nonvanishing Criterion.

2. Let n be a positive integer. Consider

$$S = \{(x, y, z) : x, y, z \in \{0, 1, \dots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

3. (Schwartz-Zippel lemma) Let $A_i \subseteq \mathbb{F}$ be finite subsets with $|A_i| = k$ for each $i = 1, \dots, n$. Let P be a non-zero polynomial over $\mathbb{F}$ on n variables and total degree at most d. Show that the number of zeros of P in $A_1 \times \dots \times A_n$ is at most dk^{n-1}.

   In particular, a nonzero polynomial of degree d vanishes on at most $d/|\mathbb{F}|$ fraction of points of $\mathbb{F}^n$.

4. Let $0 < d < |\mathbb{F}|$ and $P = \sum_{j=0}^{d} P_j$, where $P_j \in \mathbb{F}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree j and $\deg P = d$. Let $v \in \mathbb{F}^n \setminus \{0\}$ and $x \in \mathbb{F}^n$. Let $Q(t) = P(x + tv)$. Show that the coefficient of t^d in Q is $P_d(v)$.

   Show that $P_d(v) \ne 0$ for some $v \in \mathbb{F}^n$.

5. Show a polynomial $P \in \mathbb{F}_2[x_1, \dots, x_n]$ of degree d and a set $A \subseteq \mathbb{F}_2^n$ of size $|A| = \sum_{0 \le i \le d/2} \binom{n}{i}$ such that P vanishes on $A \hat{+} A$ but $P(0) \ne 0$.

6. Let us assume that a system of subsets $A(x) \subseteq \mathbb{F}_2^n$ ($x \in \mathbb{F}_2^n$) satisfies the following property:

$$\forall x \in \mathbb{F}_2^n \ (y \in x + A(x) \hat{+} A(x) \implies A(y) = \emptyset). \qquad (*)$$

   (For $A(x) = \emptyset$ we define $x + A(x) \hat{+} A(x) := \emptyset$.) Prove that the maximal possible size of $\sum_{x \in \mathbb{F}_2^n} |A(x)|$ is $r_3(\mathbb{Z}_4^n)$. (Hint: Write each element in the form $a = f + r$, where $f \in \{0, 2\}^n$ and $r \in \{0, 1\}^n$.)

7. Let us assume that for the vectors $u_1, \dots, u_n, v_1, \dots, v_n \in \mathbb{F}^r$ we have

$$\langle u_i, v_j \rangle \langle u_j, v_k \rangle \langle u_k, v_i \rangle \ne 0 \iff i = j = k.$$

   Prove that $n \le r^2$.

   * Is it true that that $n \le r^{1.5}$?

   (Here $\langle x, y \rangle = \sum x_i y_i$ is the dot product of x and y.)