Liouville quantum gravity and its spectral geometry
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Exercise sheet

**1**. (*Green function asymptotics.*)

Assuming the conformal invariance of the Green function $G_D$, and assuming given its expression in the unit disc (so that $G_{\mathbb{D}}(0,w) = -(2\pi)\log|w|$), show that if $D$ is simply connected we have

$$G_D(x,y) = -(2\pi)^{-1}\log|x-y| + \rho(x) + o(1)$$

as $y \to x$. Here $\rho(x) = \log R(x,D)$ is the logarithm of the "conformal radius" of $x$ in $D$ (which is defined as $R(x,D) = |\psi'(0)|$, where $\psi : \mathbb{D} \to D$ is a conformal isomorphism sending $\mathbb{D}$ to $D$ and 0 to $x$ – by the way, why is this conformal radius well defined?)

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**2**. (*The circle average of the GFF around a point is a BM.*)

Let $D$ be a proper, regular domain of $\mathbb{R}^2$ (so the Green function is finite). Let $h$ be a Gaussian free field on it with Dirichlet boundary conditions. Fix $z \in D$ and $\varepsilon_0 > 0$ so that $B(z,\varepsilon_0) \subset D$. For $0 < \varepsilon \le \varepsilon_0$ let $\rho_{z,\varepsilon}$ denote the uniform distribution on the circle of radius $\varepsilon$ around $z$.

(i) Check that $\rho_{z,\varepsilon} \in \mathfrak{M}$ (the space of measures which integrate the Green function).

(ii) Let $h_\varepsilon(z) = \sqrt{2\pi}(h,\rho_{z,\varepsilon})$, which is the circle average of $h$ at distance $\varepsilon$ from $z$. For $t \ge t_0 = \log(1/\varepsilon_0)$ let $B_t = h_{e^{-t}}(z)$. Check using the Markov property and conformal invariance that $(B_t)_{t\ge t_0}$ has independent and stationary increments.

(iii) Deduce that $(B_{t_0+t} - B_{t_0})_{t\ge 0}$ has the law of a multiple (say $\kappa$) of Brownian motion. Finally check that $\kappa = 1$.

(iv) Would this remain true if we were to consider the average of $h$ over a general shape scaled by $\varepsilon$ (the "potato-average process") of the field? (Formally we take a smooth contour $\gamma$ in $\mathbb{R}^2$ containing the origin, and define $\rho_{\varepsilon,z}$ to the uniform measure (with respect to arclength) on $z + \varepsilon\gamma$.) If not, do you see a way to change the definition of $\rho_{z,\varepsilon}$ so that $(h,\rho_{z,\varepsilon})$ becomes a Brownian motion when viewed as a function of $t = \log(1/\varepsilon)$?

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**3**. (*Thick points.*)

Let $\alpha > 0$. An $\alpha$-thick point of the GFF is a point $z \in D$ such that $\lim_{\varepsilon\to 0} h_\varepsilon(z)/\log(1/\varepsilon) = \alpha$. Let $\mathcal{T}_\alpha$ be the set of thick points. Show that $\mathbb{E}(\mathrm{Leb}(\mathcal{T}_\alpha)) = 0$ and thus $\mathrm{Leb}(\mathcal{T}_\alpha) = 0$, a.s. Nevertheless, $\mathcal{T}_\alpha$ is not necessarily empty. In fact, let $\mathcal{D}_\varepsilon = D \cap (\varepsilon\mathbb{Z})^2$ denote the vertices of a regular square lattice of mesh size $\varepsilon$ which lie in $D$. Let $N_\varepsilon = \sum_{z\in\mathcal{D}_\varepsilon} 1_{\{h_\varepsilon(z)\ge\alpha\log(1/\varepsilon)\}}$. Show that $\mathbb{E}(N_\varepsilon) = \varepsilon^{-(2-\alpha^2/2)+o(1)}$. Can you guess for which values of $\alpha$ is $\mathcal{T}_\alpha$ non-empty? And can you guess what the Hausdorff dimension of $\mathcal{T}_\alpha$ is?

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**4**. (*Moments of order 1 and 2 of GMC and proof of convergence in the $L^2$ regime.*)

(i) Let $D$ be a regular, proper domain. Let $\gamma > 0$. Let $M_\varepsilon(\mathrm{d}x) = \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(x)} \,\mathrm{d}x$. Show that if $A \subset D$ is a Borel set such that $A_\varepsilon = \cup_{x \in A} B(x, \varepsilon) \subset D$, then

$$\mathbb{E}(M_\varepsilon(A)) = \int_A R(x, D)^{\gamma^2/2} \,\mathrm{d}x.$$

(ii) Show that if $\gamma < \sqrt{2}$ then $\mathbb{E}(M_\varepsilon(A)^2)$ is uniformly bounded in $\varepsilon > 0$.
(iii) Try to prove that (still in the case $\gamma < \sqrt{2}$), $M_\varepsilon(A)$ is a Cauchy sequence in $L^2(\mathbb{P})$.

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**5**. (*Expected exit time for Liouville Brownian motion.*)

Let $D$ be a domain and $0 < \gamma < 2$. Let $\mathcal{M}$ denote the associated GMC measure and let $(Z_t, t \leq \tau_D)$ denote the associated Liouville Brownian motion, starting from some point $z \in D$ run until the time $\tau_D$ where it leaves $D$. We call $E_z$ the expectation over the Liouville Brownian motion trajectory, conditional on the realisation of the GFF.

Show that

$$E_z(\tau_D) = \int_D G_D(z, x) \mathcal{M}(\mathrm{d}x).$$

Deduce that $\mathbb{E}(\mathbb{E}_z(\tau_D)) = \int_D G_D(z, x) R(x, D)^{\gamma^2/2} \,\mathrm{d}x < \infty$.

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**6**. (*Non-atomicity of GMC and spectral theorem for the Green function of Liouville BM.*)

Let $D$ be a domain and $z \in D$ such that $B(z, 1) \subset D$. Let $h$ be a GFF on $D$. Let $\mathcal{M}_\varepsilon(\mathrm{d}x) = \varepsilon^2 \gamma^2/2 e^{\gamma h_\varepsilon(x)} \,\mathrm{d}x$. We admit the following formula for the mlutifractal spectrum of $\mathcal{M}$ (see Exercise 7 below for a sketch of proof):

$$\mathbb{E}[\mathcal{M}(B'_r)^q] \asymp r^{\xi(q)}; \quad \xi(q) = (2 + \frac{\gamma^2}{2})q - \frac{q^2\gamma^2}{2},$$

for all $0 < q < 4/\gamma^2$.

(i) Show that for $0 < \gamma < 2$ the measure $\mathcal{M}$ is atomless. In fact, show that there exists $C > 0$ (random but finite) and a fixed nonrandom $\eta > 0$ such that, almost surely, if $\gamma < 2$,

$$\sup_{z \in D'} \mathcal{M}(B(z, r)) \leq C r^\eta.$$

(ii) Show that $\iint_{D^2} \log^2(|x - y|) \mathcal{M}(\mathrm{d}x) \mathcal{M}(\mathrm{d}y) < \infty$. Deduce that the Green operator

$$f \in L^2(\mathcal{M}) \mapsto \mathbf{G}f(x) = \int_D G_D(x, y) \mathcal{M}(\mathrm{d}y)$$

is Hilbert–Schmidt and thus has a sequence of eigenvalues $\mu_n \to 0$.

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Probably the next exercise is too long to do in class, but I encourage you to think about it.

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**7**. The goal here is to compute the multifractal spectrum of GMC (admitting one concentration result). Let $D$ be a domain and $z \in D$ such that $B(z,1) \subset D$, and $h$ a GFF on $D$. For $0 < r < 1$ let $B_r = B(z, 2r)$ and write

$$h = h^r + u^r,$$

where $u^r$ is harmonic in $B_r$ and $h^r$ is an independent Dirichlet GFF in $B_r$. Let $B'_r = B(z,r)$. Let $\mathcal{M}_\varepsilon(\mathrm{d}x) = \varepsilon^2\gamma^2/2e^{\gamma h_\varepsilon(x)}\,\mathrm{d}x$. We will admit that for $q < 4/\gamma^2$, $\mathbb{E}[\mathcal{M}_\varepsilon(B'_1)^q] \le C = C(D,q)$ is uniformly bounded as a function of $\varepsilon$. We will also admit that $X_r = \sup_{x\in B'_r} u_r(x)$ is approximately a centered Gaussian random variable, with variance $\log(1/r)$, in the sense that for each $\lambda > 0$

$$\mathbb{E}[e^{\lambda X_r}] \asymp e^{\lambda^2 \log(1/r)/2}$$

where the implicit constants may depend on $\lambda$ but are otherwise uniform on $r > 0$.

(i) Show that

$$\mathcal{M}_\varepsilon(B'_r) \le e^{\gamma X_r} r^{2+\frac{\gamma^2}{2}} \tilde{\mathcal{M}}_{\varepsilon/r}(B'_1),$$

where $\tilde{\mathcal{M}}_\varepsilon$ is independent from $X_r$ and is the approximation of a certain GMC measure on $B_1$.

(ii) Deduce that

$$\mathbb{E}[\mathcal{M}(B'_r)^q] \asymp r^{\xi(q)}; \quad \xi(q) = (2 + \frac{\gamma^2}{2})q - \frac{q^2\gamma^2}{2}.$$