Dynamics on Sparse Networks

Kavita Ramanan, Brown University

Summer School on Mathematics of Large Networks,
Erdös Center, Rényi Institute, Budapest, Hungary
May 30-Jun 3, 2022

Outline

- Lecture I:
  Motivating Examples, The Math Model and Mean-Field Approximations

- Lecture II:
  Aggregate Dynamical Behavior on Large Sparse Graphs: Hydrodynamic Limits

- Lecture III:
  Marginal Dynamics on Large Sparse Graphs

Two complementary sessions by
G. Cocomello

Lecture I.

Motivating Examples, the Math Model, Mean-Field Approximations and Beyond

A. Motivating Examples

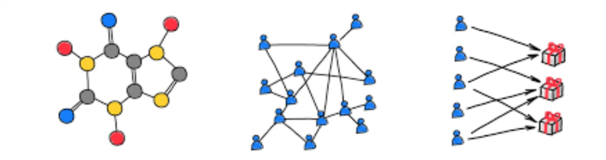
Interacting Stochastic Processes

- Study of large collections of randomly evolving interacting "particles", whose dynamic interactions are governed by an underlying network (graph), that is itself possibly random

Notation: $G = (V, E)$, $N_v = \{u \in V : uv \in E\}$, $d_v = |N_v|$

Interacting Stochastic Processes

- Study of large collections of randomly evolving interacting "particles", whose dynamic interactions are governed by an underlying network (graph), that is itself possibly random



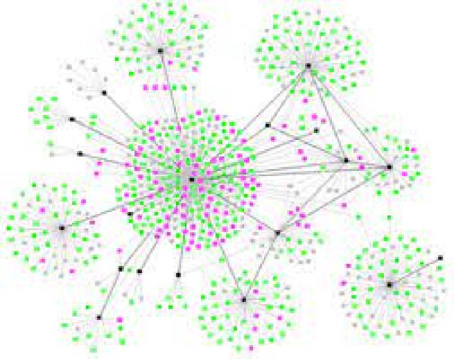
Notation: $G = (V, E)$, $N_v = \{u \in V : uv \in E\}$, $d_v = |N_v|$

- The stochastic evolution of each particle is directly influenced only by the states (or histories of states) of neighboring particles in the graph
- such stochastic dynamics model phenomena in a plethora of applications

A. Some Motivating Examples

1. Epidemiology

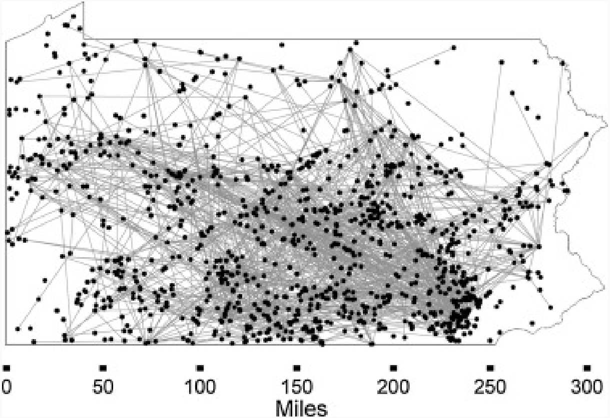
- Spread of human diseases
  Graph: human social contact networks
  (determined using location tracking technology and contact tracing)



Mason et al, (Int J. Heathcare Tech. & Management, Vol 11, No, 6, 2010)

1. Epidemiology

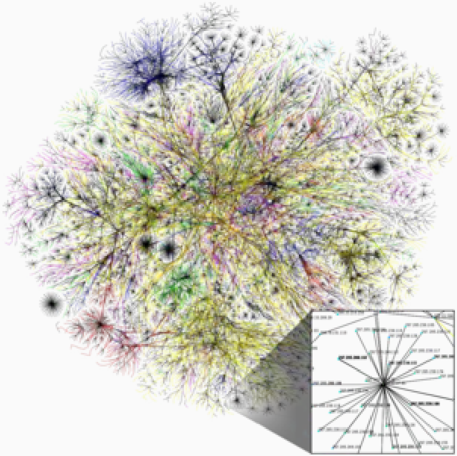
- Spread of animal diseases (Chronic Wasting Disease in 1090 farms in PA over 7.68 year period)
  Graph: Animal transport network
  (determined from data collected on shipment of farm animals)



Rorres et al, Epidemics, 23:71-75, 2018

1. Epidemiology

- Spread of viruses in computer networks
  Graph: portion of the internet
  (determined, e.g., from Bell Labs Internet Mapping or Opte Projects)



Wikipedia: routing map in a portion of the interent from Opte Project

Idealized Models of Infection Spread

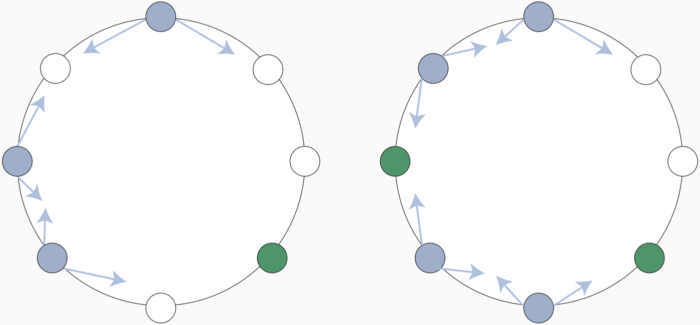
- Particles (individuals/animals/computers) are represented as nodes on a locally finite (possibly directed) graph $G = (V, E)$ that captures their "interaction structure";
- SIR model: Each vertex takes three states $\{S, I, R\}$ where
  - S (healthy but) susceptible; (white)
  - I infected; (grey)
  - R recovered (and immune); (green)

Each particle's evolution depends only on its own state and those of neighbors in the graph:

$$S \quad \xrightarrow{p \times (\text{no. infected neighbors})} \quad I \quad \xrightarrow{1} \quad R$$

SIR model

$$S \xrightarrow{p\times(\text{no. infected neighbors})} I \xrightarrow{q} R$$



SIS Model or Contact Process

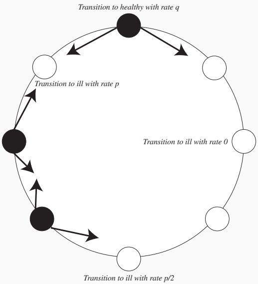
SIS model or Contact Process:

- Each particle is in one of two states.
  Represent the state space by $\{S, I\}$, where
  - S represents susceptible;
  - I represents infected;
- $X_v(t)$ is the state of particle v at time t;
- Each susceptible particle is infected at a rate proportional to the number of infected particles in their neighborhood in the graph, and each infected particle recovers at a unit rate;

$$S \xrightarrow{p \times (\text{no. infected neighbors})} I \xrightarrow{1} S$$

Discrete-Time Contact Process (with parameter p)

$$S \xrightarrow{p \times (\text{no. infected neighbors})} I \xrightarrow{1} S$$



Transition rule F: At time t for each $v \in V$,

- state $X_v(t) = I$, it switches to $X_v(t+1) = S$ w.p. 1,
- state $X_v(t) = S$, it switches to $X_v(t+1) = I$ w.p.

$$p \sum_{u \sim v} X_u(t),$$

Some Questions of Interest

- How many times will a particular individual get (re)-infected?
- What is the fraction of individuals in the population infected at any time?
- How long does it take an epidemic to die out? How does this depend on the graph structure?

Probabilistic Cellular Automata

The SIR and SIS processes are examples of synchronous Markov chains or probabilistic cellular automata

- Consider a collection of particles, indexed by the vertices of a graph $G = (V, E)$; V is the vertex set and E is the set of edges of the graph.

- For $v \in V$ and $t \in \{0, 1, \ldots\}$, let $X_v(t)$ denote the state of the particle v at time t, which takes values in a (measure) state space S.

- For $v \in V$, the evolution of X_v is as follows: for $t \in \{0, 1, \ldots, \}$,

$$\mathrm{X}_v(t+1) = F\left(\mathrm{X}_v(t), (\mathrm{X}_u(t))_{u \sim v}, \xi_v(t+1)\right),$$

where

◦ $\xi_v(t)$, $v \in V$, $t \in \{0, 1, \ldots, \}$, are independent, identically distributed random variables/noises, and

◦ F provides the transition rule at each time

◦ F depends symmetrically on the variables in the second argument

2. Opinion Dynamics

A Simple Idealized Model of Opinion Dynamics

Voter Model

- State of each particle lies in $S = \{0, 1\}$ representing two different opinions
- The set of transitions each particle can make lies in $\mathcal{J} = \{-1, 1\}$
- Each particle, after an independent exponential time, polls one of its neighbors at random and adopts that opinion

A Simple Model of Opinion Dynamics

Voter Model Dynamics

- can be described by a continuous-time Markov chain on S^V where the rate of transitions or jumps in the direction j of particle v is

$$r_j^{G,v}(x_{\bar{v}}) = \begin{cases} \mathbb{I}_{\{x_v=0\}} \frac{1}{d_v} \sum_{u \in N_v} x_u & \text{if } j = 1, \\ \mathbb{I}_{\{x_v=1\}} \left(1 - \frac{1}{d_v} \sum_{u \in N_v} x_u\right) & \text{if } j = -1, \end{cases}$$

- given a particle configuration $x \in S^V$, for $A \subset V$, denote $x_A = (x_i)_{i \in A}$
- $\bar{v} = \{v\} \cup N_v$ denotes the closure of v
- d_v is the degree of vertex v.
- Once again the jump rates of the state $X_v(t)$ of v at time t depend only on its own state and the states of the neighbors $X_{\bar{v}}(t) = (X_u(t))_{u=v, u \sim v}$

A Simple Model of Opinion Dynamics

Voter Model Dynamics

- can be described by a continuous-time Markov chain on S^V where the rate of transitions or jumps in the direction j of particle v is

$$r_j^{G,v}(x_{\overline{v}}) = \begin{cases} \mathbb{I}_{\{x_v=0\}} \frac{1}{d_v} \sum_{u \in N_v} x_u & \text{if } j = 1, \\ \mathbb{I}_{\{x_v=1\}} \left(1 - \frac{1}{d_v} \sum_{u \in N_v} x_u\right) & \text{if } j = -1, \end{cases}$$

- given a particle configuration $x \in S^V$, for $A \subset V$, denote $x_A = (x_i)_{i \in A}$
- $\bar{v} = \{v\} \cup N_v$ denotes the closure of v
- d_v is the degree of vertex v.
- Once again the jump rates of the state $X_v(t)$ of v at time t depend only on its own state and the states of the neighbors $X_{\bar{v}}(t) = (X_u(t))_{u=v,u\sim v}$
- There are numerous variants of the voter model – majority dynamics, noisy voter model, etc.

3. Load Balancing

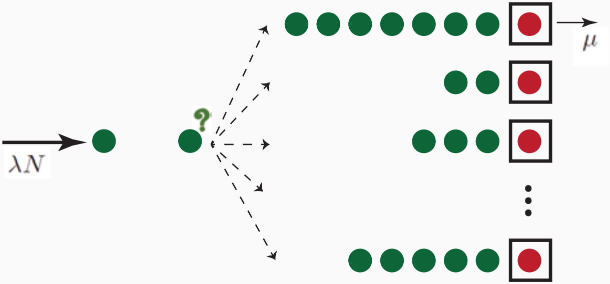
Appears in:

- Supermarkets
- Hash tables
- Distributed memory machines
- Path selection in networks
- Web Servers
- etc.

N servers; processing time distribution has unit mean ($\mu = 1$);

$N\lambda$ arrival rate;

$\lambda < 1$



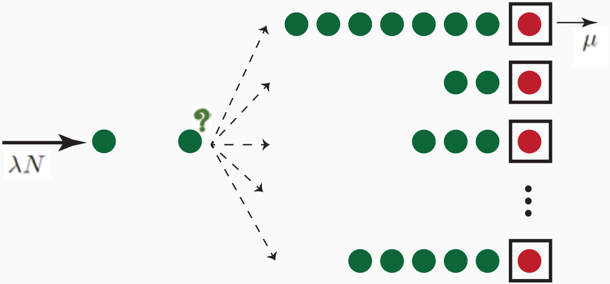
Tradeoff between performance and communication/computation

3. Load Balancing

N servers; processing time distribution has unit mean ($\mu = 1$);

$N\lambda$ arrival rate;

$\lambda < 1$



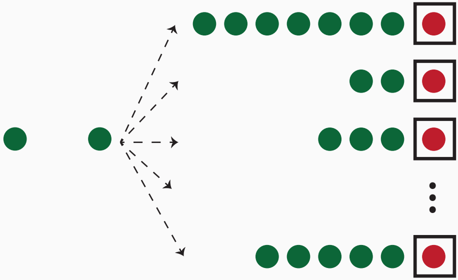
Tradeoff between performance and communication/computation

Join-the-Shortest-Queue (JSQ):

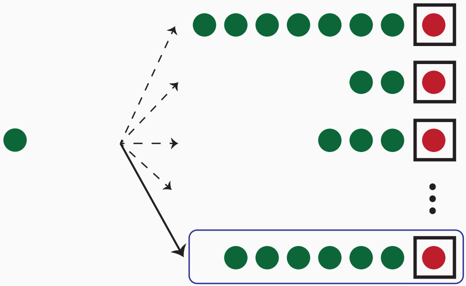
great performance $\mathbb{P}(Q > 1) \approx 0$, but in some contexts infeasible to implement

Load Balancing Algorithms

Random Routing



Random Routing

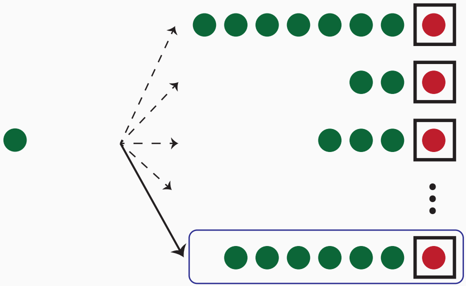


In equilibrium

$$P(Q > \ell) = \lambda^{\ell}$$

significantly worse than JSQ

Random Routing

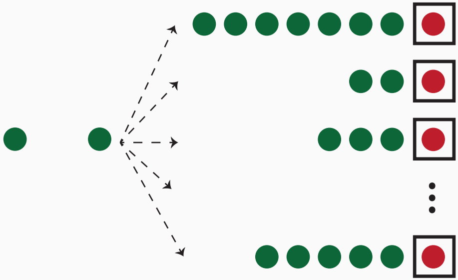


In equilibrium

$$P(Q > \ell) = \lambda^{\ell}$$

significantly worse than JSQ

Randomized Load Balancing: SQ(2) Algorithm



Q1. What is the probability of a typical queue exceeding a level?

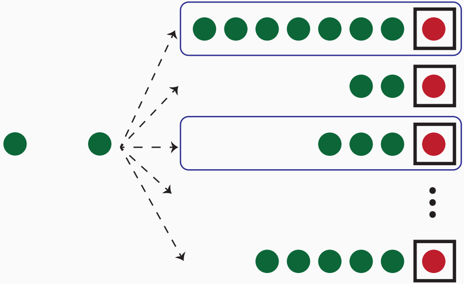
Q2. How long will an overloaded queue take to clear its backlog?

Q3. How does performance depend on the service distribution?

Q4. What if re-routing is constrained to only be to neighbors?

Load Balancing Algorithms

Randomized Load Balancing: SQ(2) Algorithm



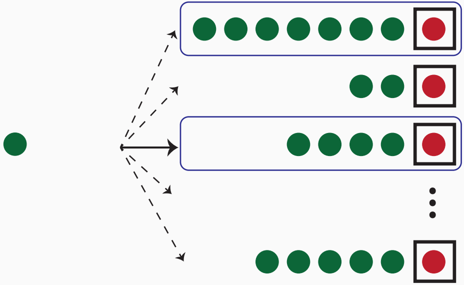
Q1. What is the probability of a typical queue exceeding a level?

Q2. How long will an overloaded queue take to clear its backlog?

Q3. How does performance depend on the service distribution?

Q4. What if re-routing is constrained to only be to neighbors?

Randomized Load Balancing: SQ(2) Algorithm



Q1. What is the probability of a typical queue exceeding a level?

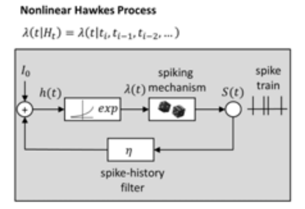
Q2. How long will an overloaded queue take to clear its backlog?

Q3. How does performance depend on the service distribution?

Q4. What if re-routing is constrained to only be to neighbors?

4. Neuronal Hawkes Model

- the neuronal Hawkes model models the firing of neurons (spike trains)
- $X(t)$ represents the number of firings of the neuron in the time $[0, t]$
- the time of the next firing of X is determined by conditional jump rate $\lambda(t)$ of X given the history of the spike train prior to t
- $\lambda(t)$ is equal to a suitable non-linear function h of the sum of a constant input I_0 and the convolution of the history of the spike train with a filter η.



- non-Markovian model: jump rates depend on the history of the process
- network models – often on a *directed* graph

1. Fix a directed graph $G = (V, E)$, and write $u \to v$ if $(u, v) \in E$
2. $X_v(t)$ represents the number of firings of vertex v up to time t
3. Given independent Poisson processes N_v on $\mathbb{R}_+^2$, $v \in V$, the evolution has the form

$$\mathrm{X}_v(t) = \int_{(0,t]\times\mathbb{R}_+} \mathbb{I}_{\{r \le r_v^G(s,\mathrm{X})\}} \, N_v(ds, dr),$$

$$r_v^G(s, f) = h\left(I_v + \sum_{u=v \text{ or } u \to v} \eta_{u\to v} * f(s-) \right),$$

where $*$ represents convolution so that $\eta_{u \to v} * \mathrm{X}$ depends on the history of the processes X_v and X_u, $u \to v$, in the interval $[0, s)$

B. The Math Model

The Basic Framework

Given a finite connected possibly random graph $G = (V, E)$, we are interested in a stochastic process

$$\{X_v(t)\}_{v \in G, t \in \mathbb{T}},$$

with $\mathbb{T} = \mathbb{N}_0$ (discrete time) or $\mathbb{T} = [0, \infty)$ (continuous time) whose dynamics is such that ...

- the instantaneous stochastic evolution of the state $X_v(t)$ of each node $v \in V$ at time t has a local dependence on the states of other particles, that is, it
  depends only on its own state and those of its neighbors in G

The Basic Framework

Given a finite connected possibly random graph $G = (V, E)$, we are interested in a stochastic process

$$\{X_v(t)\}_{v \in G, t \in \mathbb{T}},$$

with $\mathbb{T} = \mathbb{N}_0$ (discrete time) or $\mathbb{T} = [0, \infty)$ (continuous time) whose dynamics is such that ...

- the instantaneous stochastic evolution of the state $X_v(t)$ of each node $v \in V$ at time t has a local dependence on the states of other particles, that is, it
  depends only on its own state and those of its neighbors in G
- the dependence is symmetric in the neighbors (though can allow random inhomogeneities)

The Basic Framework

Given a finite connected possibly random graph $G = (V, E)$, we are interested in a stochastic process

$$\{X_v(t)\}_{v \in G, t \in \mathbb{T}},$$

with $\mathbb{T} = \mathbb{N}_0$ (discrete time) or $\mathbb{T} = [0, \infty)$ (continuous time) whose dynamics is such that ...

- the instantaneous stochastic evolution of the state $X_v(t)$ of each node $v \in V$ at time t has a local dependence on the states of other particles, that is, it
  depends only on its own state and those of its neighbors in G
- the dependence is symmetric in the neighbors (though can allow random inhomogeneities)
- the evolution itself could be described by any of the following:
  - a discrete-time process – e.g., probabilistic cellular automata
  - a continuous-time jump process
  - a (continuous-time) diffusion

Our General Focus

1. Interested in large systems
2. They suffer from the curse of dimensionality: hard to analyze exactly or even to simulate to obtain qualitative behavior

Our General Focus

1. Interested in large systems
2. They suffer from the curse of dimensionality: hard to analyze exactly or even to simulate to obtain qualitative behavior
3. Goal: To obtain mathematically rigorous statements about dynamics – specifically want to establish asymptotic limit theorems as the graph size goes to infinity
   leading to principled approximations for dynamics on large graphs

Our General Focus

1. Interested in large systems
2. They suffer from the curse of dimensionality: hard to analyze exactly or even to simulate to obtain qualitative behavior
3. Goal: To obtain mathematically rigorous statements about dynamics – specifically want to establish asymptotic limit theorems as the graph size goes to infinity
   leading to principled approximations for dynamics on large graphs
4. Note: There are many underlying graph models that fit the data – I will not provide details into the panoply of random graph models
   Remco van der Hofstad's lectures

Specific Questions

Given (G, V) and dynamics of the interacting processes, e.g., F in

$$X_v^{G,x}(t+1) = F\left(X_v(t), (X_{N_v}(t)), \xi_v(t+1)\right),$$

where the dependence on $(X_{N_v}(t))$ is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}(t)}$$

Specific Questions

Given (G, V) and dynamics of the interacting processes, e.g., F in

$$\mathsf{X}_v^{G,x}(t+1) = F\left(\mathsf{X}_v(t), (\mathsf{X}_{N_v}(t)), \xi_v(t+1)\right),$$

where the dependence on $(X_{N_v}(t))$ is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|}\sum_{v\in G}\delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|}\sum_{v\in G}\delta_{X_v^{G,x}(t)}$$

Key questions: Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, and appropriate initial conditions $x^n \in \mathcal{X}^{V_n}$,

Q1. Do the processes X^{G_n, x^n} converge in a suitable sense?

Specific Questions

Given (G, V) and dynamics of the interacting processes, e.g., F in

$$\mathsf{X}_{\mathsf{v}}^{\mathsf{G},\mathsf{x}}(\mathsf{t}+1) = \mathsf{F}\left(\mathsf{X}_{\mathsf{v}}(\mathsf{t}), (\mathsf{X}_{\mathsf{N}_{\mathbf{v}}}(\mathsf{t})), \xi_{\mathsf{v}}(\mathsf{t}+1)\right),$$

where the dependence on $(X_{N_v}(t))$ is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}(t)}$$

Key questions: Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, and appropriate initial conditions $x^n \in \mathcal{X}^{V_n}$,

Q1. Do the processes $\mathsf{X}^{\mathsf{G}_{\mathbf{n}},\mathsf{x}^{\mathbf{n}}}$ converge in a suitable sense?

Q2. Do the global empirical measures μ^{G_n,x^n} converge (hydrodynamic limits) ?

Specific Questions

Given (G, V) and dynamics of the interacting processes, e.g., F in

$$\mathrm{X}_{\mathrm{v}}^{\mathrm{G},\mathrm{x}}(\mathrm{t}+1) = \mathrm{F}\left(\mathrm{X}_{\mathrm{v}}(\mathrm{t}), (\mathrm{X}_{\mathrm{N}_{\mathbf{v}}}(\mathrm{t})), \xi_{\mathrm{v}}(\mathrm{t}+1)\right),$$

where the dependence on $(X_{N_v}(t))$ is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}(t)}$$

Key questions: Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, and appropriate initial conditions $x^n \in \mathcal{X}^{V_n}$,

Q1. Do the processes $\mathrm{X}^{\mathrm{G}_{\mathbf{n}},\mathrm{x}^{\mathbf{n}}}$ converge in a suitable sense?

Q2. Do the global empirical measures μ^{G_n,x^n} converge (hydrodynamic limits) ?

Q3. can one autonomously characterize the limiting dynamics of a fixed or "typical particle" $\mathrm{X}_{\mathrm{v}}^{\mathrm{G}_{\mathbf{n}},\mathrm{x}_{\mathbf{n}}}(\mathrm{t})$, $t \in [0, T]$?

C. Mean-Field Limits

Well-studied case: $G = K_n$, the complete graph on n-vertices. Set $X^{n,x} = X^{G_n,x}$.

Discrete-time Evolution:

$$X_v^{n,x}(t+1) = F\left(X_v^n(t), (X_{N_v}^n(t)), \xi_v(t+1)\right),$$

with $X^{n,x}(0) = x$, and $\{\xi_v(t)\}_{v \in V, t \in \mathbb{N}}$ iid noises.

Assume F depends on $X_{N_v}^n$ symmetrically, and can be written as a nice function of "average quantities":

$$\mu^n(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_v^n(t)} \qquad m_g^n(t) = \frac{1}{n} \sum_{i=1}^{n} g(X_v^n(t))$$

Mean-field limits

$$X_v^{n,x}(t+1) = F\left(X_v^n(t), (X_{N_v}^n(t)), \xi_v(t+1)\right),$$

$$\mu^n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_v^n(t)} \qquad m_g^n(t) = \frac{1}{n}\sum_{i=1}^{n} g(X_v^n(t))$$

Mean-field limits

$$X_v^{n,x}(t+1) = F\left(X_v^n(t), (X_{N_v}^n(t)), \xi_v(t+1)\right),$$

$$\mu^n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_v^n(t)} \qquad m_g^n(t) = \frac{1}{n}\sum_{i=1}^{n} g(X_v^n(t))$$

Under our simplifying assumption, we can rewrite the dynamics as

$$X_v^{n,x}(t+1) = \bar{F}\left(X_v^n(t), (\mu^n(t)), \xi_v(t+1)\right)$$

$$X_v^{n,x}(t+1) = \bar{F}\left(X_v^n(t), (\mu^n(t)), \xi_v(t+1)\right)$$

Classical Theorem

Under mild continuity conditions on F, (equivalently, $\bar{F}$), one can show that X_1^n converges weakly to the process $\bar{X}$, which evolves for $t \in \mathbb{N}$ like

$$\bar{X}(t+1) = \bar{F}(\bar{X}(t), \bar{\mu}(t), \xi(t+1)), \quad \bar{\mu}(t) = \mathrm{Law}(X(t)),$$

where $\{\xi(t)\}_{t\in\mathbb{N}}$ is an iid sequence with the same distribution as $\xi_v(t)$. Moreover, $\bar{\mu}(t) = \mathrm{Law}(\bar{X}(t))$ is the (weak) limit of $\mu^n(t)$

Mean-Field Limits

$$X_v^{n,x}(t+1) = \bar{F}\left(X_v^n(t), \mu^n(t), \xi_v(t+1)\right)$$

$$\mu^n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_v^n(t)} \qquad m_g^n(t) = \frac{1}{n}\sum_{i=1}^{n} g(X_v^n(t))$$

Classical Theorem (abridged)

X_1^n converges weakly to the process $\bar{X}$, which evolves for $t \in \mathbb{N}$ like

$$\bar{X}(t+1) = \bar{F}(\bar{X}(t), \bar{\mu}(t), \xi(t+1)), \quad \bar{\mu}(t) = \mathrm{Law}(X(t)),$$

Moreover, $\bar{\mu}(t) = \mathrm{Law}(\bar{X}(t))$ is the (weak) limit of $\mu^n(t)$

- The trivial setting: Suppose $\bar{F}$ does not depend on $\mu^n(t)$ (respy, $m_g^n(t)$). Then you have independent particles, whose law is the same with every n, and by the strong law of large numbers $m_g^n(t)$ converges (as $n \to \infty$) to $\mathbb{E}[g(X_v^1(t))]$ for any chosen v.
- Intuition for the general case: Due to nature of interactions, you still have decay of correlations and particles are weakly interacting, and μ^n converges to a deterministic limit. Then invoke continuity of $\bar{F}$.

Mean-Field Limits

- The phenomenon of asymptotic independence is also referred to as propagation of chaos
- In the continuous-time setting, where you have a collection of particles $\{X_v^n\}_{v \in K_n}$ evolving according to a continuous-time Markov chain, where each vertex X_v^n evolves "like" a Markov chain with the jump rate from state i to j being $\Gamma_{ij}(\mu^n)$ (for example, it could depending on the mean of some function of the particles: $\Gamma_{ij}(m_g^n)$.
- In this case the typical particle evolves according to a time-inhomogeneous Markov chain, where the time-inhomogeneity arises due to dependence on the law.
- Thus, in the continuous time setting you get an ODE rather than a nonlinear recursion - if the state space is discrete with k values, then each $\Gamma(p))$ is a $k \times k$ matrix and $\bar{\mu}$ evolves according to the nonlinear equation

$$\frac{d\bar{\mu}}{dt}(t) = \bar{\mu}(t)\Gamma(\bar{\mu}(t)),$$

  which describes the law of the the typical particle (Kolmogorov forward equation or Fokker Planck equation)

Efficacy of Mean-Field Limits

Mean-field Models work well on complete graphs

- SIR model – Kermack-McKendrick model
- Opinion dynamics – Curie-Weiss model
- Load Balancing networks – used to analyze the power of two choices

  Dobrushin-Vvedenskalya-Karpelevich, Mitzenmacher, Budhiraja et al, Borst et al

- Neuronal Models
  T. Austin, E. Löcherabch, J. Touboul, ...

What about dense graph sequences?

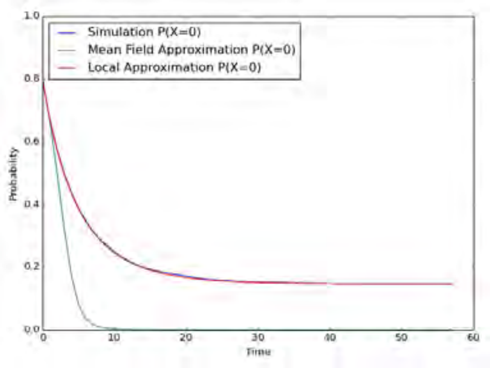
- If you have a sequence of dense graphs $G_n, n \in \mathbb{N}$, in the sense that the degrees of all vertices diverge (with additional conditions imposed), then one can in many cases still show correlation decay and argue similarly to same intuition of asymptotic independence
- The proofs are more complicated.
- Caveat. Dense regime more subtle – can fail without additional conditions.

However ...

- Many real-world networks are not complete (or dense)
- In fact, many are sparse or heterogeneous – (see also R.van der Hofstad lectures)

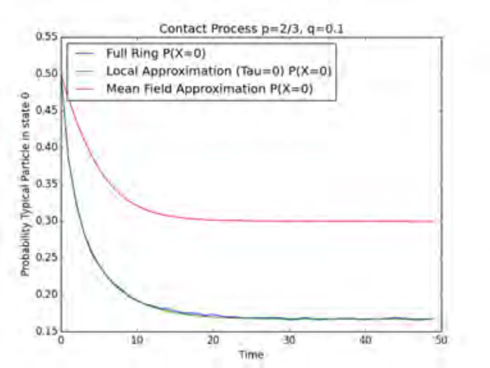
Do Mean-Field Approximations Work Well on any Graph?

Discrete-time SIR Process



Plot of probability of being healthy vs. time

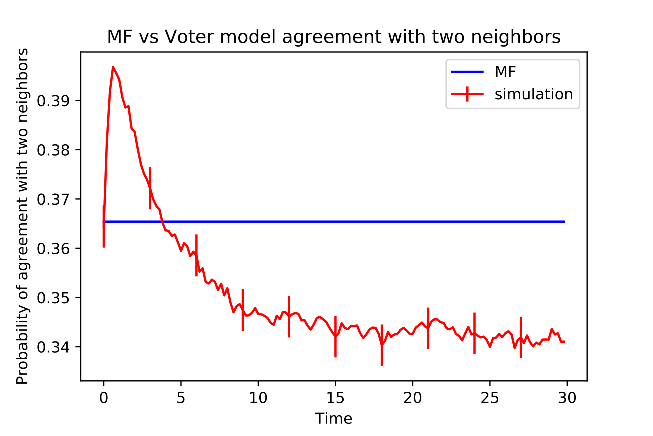
Discrete-time Contact Process



Plot of probability of being healthy vs. time

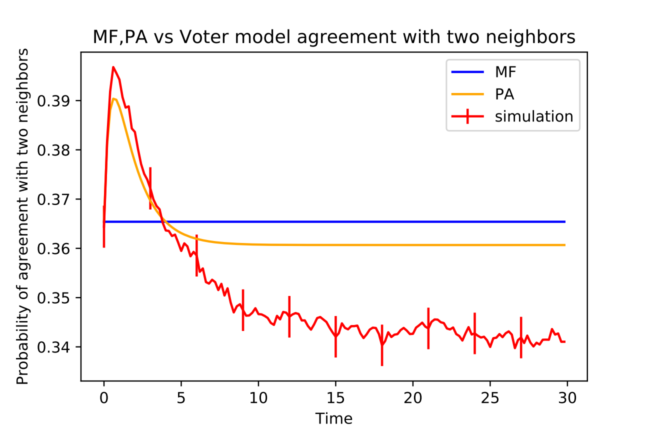
Do mean-field (MF) approximations work well on any graph?

Continuous-time voter model on $G = \mathbb{T}_3$, a rooted 3-regular tree ...



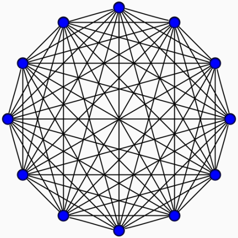
Do common refinements of mean-field (MF) approximations work better?
For example, consider pair approximations (PA)

Voter model on $G = \mathbb{T}_3$, a rooted 3-regular tree

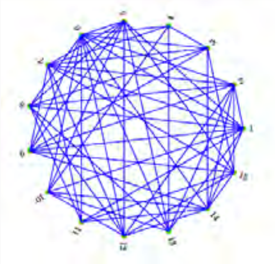


Lecture II:

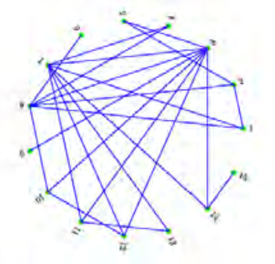
Aggregate Dynamical Behavior on Large Sparse Graphs: Hydrodynamic Limits



(a) Complete graph



(b) Dense graph



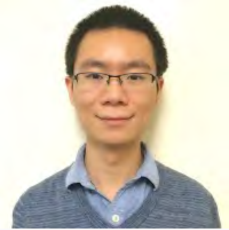
(c) Sparse Graph

OUR FOCUS
(SPARSE GRAPH SEQUENCES)

To characterize asymptotic limits of typical node dynamics for sequences of (possibly random) graphs G_n whose maximal (average) degree is bounded.

Includes Discussion of Joint Work with ...

Daniel Lacker
Columbia University



Ruoyu Wu
Iowa State University

Ankan Ganguly
Brown University

Yin-Ting Liao
Brown University

Brown University Undergraduate Student Projects

Mitchell Wortsman
University of Washington

Timothy-Sudijono
Stanford University

Mira Gordin
Princeton

What is a reasonable asymptotic regime to consider?

- First step: for what sequences G_n can we expect to obtain limit theorems?
- Can't just take the number of vertices to infinity – since graph topology matters!

What is a reasonable asymptotic regime to consider?

- First step: for what sequences G_n can we expect to obtain limit theorems?
- Can't just take the number of vertices to infinity – since graph topology matters!
- Instead, consider graphs G_n that converge in the local topology

A. Notion of local convergence

Local weak convergence of graphs

Idea: Encode sparsity via local weak convergence of graphs.
(a.k.a. Benjamini-Schramm convergence; also see Aldous-Steele '03, Bordenave '16, van der Hofstad '22+)

Local weak convergence of graphs

Idea: Encode sparsity via local weak convergence of graphs. (a.k.a. Benjamini-Schramm convergence; also see Aldous-Steele '03, Bordenave '16, van der Hofstad '22+)

Definition: A graph $G = (V, E, \rho)$ is assumed to be rooted, finite or countable, locally finite, and connected.

Local weak convergence of graphs

Idea: Encode sparsity via local weak convergence of graphs.
(a.k.a. Benjamini-Schramm convergence; also see Aldous-Steele '03, Bordenave '16, van der Hofstad '22+)

Definition: A graph $G = (V, E, \rho)$ is assumed to be rooted, finite or countable, locally finite, and connected.

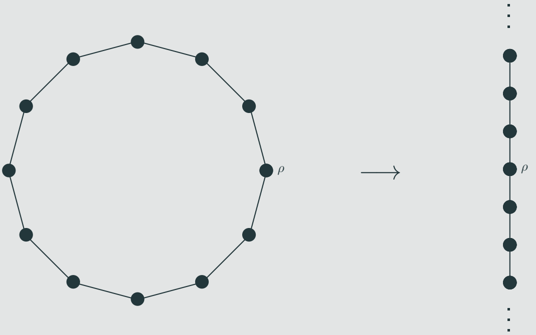
Definition: Rooted graphs G_n converge locally to G if:

$$\forall k \; \exists N \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq N,$$

where $B_k(\cdot)$ is ball of radius k at root, and $\cong$ means isomorphism.

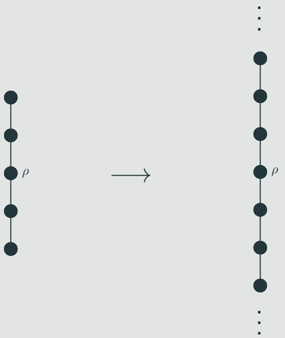
Examples of local weak convergence

1. Cycle graph converges to infinite line



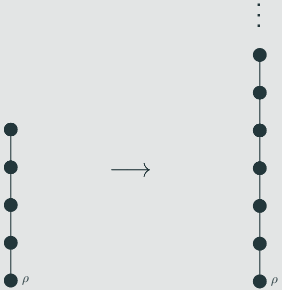
Examples of local weak convergence

2. Line graph converges to infinite line



Examples of local weak convergence

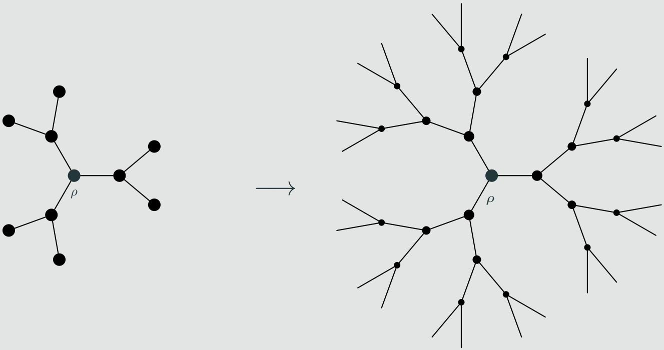
3. Line graph rooted at end converges to semi-infinite line



Examples of local weak convergence

4. Finite to infinite d-regular trees

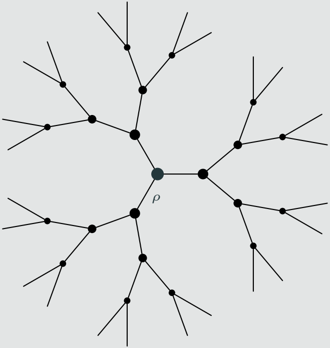
(A graph is d-regular if ever vertex has degree d.)



Examples of local weak convergence

5. Uniformly random regular graph to infinite regular tree

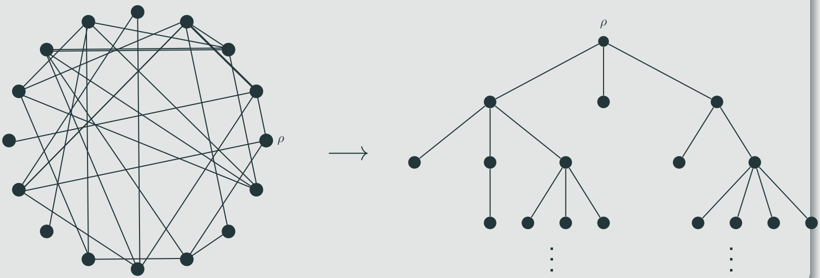
Fix d. Among all d-regular graphs on n vertices, select one uniformly at random. Place the root at a (uniformly) random vertex. When $n \to \infty$, this converges (in law) to the infinite d-regular tree. (McKay '81) $\longrightarrow$



Examples of local weak convergence

6. Erdős-Rényi to Galton-Watson

If $G_n = G(n, p_n)$ with $np_n \to p \in (0, \infty)$, then G_n converges in law to the Galton-Watson tree with offspring distribution $\mathrm{Poisson}(p)$.



7. Preferential Attachment Graphs to a Random Tree

A result by Berger-Borgs-Chayes-Saberi ('14) shows convergence of preferential attachment graphs to a random tree

Key Point

Local limits of many classes of random graphs are often trees

Examples of Local weak convergence

7. Preferential Attachment Graphs to a Random Tree

A result by Berger-Borgs-Chayes-Saberi ('14) shows convergence of preferential attachment graphs to a random tree

Key Point

Local limits of many classes of random graphs are often trees

8. Convergence of Finite Lattices

$\mathbb{Z}^\kappa \cap [-n, n]^\kappa$ converges to $\mathbb{Z}^\kappa$

Local convergence of marked graphs

Recall: $G_n = (V_n, E_n, \rho_n)$ converges locally to $G = (V, E, \rho)$ if

$$\forall k \ \exists N \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq N.$$

Local convergence of marked graphs

Recall: $G_n = (V_n, E_n, \rho_n)$ converges locally to $G = (V, E, \rho)$ if

$$\forall k \; \exists N \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq N.$$

Definition: With G_n, G as above: Given a metric space (E, d_E) and a sequence $\boldsymbol{x}^n = (x_v^n)_{v \in G_n} \in E^{G_n}$, say that $(G_n, \boldsymbol{x}^n)$ converges locally to $(G, \boldsymbol{x})$ if

$$\forall k, \epsilon > 0 \; \exists N \text{ s.t. } \forall n \geq N \; \exists \varphi : B_k(G_n) \to B_k(G) \text{ isomorphism}$$
$$\text{s.t. } \max_{v \in B_k(G_n)} d_E(x_v^n, x_{\varphi(v)}) < \epsilon.$$

Lemma

The set $\mathcal{G}_*[E]$ of (isomorphism classes of) $(G, \boldsymbol{x})$ admits a Polish topology compatible with the above convergence.

B. Results on convergence

Results on Convergence

$$X_v^{G_n,x^n}(t+1) = \bar{F}\left(X_v^{G_n,x^n}(t), \mu_v^{G_n,x^n}(t), \xi_v(t+1)\right)$$

where $\mu_v^{G_n}$ is the local empirical measure of the neighborhood of v:

$$\mu_v^{G_n}(t) = \frac{1}{n}\sum_{u\in N_v}\delta_{X_u^{G_n,x^n}(t)} \qquad m_v^{G_n}(t) = \frac{1}{n}\sum_{u\in N_v} g(X_u^{G_n,x^n}(t)).$$

Results on Convergence

$$\mathsf{X}_{\mathsf{v}}^{\mathsf{G_n},\mathsf{x^n}}(\mathsf{t}+1) = \overline{\mathsf{F}}\left(\mathsf{X}_{\mathsf{v}}^{\mathsf{G_n},\mathsf{x^n}}(\mathsf{t}), \mu_{\mathsf{v}}^{\mathsf{G_n},\mathsf{x^n}}(\mathsf{t}), \xi_{\mathsf{v}}(\mathsf{t}+1)\right)$$

where $\mu_v^{G_n}$ is the local empirical measure of the neighborhood of v:

$$\mu_v^{G_n}(t) = \frac{1}{n}\sum_{u \in N_v} \delta_{X_u^{G_n,x^n}(t)} \qquad m_v^{G_n}(t) = \frac{1}{n}\sum_{u \in N_v} g(X_u^{G_n,x^n}(t)).$$

Key questions: Given $G_n = (V_n, E_n)$, $x^n \in \mathcal{X}^{V_n}$, with $|V_n| \to \infty$.

$$X_v^{G_n,x^n}(t+1) = \bar{F}\left(X_v^{G_n,x^n}(t), \mu_v^{G_n,x^n}(t), \xi_v(t+1)\right)$$

where $\mu_v^{G_n}$ is the local empirical measure of the neighborhood of v:

$$\mu_v^{G_n}(t) = \frac{1}{n}\sum_{u\in N_v} \delta_{X_u^{G_n,x^n}(t)} \qquad m_v^{G_n}(t) = \frac{1}{n}\sum_{u\in N_v} g(X_u^{G_n,x^n}(t)).$$

Key questions: Given $G_n = (V_n, E_n)$, $x^n \in \mathcal{X}^{V_n}$, with $|V_n| \to \infty$.

Q1. Do the processes X^{G_n,x^n} converge in a suitable sense?

A1: Theorem 1: Lacker-R-Wu '19

Under mild continuity conditions on F, if (G_n, x^n) converges locally to (G, x) in distribution, then (G_n, X^{G_n,x^n}) converges locally in distribution to the dynamics on the limit graph $(G, X^{G,x})$

Results on Convergence

$$X_v^{G_n,x^n}(t+1) = \bar{F}\left(X_v^{G_n,x^n}(t), \mu_v^{G_n,x^n}(t), \xi_v(t+1)\right)$$

where $\mu_v^{G_n}$ is the local empirical measure of the neighborhood of v:

$$\mu_v^{G_n}(t) = \frac{1}{n}\sum_{u \in N_v} \delta_{X_u^{G_n,x^n}(t)} \qquad m_v^{G_n}(t) = \frac{1}{n}\sum_{u \in N_v} g(X_u^{G_n,x^n}(t)).$$

Key questions: Given $G_n = (V_n, E_n)$, $x^n \in \mathcal{X}^{V_n}$, with $|V_n| \to \infty$.

Q1. Do the processes X^{G_n,x^n} converge in a suitable sense?

A1: Theorem 1: Lacker-R-Wu '19

Under mild continuity conditions on F, if (G_n, x^n) converges locally to (G, x) in distribution, then (G_n, X^{G_n,x^n}) converges locally in distribution to the dynamics on the limit graph $(G, X^{G,x})$

Comment: *Incomplete* list of related works on discrete processes on directed graphs (Olvera-Cravioto-Chen-Litvak '17; Garavaglia-van der Hofstad-Litvak '18); pairwise-interacting diffusions

Convergence Result in Discrete Time

Basic Idea of Proof

- Need to show that the map $(G, \boldsymbol{x}) \mapsto (G, \boldsymbol{X}^{G,\boldsymbol{x}})$ from graphs marked with initial conditions to graphs marked with IPS trajectories

$$\mathcal{G}_*[\mathcal{S}] \mapsto \mathcal{G}_*[\mathcal{S}^\infty]$$

Basic Idea of Proof

- Need to show that the map $(G, \boldsymbol{x}) \mapsto (G, \boldsymbol{X}^{G,\boldsymbol{x}})$ from graphs marked with initial conditions to graphs marked with IPS trajectories

$$\mathcal{G}_*[\mathcal{S}] \mapsto \mathcal{G}_*[\mathcal{S}^\infty]$$

- This is equivalent to showing that for every $t, k \in \mathbb{N}_0$ and every bounded, cont. map φ on truncated paths $\mathcal{G}_*[\mathcal{S}^{k+1}]$, the function

$$\psi(G, \boldsymbol{x}) = \mathbb{E}[\varphi(G, \boldsymbol{X}_t^{G,\boldsymbol{x}})].$$

is continuous on $\mathcal{G}_*[\mathcal{S}]$.

Convergence Result in Discrete Time

Basic Idea of Proof

- Need to show that the map $(G, \boldsymbol{x}) \mapsto (G, \boldsymbol{X}^{G,\boldsymbol{x}})$ from graphs marked with initial conditions to graphs marked with IPS trajectories

$$\mathcal{G}_*[\mathcal{S}] \mapsto \mathcal{G}_*[\mathcal{S}^\infty]$$

- This is equivalent to showing that for every $t, k \in \mathbb{N}_0$ and every bounded, cont. map φ on truncated paths $\mathcal{G}_*[\mathcal{S}^{k+1}]$, the function

$$\psi(G, \boldsymbol{x}) = \mathbb{E}[\varphi(G, \boldsymbol{X}_t^{G,\boldsymbol{x}})].$$

  is continuous on $\mathcal{G}_*[\mathcal{S}]$.
- Fix $r \in \mathbb{N}$. It suffices to verify that the following "r-localized" function is continuous:

$$\psi_r(G, \boldsymbol{x}) := \varphi(B_r(G), \boldsymbol{X}_t^{B_r(G), \boldsymbol{x}_{B_r(G)}}).$$

Convergence Result in Discrete Time

Basic Idea of Proof

- Need to show that the map $(G, \boldsymbol{x}) \mapsto (G, \boldsymbol{X}^{G,\boldsymbol{x}})$ from graphs marked with initial conditions to graphs marked with IPS trajectories

$$\mathcal{G}_*[\mathcal{S}] \mapsto \mathcal{G}_*[\mathcal{S}^\infty]$$

- This is equivalent to showing that for every $t, k \in \mathbb{N}_0$ and every bounded, cont. map φ on truncated paths $\mathcal{G}_*[\mathcal{S}^{k+1}]$, the function

$$\psi(G, \boldsymbol{x}) = \mathbb{E}[\varphi(G, \boldsymbol{X}_t^{G,\boldsymbol{x}})].$$

is continuous on $\mathcal{G}_*[\mathcal{S}]$.

- Fix $r \in \mathbb{N}$. It suffices to verify that the following "r-localized" function is continuous:

$$\psi_r(G, \boldsymbol{x}) := \varphi(B_r(G), \boldsymbol{X}_t^{B_r(G), \boldsymbol{x}_{B_r(G)}}).$$

- **Key realization**: on any interval $[0, t]$, the behavior of particles in a B_r-neighborhood of the root only depend on the initial conditions in a B_{r+t}-neighborhood of the root. This dependence is continuous due to the continuity of each F_s, $s \leq t$.

Results on Convergence (contd.)

- In the setting of continuous time jump processes, with jump rates $\{r_j^v\}_{j \in \mathcal{J}, v \in V}$, the range of influence of dynamics at a node is potentially infinite at all positive times – this question is more subtle

Assumption. For any fixed time $T > 0$, there are finite constants $C_{k,T}, k \in \mathbb{N}$, such that the rates $\sup_{x,j} r_j^v(x) \leq C_{d_v,T}$.

Results on Convergence (contd.)

- In the setting of continuous time jump processes, with jump rates $\{r_j^v\}_{j \in \mathcal{J}, v \in V}$, the range of influence of dynamics at a node is potentially infinite at all positive times – this question is more subtle

Assumption. For any fixed time $T > 0$, there are finite constants $C_{k,T}, k \in \mathbb{N}$, such that the rates $\sup_{x,j} r_j^v(x) \leq C_{d_v,T}$.

A1': Theorem 2: Ganguly-R; '22

If (G_n, x^n) converges locally to (G, x) in distribution, **and the limit graph G is finitely dissociable**, $(G_n, \mathrm{X}^{\mathbf{G_n}, \mathbf{x^n}})$ converges locally in distribution to $(G, \mathrm{X}^{\mathrm{G,x}})$

Results on Convergence (contd.)

- In the setting of continuous time jump processes, with jump rates $\{r_j^v\}_{j \in \mathcal{J}, v \in V}$, the range of influence of dynamics at a node is potentially infinite at all positive times – this question is more subtle

Assumption. For any fixed time $T > 0$, there are finite constants $C_{k,T}, k \in \mathbb{N}$, such that the rates $\sup_{x,j} r_j^v(x) \leq C_{d_v,T}$.

A1': Theorem 2: Ganguly-R; '22

If (G_n, x^n) converges locally to (G, x) in distribution, **and the limit graph G is finitely dissociable**, $(G_n, \mathrm{X}^{\mathrm{G_n}, \mathrm{x^n}})$ converges locally in distribution to $(G, \mathrm{X}^{\mathrm{G}, \mathrm{x}})$

- All graphs with maximally bounded degree and Galton-Watson trees whose offspring distribution have a finite moment are finitely dissociable (Refer to [G-R '22] for the definition of finitely dissociable graphs)
- Result in fact holds for non-Markovian jump processes under an additional continuity condition on the path-dependent jump rates.

Comments on the Proof

- Much more subtle

Comments on the Proof

- Much more subtle
- First prove this for finite graphs, exploiting continuity of jump rates

Comments on the Proof

- Much more subtle
- First prove this for finite graphs, exploiting continuity of jump rates
- Show "spatial localization" property (in effect says influence is restricted to some finite random graph)

Comments on the Proof

- Much more subtle
- First prove this for finite graphs, exploiting continuity of jump rates
- Show "spatial localization" property (in effect says influence is restricted to some finite random graph)
- Establish a consistent spatial localization property (across convergent graph sequences)

Comments on the Proof

- Much more subtle
- First prove this for finite graphs, exploiting continuity of jump rates
- Show "spatial localization" property (in effect says influence is restricted to some finite random graph)
- Establish a consistent spatial localization property (across convergent graph sequences)
- Use a percolation argument to show that finite dissociability ensures spatial localization
- Analyze an inhomogeneous site percolation on trees to show Galton-Watson trees are finitely dissociable
- A mathematical subtlety: Well-posedness of the IPS on the infinite graph is not automatic; need restrictions
  Recent related work on well-posedness for SEP: Gantert-Schmid '21

C. Hydrodynamic Limits

Hydrodynamic Limits

- Sequence of marked graphs $(G_n, \boldsymbol{x}^n) \to (G, \boldsymbol{x})$ (weakly) in $\mathcal{G}_*[\mathcal{S}]$
- Recall original form of dynamics:

$$X_v^{G_n,x^n}(t+1) = F\left(X_v^{G_n,x^n}(t), \mu_v^{G_n,x^n}(t), \xi_{t+1}^v\right), \quad v \in V_n,$$

with $X(0) = \boldsymbol{x}^n$ and

$$\mu_v^{G_n,x^n}(t) = \frac{1}{|\mathcal{N}_v^{G_n}|} \sum_{u \in \mathcal{N}_v^{G_n}} \delta_{X_v^{G_n,x^n}(t)}$$

Hydrodynamic Limits

- Sequence of marked graphs $(G_n, \boldsymbol{x}^n) \to (G, \boldsymbol{x})$ (weakly) in $\mathcal{G}_*[\mathcal{S}]$
- Recall original form of dynamics:

$$X_v^{G_n,x^n}(t+1) = F\left(X_v^{G_n,x^n}(t), \mu_v^{G_n,x^n}(t), \xi_{t+1}^v\right), \quad v \in V_n,$$

with $X(0) = \boldsymbol{x}^n$ and

$$\mu_v^{G_n,x^n}(t) = \frac{1}{|\mathcal{N}_v^{G_n}|} \sum_{u \in \mathcal{N}_v^{G_n}} \delta_{X_v^{G_n,x^n}(t)}$$

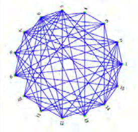
Key Question 2: Does the (global) empirical measure

$$\mu^{G_n,x^n}(t) := \frac{1}{|V_n|} \sum_{v \in V_n} \delta_{X_v^{G_n,x^n}(t)}$$

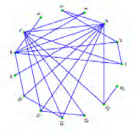
have a limit?

Hydrodynamic Limits

- Let $\mathcal{G}(n, p_n)$ be a (sparse) Erdös-Rényi graph; o a uniform random vertex

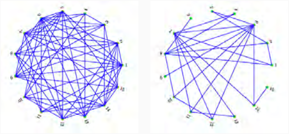


(a) Dense graph



(b) Sparse Graph

- Let $\mathcal{G}(n, p_n)$ be a (sparse) Erdös-Rényi graph; o a uniform random vertex



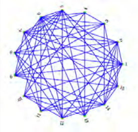
(a) Dense graph (b) Sparse Graph

Theorem 3: Lacker-R-Wu '19; Ganguly-R '22

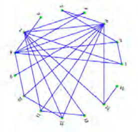
$G_n \sim \mathcal{G}(n, p_n)$ with $np_n \to \bar{c} \in (0, \infty)$.

Hydrodynamic Limits

- Let $\mathcal{G}(n, p_n)$ be a (sparse) Erdös-Rényi graph; o a uniform random vertex



(a) Dense graph



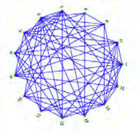
(b) Sparse Graph

Theorem 3: Lacker-R-Wu '19; Ganguly-R '22

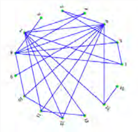
$G_n \sim \mathcal{G}(n, p_n)$ with $np_n \to \bar{c} \in (0, \infty)$. Then $(\mu^{G_n})_{n\in\mathbb{N}}$ converges weakly to the deterministic measure $\mathrm{Law}(X_o^{\mathcal{T}})$, where $\mathcal{T} \sim UGW(Poisson(\bar{c})$.

Hydrodynamic Limits

- Let $\mathcal{G}(n, p_n)$ be a (sparse) Erdös-Rényi graph; o a uniform random vertex



(a) Dense graph



(b) Sparse Graph

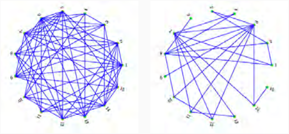
Theorem 3: Lacker-R-Wu '19; Ganguly-R '22

$G_n \sim \mathcal{G}(n, p_n)$ with $np_n \to \bar{c} \in (0, \infty)$. Then $(\mu^{G_n})_{n \in \mathbb{N}}$ converges weakly to the deterministic measure $\mathrm{Law}(X_o^{\mathcal{T}})$, where $\mathcal{T} \sim UGW(Poisson(\bar{c})$.

Note: Get a deterministic limit even though neighboring vertices remain strongly correlated

Hydrodynamic Limits

- Let $\mathcal{G}(n, p_n)$ be a (sparse) Erdös-Rényi graph; o a uniform random vertex



(a) Dense graph (b) Sparse Graph

Theorem 3: Lacker-R-Wu '19; Ganguly-R '22

$G_n \sim \mathcal{G}(n, p_n)$ with $np_n \to \bar{c} \in (0, \infty)$. Then $(\mu^{G_n})_{n \in \mathbb{N}}$ converges weakly to the deterministic measure $\mathrm{Law}(X_o^{\mathcal{T}})$, where $\mathcal{T} \sim UGW(Poisson(\bar{c})$.

Note: Get a deterministic limit even though neighboring vertices remain strongly correlated

Note: In fact, the result applies more broadly as long as (G_n, x^n) converges to a limit (G, x) locally (in probability), a notion that applies to possibly disconnected graphs and is stronger than the local convergence in distribution defined earlier.

Note: Luckily, this stronger convergence continues hold for many models of interest, including random regular graphs, configuration model,

Outline of the Proof of Theorem 3

1. Let ρ_1^n and ρ_2^n be two independent vertices, both chosen uniformly at random from the graph G_n.
2. For any $k \in \mathbb{N}$, let $C_1^n[k]$ be the connected k-neighborhood of the graph G_n around the "root" ρ_1^n.

Outline of the Proof of Theorem 3

1. Let ρ_1^n and ρ_2^n be two independent vertices, both chosen uniformly at random from the graph G_n.
2. For any $k \in \mathbb{N}$, let $C_1^n[k]$ be the connected k-neighborhood of the graph G_n around the "root" ρ_1^n.
3. Then local convergence of G_n ensures $C_1^n[k]$ and $C_2^n[k]$ are asymptotically independent

Outline of the Proof of Theorem 3

1. Let ρ_1^n and ρ_2^n be two independent vertices, both chosen uniformly at random from the graph G_n.
2. For any $k \in \mathbb{N}$, let $C_1^n[k]$ be the connected k-neighborhood of the graph G_n around the "root" ρ_1^n.
3. Then local convergence of G_n ensures $C_1^n[k]$ and $C_2^n[k]$ are asymptotically independent
4. We correspondingly establish asymptotic independence of $(X^{G_n,x^n}_{C_1^n[k]}, X^{G_n,x^n}_{C_2^n[k]})$.
5. This requires analysis of the dynamics and is done by establishing correlation decay estimates, which are obtained using coupling techniques or (in the continuous time jump process setting) consistent spatial localization

Hydrodynamic Limits (contd.)

For a finite graph $G = (V, E)$, let $C_\rho(G)$ be the random connected rooted graph obtained by assigning a root uniformly at random and then isolating the corresponding connected component.

Hydrodynamic Limits (contd.)

For a finite graph $G = (V, E)$, let $C_\rho(G)$ be the random connected rooted graph obtained by assigning a root uniformly at random and then isolating the corresponding connected component.

Theorem 4: Lacker-R-Wu '18

$G_n \sim \mathcal{G}(n, p_n)$ with $np_n \to \bar{c} \in (0, \infty)$. Let $\mathcal{T} \sim UGW(Poisson(\bar{c}))$.

- If $\bar{c} \leq 1$ then $(\bar{\mu}^{C_\rho(G_n)})_{n\in\mathbb{N}}$ converges in law to the random empirical measure $\bar{\mu}^{\mathcal{T}}$.
- If $\bar{c} > 1$, then $(\bar{\mu}^{C_\rho(G_n)})_{n\in\mathbb{N}}$ converges in law to the random empirical measure $\tilde{\mu}^{\mathcal{T}}$ defined by

$$\tilde{\mu}^{\mathcal{T}} = \begin{cases} \bar{\mu}^{\mathcal{T}} & \text{on } \{|\mathcal{T}| < \infty\} \\ \mathcal{L}(X_o^{\mathcal{T}} \,||\mathcal{T}| = \infty) & \text{on } \{|\mathcal{T}| = \infty\} \end{cases}$$

Outline of the Proof of Theorem 4

1. Establish a quenched concentration estimate for the empirical measure (holds in greater generality) – that is show that given a realization of the graph, the empirical measure is very close to its expectation on that graph
2. Write out the expectation of the empirical measure, by condition on being in the maximal component of the graph and not.
3. Uses various properties about the Erdös-Rényi graph at criticality and subcriticality to show that each term converges to the corresponding limit term

Outline of the Proof of Theorem 4

1. Establish a quenched concentration estimate for the empirical measure (holds in greater generality) – that is show that given a realization of the graph, the empirical measure is very close to its expectation on that graph
2. Write out the expectation of the empirical measure, by condition on being in the maximal component of the graph and not.
3. Uses various properties about the Erdös-Rényi graph at criticality and subcriticality to show that each term converges to the corresponding limit term
4. Once again, this result can be extended to a large class of random graph sequences, using nice duality properties of random graphs (van der Hofstad, 22+)

Hydrodynamic Limits

- Thus we see that in the sparse regime the empirical measure limit can be both deterministic and random.
- Which one depends on the nature of local convergence, and you get deterministic limits only with the stronger local convergence in probability (which I did not define) rather than local convergence in distribution ensures deterministic limits.

Hydrodynamic Limits

- Thus we see that in the sparse regime the empirical measure limit can be both deterministic and random.
- Which one depends on the nature of local convergence, and you get deterministic limits only with the stronger local convergence in probability (which I did not define) rather than local convergence in distribution ensures deterministic limits.

Hydrodynamic Limits

- Thus we see that in the sparse regime the empirical measure limit can be both deterministic and random.
- Which one depends on the nature of local convergence, and you get deterministic limits only with the stronger local convergence in probability (which I did not define) rather than local convergence in distribution ensures deterministic limits.
- This raises the natural question of whether at least whenever the empirical measure is deterministic, it coincides with the law of a typical particle?

Other related convergence results

1. Convergence of empirical measure to the law of the root particle for lattices.
2. Convergence of empirical meausure to a deterministic limit for regular trees, but *does not equal* the law of the root particle.

Prelude to Lecture 3

Dynamics of a Typical Particle

A1. Answer to the first question showed that the typical particle $X_v^{G_n,x}$ in a large system can be approximated by the marginal of the dynamical system on an infinite graph G:

$$X_v^G(t+1) = F\left(X_v^G(t), \mu_v(t), \xi_v(t+1)\right), \quad v \in V,$$

where $F : \mathcal{S} \times \mathcal{P}(\mathcal{S}) \times \mathcal{S} \mapsto \mathcal{S}$,
with $\mu_v^G(t)$ is the local empirical measure at v:

$$\mu_v^G(t) = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_u^G(t)}, \quad v \in V.$$

Key Questions:
(Q3) Is there an autonomous description of the limiting dynamics of a typical particle?

Lecture III:

Marginal Dynamics on Large Sparse Graphs

Recap of Lectures 1 and 2

Given (G, V) and locally interacting processes:

$$X_v^{G,x}(t+1) = F\left(X_v(t), (X_{N_v}(t)), \xi_v(t+1)\right), \quad v \in V, t \in \mathbb{N}_0$$

where $X^{G,x}(0) = x$ and the dependence of F on X_{N_v} is symmetric.

Recap of Lectures 1 and 2

Given (G, V) and locally interacting processes:

$$X_v^{G,x}(t+1) = F\left(X_v(t), (X_{N_v}(t)), \xi_v(t+1)\right), \quad v \in V, t \in \mathbb{N}_0$$

where $X^{G,x}(0) = x$ and the dependence of F on X_{N_v} is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}(t)}$$

Given (G, V) and locally interacting processes:

$$\mathrm{X}_{\mathrm{v}}^{\mathrm{G,x}}(\mathrm{t}+1) = \mathrm{F}\left(\mathrm{X}_{\mathrm{v}}(\mathrm{t}), (\mathrm{X}_{\mathrm{N}_{\mathbf{v}}}(\mathrm{t})), \xi_{\mathrm{v}}(\mathrm{t}+1)\right), \quad v \in V, t \in \mathbb{N}_0$$

where $X^{G,x}(0) = x$ and the dependence of F on X_{N_v} is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|}\sum_{v\in G}\delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|}\sum_{v\in G}\delta_{X_v^{G,x}(t)}$$

Key questions: Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, and appropriate initial conditions $x^n \in \mathcal{X}^{V_n}$,

Q1. Do the processes $\mathrm{X}^{\mathrm{G}_{\mathbf{n}},\mathrm{x}^{\mathbf{n}}}$ converge in a suitable sense?

Recap of Lectures 1 and 2

Given (G, V) and locally interacting processes:

$$X_v^{G,x}(t+1) = F\left(X_v(t), (X_{N_v}(t)), \xi_v(t+1)\right), \quad v \in V, t \in \mathbb{N}_0$$

where $X^{G,x}(0) = x$ and the dependence of F on X_{N_v} is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}(t)}$$

Key questions: Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, and appropriate initial conditions $x^n \in \mathcal{X}^{V_n}$,

Q1. Do the processes X^{G_n, x^n} converge in a suitable sense?

Q2. Do the global empirical measures μ^{G_n, x^n} converge (hydrodynamic limits) ?

Recap of Lectures 1 and 2

Given (G, V) and locally interacting processes:

$$X_v^{G,x}(t+1) = F\left(X_v(t), (X_{N_v}(t)), \xi_v(t+1)\right), \quad v \in V, t \in \mathbb{N}_0$$

where $X^{G,x}(0) = x$ and the dependence of F on X_{N_v} is symmetric.

Quantity of interest: global empirical measure and empirical measure flow:

$$\mu^{G,x} := \frac{1}{|V|}\sum_{v \in G} \delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|V|}\sum_{v \in G} \delta_{X_v^{G,x}(t)}$$

Key questions: Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, and appropriate initial conditions $x^n \in \mathcal{X}^{V_n}$,

Q1. Do the processes X^{G_n,x^n} converge in a suitable sense?

Q2. Do the global empirical measures μ^{G_n,x^n} converge (hydrodynamic limits) ?

Throughout, assume F is continuous

Recall First Convergence Result

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad v \in V$$

Theorem: Lacker-R-Wu '19

The graphs marked with initial conditions (G_n, x^n) converge locally to a limit graph (G, x) in distribution, then the graphs marked with the solution trajectories (G_n, X^{G_n, x^n}) converge locally in distribution to the dynamics on the limit graph $(G, X^{G,x})$.

Recall First Convergence Result

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad v \in V$$

Theorem: Lacker-R-Wu '19

The graphs marked with initial conditions (G_n, x^n) converge locally to a limit graph (G, x) in distribution, then the graphs marked with the solution trajectories (G_n, X^{G_n, x^n}) converge locally in distribution to the dynamics on the limit graph $(G, X^{G,x})$.

- **Examples:** G^n - E-R, connected component of the root in E-R, CM, PAM (under second moment assumptions) and random regular

Recall First Convergence Result

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad v \in V$$

Theorem: Lacker-R-Wu '19

The graphs marked with initial conditions (G_n, x^n) converge locally to a limit graph (G, x) in distribution, then the graphs marked with the solution trajectories (G_n, X^{G_n,x^n}) converge locally in distribution to the dynamics on the limit graph $(G, X^{G,x})$.

- **Examples:** G^n - E-R, connected component of the root in E-R, CM, PAM (under second moment assumptions) and random regular
  x^n - i.i.d. initial conditions or Gibbs measures

Recall First Convergence Result

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad v \in V$$

Theorem: Lacker-R-Wu '19

The graphs marked with initial conditions (G_n, x^n) converge locally to a limit graph (G, x) in distribution, then the graphs marked with the solution trajectories (G_n, X^{G_n, x^n}) converge locally in distribution to the dynamics on the limit graph $(G, X^{G,x})$.

- **Examples:** G^n - E-R, connected component of the root in E-R, CM, PAM (under second moment assumptions) and random regular x^n - i.i.d. initial conditions or Gibbs measures
- **Analogous results** for
  - discrete-time on directed graphs for PageRank (Olvera-Cravioto-et al '17; Garavaglia-van der Hofstad-Litvak '18, ...)
  - diffusions (Oliveira-Reis-Stolerman '18, Lacker-R-Wu '19)
  - jump processes: also requires G be finitely dissociable (Ganguly-'R '22)

The Second Convergence Result

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad \mu^{G,x} := \frac{1}{|V|}\sum_{v \in G} \delta_{X_v^{G,x}}$$

Theorem: Lacker-R-Wu '19

If (G_n, x^n) converges locally in probability to (G, x), then (G_n, X^{G_n,x^n}) converges locally in probability to the dynamics on the limit graph $(G, X^{G,x})$. Furthermore, then the global empirical measure μ^{G_n,x^n} converges weakly to $\mathrm{Law}(X_o^{G,x})$, the law of the root in the limit graph.

The Second Convergence Result

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad \mu^{G,x} := \frac{1}{|V|} \sum_{v \in G} \delta_{X_v^{G,x}}$$

Theorem: Lacker-R-Wu '19

If (G_n, x^n) converges locally in probability to (G, x), then (G_n, X^{G_n, x^n}) converges locally in probability to the dynamics on the limit graph $(G, X^{G,x})$. Furthermore, then the global empirical measure μ^{G_n, x^n} converges weakly to $\mathrm{Law}(X_o^{G,x})$, the law of the root in the limit graph.

- **Examples:** G^n - E-R, ~~connected component of the root in E-R~~, CM (under second moment assumptions) and random regular

The Second Convergence Result

$$X_v^{G,x}(t+1) = F(X_v(t), X_{N_v}(t), \xi_v(t+1)), \quad \mu^{G,x} := \frac{1}{|V|}\sum_{v \in G} \delta_{X_v^{G,x}}$$

Theorem: Lacker-R-Wu '19

If (G_n, x^n) converges locally in probability to (G, x), then (G_n, X^{G_n, x^n}) converges locally in probability to the dynamics on the limit graph $(G, X^{G,x})$. Furthermore, then the global empirical measure μ^{G_n, x^n} converges weakly to $\mathrm{Law}(X_o^{G,x})$, the law of the root in the limit graph.

- **Examples:** G^n - E-R, ~~connected component of the root in E-R~~, CM (under second moment assumptions) and random regular
  x^n - i.i.d. initial conditions or Gibbs measures

The Second Convergence Result

$$X_v^{G,x}(t+1) = F(X_v(t), X_{N_v}(t), \xi_v(t+1)), \quad \mu^{G,x} := \frac{1}{|V|}\sum_{v \in G} \delta_{X_v^{G,x}}$$

Theorem: Lacker-R-Wu '19

If (G_n, x^n) converges locally in probability to (G, x), then (G_n, X^{G_n, x^n}) converges locally in probability to the dynamics on the limit graph $(G, X^{G,x})$. Furthermore, then the global empirical measure μ^{G_n, x^n} converges weakly to $\mathrm{Law}(X_o^{G,x})$, the law of the root in the limit graph.

- **Examples:** G^n - E-R, ~~connected component of the root in E-R~~, CM (under second moment assumptions) and random regular
  x^n - i.i.d. initial conditions or Gibbs measures
- **Analogous results** for
  - diffusions (Lacker-R-Wu '19)
  - jump processes – with G finitely dissociable (Ganguly-R '22)
- Subtleties: If (G^n, x^n) only converges locally in law, then the limit of μ^{G_n, x^n} could be stochastic, or may fail to coincide with the law of the root particle (Prop 7.7, DRW - canopy tree).

Focus of this Lecture

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad v \in V, t \in \mathbb{N}_0,$$

Key Questions:

(Q3) Is there an autonomous description of the (law of the) limiting dynamics of the root particle, $X_{\varnothing}^{G,x}$?

Focus of this Lecture

$$X_v^{G,x}(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right), \quad v \in V, t \in \mathbb{N}_0,$$

Key Questions:

(Q3) Is there an autonomous description of the (law of the) limiting dynamics of the root particle, $X_{\varnothing}^{G,x}$?

- It turns out the right question to ask is whether there is an autonomous description of the law of the root and its neighborhood:

$$X_{\bar{\varnothing}}^{G,x} = X_{\varnothing, u \sim \varnothing}^{G,x}$$

- We will answer this question for the case when G is a (random) tree.
- For simplicity we will focus on the case when x is i.i.d. with law λ (though extensions are considered in Ganguly-'R '22)
- Recall that in the mean-field case, the typical dynamics is described by a nonlinear Markov chain

Background on Markov chains

- Suppose $\mathcal{X} = \{1, \dots, m\}$ is a finite state space
- $\{Z(t)\}_{t \in \mathbb{N}}$ is a homogeneous Markov chain if there exists a $m \times m$ (stochastic) matrix $P = \{P_{zz'}\}_{z,z' \in \mathcal{X}}$ such that

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}.$$

Background on Markov chains

- Suppose $\mathcal{X} = \{1, \ldots, m\}$ is a finite state space
- $\{Z(t)\}_{t \in \mathbb{N}}$ is a homogeneous Markov chain if there exists a $m \times m$ (stochastic) matrix $P = \{P_{zz'}\}_{z,z' \in \mathcal{X}}$ such that

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}.$$

- Let $\mathcal{P}(\mathcal{X})$ be the space of probability vectors on $\mathcal{X}$: $(\nu_1, \ldots, \nu_m)$ with $\nu_j \in [0,1]$ and $\sum_{j=1}^m \nu_j = 1$.
- The law $\nu(t) = \mathcal{L}aw(Z(t)) \in \mathcal{P}(\mathcal{X})$ of the process evolves according to the (autonomous) linear matrix equation

$$\nu(t+1) = \nu(t) P$$

Background on Markov chains

- Suppose $\mathcal{X} = \{1, \dots, m\}$ is a finite state space
- $\{Z(t)\}_{t\in\mathbb{N}}$ is a homogeneous Markov chain if there exists a $m \times m$ (stochastic) matrix $P = \{P_{zz'}\}_{z,z'\in\mathcal{X}}$ such that

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}.$$

- Let $\mathcal{P}(\mathcal{X})$ be the space of probability vectors on $\mathcal{X}$: $(\nu_1, \dots, \nu_m)$ with $\nu_j \in [0,1]$ and $\sum_{j=1}^m \nu_j = 1$.
- The law $\nu(t) = \mathcal{L}aw(Z(t)) \in \mathcal{P}(\mathcal{X})$ of the process evolves according to the (autonomous) linear matrix equation

$$\nu(t+1) = \nu(t)P$$

- **Note.** If $Z(t+1)$ depends on $Z(t)$ and $Z(t-1)$, then $Y(t) = (Z(t-1), Z(t))$ forms a Marov chain on the state space $\mathcal{X}^2$.
- $\{Z(t)\}_{t\in\mathbb{N}}$ is an inhomogeneous Markov chain if there exists a family of (stochastic) matrices $P(t) = \{P_{zz'}(t)\}_{z,z'\in\mathcal{X},t\in\mathbb{N}_0}$ such that

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}(t), \qquad \nu(t+1) = \nu(t)P(t).$$

Background on Markov chains

- homogeneous Markov chain:

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}, \qquad \nu(t+1) = \nu(t)P$$

- inhomogeneous Markov chain:

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}(t), \qquad \nu(t+1) = \nu(t)P(t).$$

Background on Markov chains

- homogeneous Markov chain:

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}, \qquad \nu(t+1) = \nu(t)P$$

- inhomogeneous Markov chain:

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = P_{z,z'}(t), \qquad \nu(t+1) = \nu(t)P(t).$$

- A nonilinear Markov chain is a particular kind of inhomogeneous Markov chain in which the dependence on t is only through the law of the process at time t:

$$P_{zz'}(t) = \bar{P}(\nu(t))$$

for a suitable family of (stochastic) matrices $\{\bar{P}(\nu)\}_{\nu \in \mathcal{P}(\mathcal{X})}$.

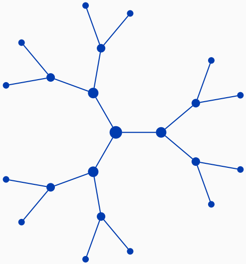
- $\{Z(t)\}_{t \in \mathbb{N}}$ is a nonlinear Markov chain if there exists a family of stochastic matrices $\bar{P}(\nu) = \{\bar{P}_{zz'}(\nu)\}_{z,z' \in \mathcal{X}, \nu \in \mathcal{P}(\mathcal{X})}$, such that

$$\mathbb{P}(Z(t+1) = z' | Z(t) = z) = \bar{P}_{z,z'}(\nu(t)), \quad \nu(t+1) = \nu(t)\bar{P}(\nu(t)).$$

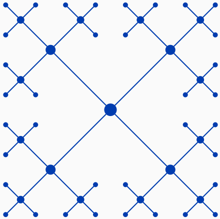
Evolution of the law is through the nonlinear matrix equation.

Back to Marginal Dynamics on Regular Trees

Suppose the limiting graph G is an infinite d-regular tree with fixed root.



$d = 3$



$d = 4$

Simplest Case: Marginal Dynamics on the Line

- For simplicity consider the case $d = 2$, identify $\mathcal{T}_2$ with $\mathbb{Z}$, set $\varnothing = 0$.
- Note that Theorem 1 implies that for a typical vertex $\varnothing$, $\{X_\varnothing, (X_v)_{v\sim\varnothing}\}$ can be obtained as the marginal of the infinite coupled system of Markov chains:

$$\mathrm{X_i(t+1) = F\,(X_i(t), X_{N_i}(t), \xi_i(t+1))}, \quad i \in \mathbb{Z}, t \in \mathbb{N}_0,$$

- Denote $X := X^G$ and note $X_{N_i} = X_{\{i-1,i,i+1\}}$
- So we are interested in an autonomous characterization of the marginal law of

$$X_{\{-1,0,1\}} = (X_{-1}, X_0, X_1).$$

Simplest Case: Marginal Dynamics on the Line

- For simplicity consider the case $d = 2$, identify $\mathcal{T}_2$ with $\mathbb{Z}$, set $\varnothing = 0$.
- Note that Theorem 1 implies that for a typical vertex $\varnothing$, $\{X_\varnothing, (X_v)_{v\sim\varnothing}\}$ can be obtained as the marginal of the infinite coupled system of Markov chains:

$$\mathsf{X_i(t+1) = F\left(X_i(t), X_{N_i}(t), \xi_i(t+1)\right)}, \quad i \in \mathbb{Z}, t \in \mathbb{N}_0,$$

- Denote $X := X^G$ and note $X_{N_i} = X_{\{i-1,i,i+1\}}$
- So we are interested in an autonomous characterization of the marginal law of

$$X_{\{-1,0,1\}} = (X_{-1}, X_0, X_1).$$

- Given $t \in \mathbb{N}$, $A \subset \mathbb{Z}$, sequence $y \in \mathcal{X}^\infty$, denote the history upto time t by

$$y_A[t] = (y_A(0), y_A(1), \ldots, y_A(t))$$

A "Ghost" Particle System on the Root Neighborhood

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.

A “Ghost” Particle System on the Root Neighborhood

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.
- At each time $t \in \mathbb{N}_0$, given the past $Y_{-1,0,1}[t]$ and its law, recursively define the family of conditional laws: for $z \in \mathcal{X}$,

$$\gamma_t(z | y_0, y_1) = \mathbb{P}\Big(Y_{-1}(t) = z \,|\, Y_0[t] = y_0[t], Y_1[t] = y_1[t] \Big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

A "Ghost" Particle System on the Root Neighborhood

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.
- At each time $t \in \mathbb{N}_0$, given the past $Y_{-1,0,1}[t]$ and its law, recursively define the family of conditional laws: for $z \in \mathcal{X}$,

$$\gamma_t(z|\, y_0, y_1) = \mathbb{P}(Y_{-1}(t) = z \mid Y_0[t] = y_0[t], Y_1[t] = y_1[t]\Big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

- Sample independent ghost particles $Y_{-2}(t)$ and $Y_2(t)$ such that

$$\mathbb{P}\Big(Y_{-2}(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_{-1}[t], Y_0[t])$$

$$\mathbb{P}\Big(Y_2(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_1[t], Y_0[t])$$

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.
- At each time $t \in \mathbb{N}_0$, given the past $Y_{-1,0,1}[t]$ and its law, recursively define the family of conditional laws: for $z \in \mathcal{X}$,

$$\gamma_t(z|\, y_0, y_1) = \mathbb{P}\big(Y_{-1}(t) = z \mid Y_0[t] = y_0[t],\, Y_1[t] = y_1[t]\Big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

- Sample independent ghost particles $Y_{-2}(t)$ and $Y_2(t)$ such that

$$\mathbb{P}\Big(Y_{-2}(t) = z| Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_{-1}[t],\, Y_0[t])$$

$$\mathbb{P}\Big(Y_{2}(t) = z| Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_{1}[t],\, Y_0[t])$$

- Sample new iid noises $(\xi_{-1}(t+1), \xi_0(t+1), \xi_1(t+1))$, and update:

$$\begin{aligned}
Y_{-1}(t+1) &= F\Big(Y_{-1}(t), (Y_0(t), Y_{-2}(t)), \xi_{-1}(t+1)\Big) \\
Y_{1}(t+1) &= F\Big(Y_{1}(t), (Y_0(t), Y_{2}(t)), \xi_{1}(t+1)\Big) \\
Y_{0}(t+1) &= F\Big(Y_{i}(t), Y_{i-1,i+1}(t), \xi_{i}(t+1)\Big)
\end{aligned}$$

A "Ghost" Particle System on the Root Neighborhood

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.
- At each time $t \in \mathbb{N}_0$, given the past $Y_{-1,0,1}[t]$ and its law, recursively define the family of conditional laws: for $z \in \mathcal{X}$,

$$\gamma_t(z | y_0, y_1) = \mathbb{P}\Big(Y_{-1}(t) = z \mid Y_0[t] = y_0[t], Y_1[t] = y_1[t]\Big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

- Sample independent ghost particles $Y_{-2}(t)$ and $Y_2(t)$ such that

$$\mathbb{P}\Big(Y_{-2}(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z | Y_{-1}[t], Y_0[t])$$

$$\mathbb{P}\Big(Y_2(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z | Y_1[t], Y_0[t])$$

- Sample new iid noises $(\xi_{-1}(t+1), \xi_0(t+1), \xi_1(t+1))$, and update:

$$\begin{aligned} Y_{-1}(t+1) &= F\Big(Y_{-1}(t), (Y_0(t), Y_{-2}(t)), \xi_{-1}(t+1)\Big) \\ Y_1(t+1) &= F\Big(Y_1(t), (Y_0(t), Y_2(t)), \xi_1(t+1)\Big) \\ Y_0(t+1) &= F\Big(Y_i(t), Y_{i-1,i+1}(t), \xi_i(t+1)\Big) \end{aligned}$$

- **Note**: $Y_{-1,0,1}(t+1)$ only depends on past $Y_{-1,0,1}[t]$ **and** its law

Result in the Case of $\mathbb{Z}$

$$\mathsf{X_i(t+1) = F\,(X_i(t), X_{N_i}(t), \xi_i(t+1))}, \quad i \in \mathbb{Z}, t \in \mathbb{N}_0.$$

Summary of Ghost particle evolution – Local Equations:

$$Y_i(t+1) = F\Big(Y_i(t), Y_{i-1,i+1}(t), \xi_i(t+1)\Big), \quad i = \{-1, 0, 1\}, \quad \text{where}$$

$$\mathbb{P}\Big(Y_2(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z| \, Y_1[t], Y_0[t])$$

$$X_i(t+1) = F\,(X_i(t), X_{N_i}(t), \xi_i(t+1)), \quad i \in \mathbb{Z}, t \in \mathbb{N}_0.$$

Summary of Ghost particle evolution – Local Equations:

$$Y_i(t+1) = F\Big(Y_i(t), Y_{i-1,i+1}(t), \xi_i(t+1)\Big), \quad i = \{-1,0,1\}, \quad \text{where}$$

$$\mathbb{P}\Big(Y_2(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_1[t], Y_0[t]) = F(z, Y_{-1,0,1}[t], \mathrm{Law}(Y[t]))$$

with γ_t only depending on $\mathrm{Law}(Y[t])$ in the following explicit manner:

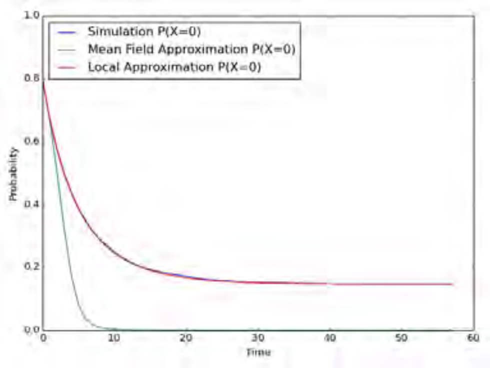
$$\gamma_t(z|\, y_0, y_1) = \mathbb{P}\Big(Y_{-1}(t) = z \,|\, Y_0[t] = y_0[t], Y_1[t] = y_1[t]\Big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

Theorem 5 (Lacker-R-Wu '19, '22)

Let $\{Y_{-1,0,1}(t)\}_{t\in\mathbb{N}}$ be the "ghost" particles starting at $X_{\{-1,0,1\}}(0)$. Then $\mathrm{Law}(Y_{-1,0,1}) = \mathrm{Law}(X_{-1,0,1})$. **This provides an autonomous characterization of the marginal law!**

How Good are the (Markov) Local Equations Approximations?

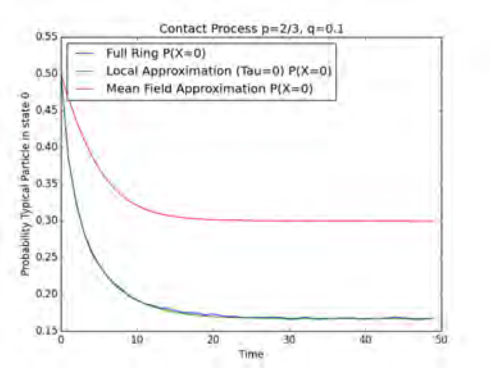
SIR Model



Plot of probability of being healthy vs. time

How Good are the (Markov) Local Equations Approximations?

Discrete-time Contact Process



Plot of probability of being healthy vs. time

Why is the theorem true?

- In the complete graph, the autonomous characterization arises due to asymptotic independence and exchangeability (permutation symmetry)

Why is the theorem true?

- In the complete graph, the autonomous characterization arises due to asymptotic independence and exchangeability (permutation symmetry)
- In this setting, we show that one can exploit conditional independence properties and graph symmetries

Why is the theorem true?

- In the complete graph, the autonomous characterization arises due to asymptotic independence and exchangeability (permutation symmetry)
- In this setting, we show that one can exploit conditional independence properties and graph symmetries
- But let's see precisely how this is done ...

Describing Dependencies of High-Dim Random Vectors

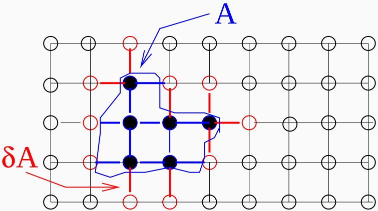
Notation: For a set A of vertices in a graph $G = (V, E)$, define

Boundary: $\partial A = N_A(G) = \{u \in V \backslash A : \exists u \in A \text{ s.t. } u \sim v\}$

State space $\mathbb{S}$; $Z = \{Z_v, v \in V\}$ high-dimensional vector in $\mathbb{S}^V$

Defn. $Z = \{Z_v\}_{v \in V}$ is said to be a Markov Random Field (MRF) with respect to G if for every partition A, B, C of V with $C = \partial A = \mathcal{N}_A(G)$

$$(Z_v)_{v \in A} \perp (Z_v)_{v \in B} \mid (Z_v)_{v \in C}, \qquad (1)$$



Defn. On an infinite graph G, $Z = (Z_v)_{v \in V}$ is a

- Local MRF if the same holds only for A finite
- Semi-global MRF (SGMRF) if the same holds only for C finite

Searching for a Conditional Independence Property

Fix (G, x) infinite. Denote $X = X^{G,x}$. Assume $(X_v(0))_{v \in V}$ iid.

$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Question A:

For $t > 0$, will $(X_v(t))_{v \in V}$ form a SGMRF wrt G?

In other words, for $A \subset V$ with $|\partial A| < \infty$ and $B \subset V \setminus [A \cup \partial A]$,

is $X_A(t) \perp X_B(t) | X_{\partial A}(t)$?

Searching for a Conditional Independence Property

Fix (G, x) infinite. Denote $X = X^{G,x}$. Assume $(X_v(0))_{v \in V}$ iid.

$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Question A:

For $t > 0$, will $(X_v(t))_{v \in V}$ form a SGMRF wrt G?

In other words, for $A \subset V$ with $|\partial A| < \infty$ and $B \subset V \setminus [A \cup \partial A]$,

is $X_A(t) \perp X_B(t) | X_{\partial A}(t)$?

$$G = \mathbb{Z}, A = \{-1, -2, \dots, \}, \partial A = \{0\} B = \{1\}.$$

Searching for a Conditional Independence Property

Fix (G, x) infinite. Denote $X = X^{G,x}$. Assume $(X_v(0))_{v \in V}$ iid.

$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

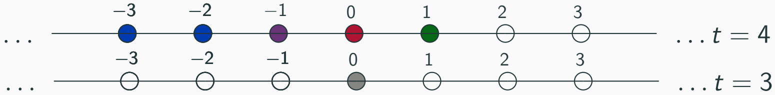
Question A:

For $t > 0$, will $(X_v(t))_{v \in V}$ form a SGMRF wrt G?

In other words, for $A \subset V$ with $|\partial A| < \infty$ and $B \subset V \setminus [A \cup \partial A]$,

is $X_A(t) \perp X_B(t) | X_{\partial A}(t)$?

$$G = \mathbb{Z}, A = \{-1, -2, \dots, \}, \partial A = \{0\} B = \{1\}.$$



Searching for a Conditional Independence Property

Fix (G, x) infinite. Denote $\mathsf{X} = \mathsf{X}^{G,x}$. Assume $(\mathsf{X}_v(0))_{v \in V}$ iid.

$$\mathsf{X}_v(t+1) = F\left(\mathsf{X}_v(t), \mathsf{X}_{N_v}(t), \xi_v(t+1)\right),$$

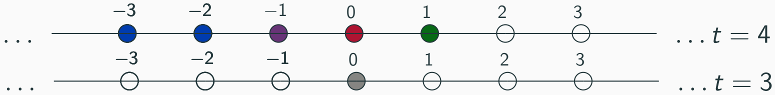
Question A:

For $t > 0$, will $(\mathsf{X}_v(t))_{v \in V}$ form a SGMRF wrt G?

In other words, for $A \subset V$ with $|\partial A| < \infty$ and $B \subset V \setminus [A \cup \partial A]$,

is $\mathsf{X}_A(t) \perp \mathsf{X}_B(t) | \mathsf{X}_{\partial A}(t)$?

$$G = \mathbb{Z}, A = \{-1, -2, \ldots,\}, \partial A = \{0\} B = \{1\}.$$



Answer A:

No!

In Search of a Conditional Independence Property

$$\mathsf{X_v(t+1) = F\,(X_v(t), X_{N_v}(t), \xi_v(t+1))},$$

Question B:

For $t > 0$, do the particle histories $(X_v[t])_{v \in V}$ form a SGMRF wrt G?

Recall notation for path history: $x[t] := (x(s), s = 0, 1, \dots, t)$.

In Search of a Conditional Independence Property

$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Reformulation of Question B:

Given $t > 0$, for any $A \subset V$ with finite ∂A and $B \subset V \setminus [A \cup \partial A]$,

Is $X_A[t] \perp X_B[t] | X_{\partial A}[t]$?

In Search of a Conditional Independence Property

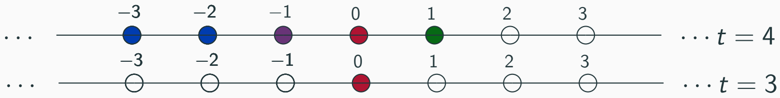
$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Reformulation of Question B:

Given $t > 0$, for any $A \subset V$ with finite ∂A and $B \subset V \setminus [A \cup \partial A]$,

Is $X_A[t] \perp X_B[t] | X_{\partial A}[t]$?

$$G = \mathbb{Z}, A = \{-1, -2, -3, \ldots, \}, \partial A = \{0\}, B = \{1, 2, \ldots\}.$$



In Search of a Conditional Independence Property

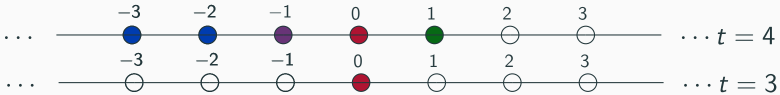
$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Reformulation of Question B:

Given $t > 0$, for any $A \subset V$ with finite ∂A and $B \subset V \setminus [A \cup \partial A]$,

Is $X_A[t] \perp X_B[t] | X_{\partial A}[t]$?

$$G = \mathbb{Z}, A = \{-1, -2, -3, \ldots, \}, \partial A = \{0\}, B = \{1, 2, \ldots\}.$$



$$\begin{aligned} X_{-1}(t+1) &= F(X_{-1}(t), (X_0(t), X_{-2}(t+1)), \xi_{-1}(t+1)) \\ \Rightarrow X_A[t+1] &= \text{a function of } (X_A[t], X_0[t], \xi_A[t+1]) \\ \text{Likewise } X_B[t+1] &= \text{a function of } (X_B[t], X_0[t], \xi_B[t+1]) \end{aligned}$$

Is $X_A[t] \perp X_B[t] | X_{\partial A}[t]$?

In Search of a Conditional Independence Property

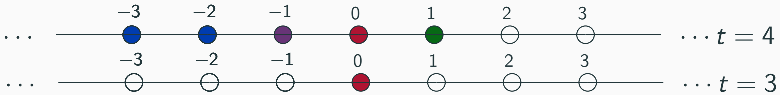
$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Reformulation of Question B:

Given $t > 0$, for any $A \subset V$ with finite ∂A and $B \subset V \setminus [A \cup \partial A]$,

Is $X_A[t] \perp X_B[t] | X_{\partial A}[t]$?

$$G = \mathbb{Z}, A = \{-1, -2, -3, \ldots, \}, \partial A = \{0\}, B = \{1, 2, \ldots\}.$$



$$\begin{aligned} X_{-1}(t+1) &= F(X_{-1}(t), (X_0(t), X_{-2}(t+1)), \xi_{-1}(t+1)) \\ \Rightarrow X_A[t+1] &= \text{a function of } (X_A[t], X_0[t], \xi_A[t+1]) \\ \text{Likewise } X_B[t+1] &= \text{a function of } (X_B[t], X_0[t], \xi_B[t+1]) \end{aligned}$$

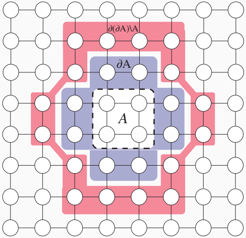
Is $X_A[t] \perp X_B[t] | X_{\partial A}[t]$?

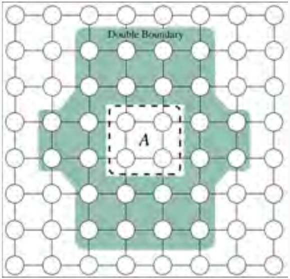
Answer B:

No!

Double Boundary

$$\partial^2 A = \partial A \cup [\partial(\partial A) \setminus A]$$





Definition: A family of random variables $(Y_v)_{v \in V}$ is a 2nd-order Markov random field if

$$Y_A \perp Y_B \;\mid\; Y_{\partial^2 A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Trying again ...

$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

Question C:

Given $t > 0$, for any $A \subset V$ with ∂A finite and $B \subset V \setminus [A \cup \partial^2 A]$, is

$$X_A[t] \perp X_B[t] | X_{\partial^2 A}[t]?$$

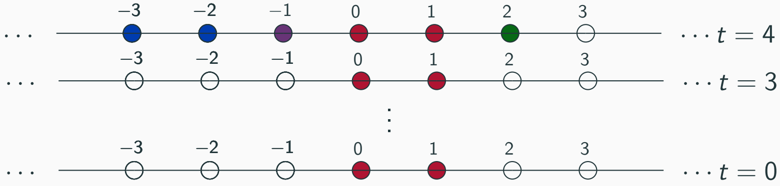
Trying again ...

$$X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right),$$

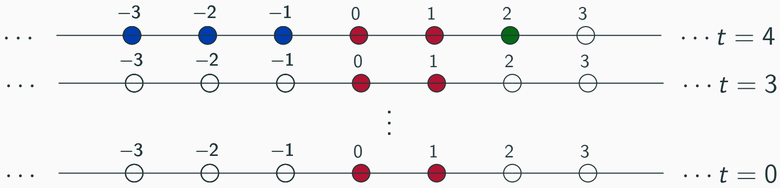
Question C:

Given $t > 0$, for any $A \subset V$ with ∂A finite and $B \subset V \setminus [A \cup \partial^2 A]$, is

$$X_A[t] \perp X_B[t] | X_{\partial^2 A}[t]?$$



Trying again ...

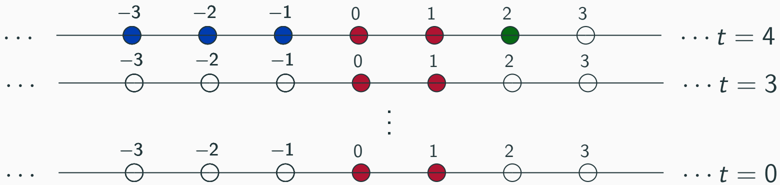


Theorem 5: (Lacker, R, Wu '18, Ganguly-R '22) YES!

$$X_A[t] \perp X_B[t] | X_{\partial^2 A}[t]$$

for all $A \subset V$ with ∂A finite and $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Trying again ...



Theorem 5: (Lacker, R, Wu '18, Ganguly-R '22) YES!

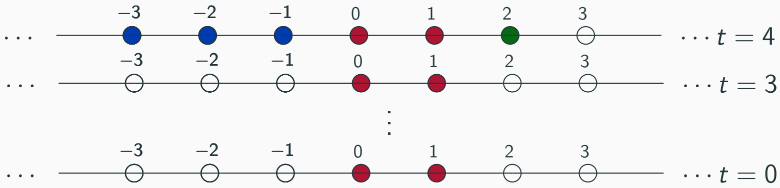
$$X_A[t] \perp X_B[t] | X_{\partial^2 A}[t]$$

for all $A \subset V$ with ∂A finite and $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Generalizations: In fact,

- this result holds even when $(X_v(0))_{v \in V}$ is just **a second-order MRF** – does not require $(X_v(0))_{v \in V}$ i.i.d.

Trying again ...



Theorem 5: (Lacker, R, Wu '18, Ganguly-R '22) YES!

$$X_A[t] \perp X_B[t] | X_{\partial^2 A}[t]$$

for all $A \subset V$ with ∂A finite and $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Generalizations: In fact,

- this result holds even when $(X_v(0))_{v \in V}$ is just **a second-order MRF** – does not require $(X_v(0))_{v \in V}$ i.i.d.

Related work for gradient diffusions: Deuschel ('87); Cattiaux, Roelly, Zessin ('96); ...

For non-gradient diffusions on $\mathbb{Z}^d$ with shift-invariant initial conditions: Dereudre and Roelly (2017)

From Conditional Independence to Local Equations

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.
- At each time $t \in \mathbb{N}_0$, given the past $Y_{-1,0,1}[t]$ and its law, recursively define the family of conditional laws: for $z \in \mathcal{X}$,

$$\gamma_t(z|\, y_0, y_1) = \mathbb{P}\big(Y_{-1}(t) = z \,|\, Y_0[t] = y_0[t],\, Y_1[t] = y_1[t]\big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

From Conditional Independence to Local Equations

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$.
- At each time $t \in \mathbb{N}_0$, given the past $Y_{-1,0,1}[t]$ and its law, recursively define the family of conditional laws: for $z \in \mathcal{X}$,

$$\gamma_t(z|\, y_0, y_1) = \mathbb{P}\big(Y_{-1}(t) = z \mid Y_0[t] = y_0[t],\, Y_1[t] = y_1[t]\big), \quad y_0, y_1 \in \mathcal{X}^\infty$$

- Sample independent ghost particles $Y_{-2}(t)$ and $Y_2(t)$ such that

$$\mathbb{P}\Big(Y_{-2}(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_{-1}[t],\, Y_0[t])$$

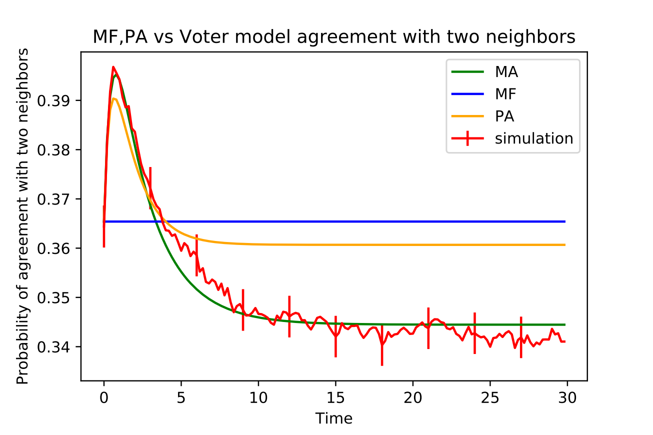
$$\mathbb{P}\Big(Y_2(t) = z | Y_{-1,0,1}[t]\Big) = \gamma_t(z|\, Y_1[t],\, Y_0[t])$$

- Sample new iid noises $(\xi_{-1}(t+1), \xi_0(t+1), \xi_1(t+1))$, and update:

$$\begin{aligned} Y_{-1}(t+1) &= F\Big(Y_{-1}(t), (Y_0(t), Y_{-2}(t)), \xi_{-1}(t+1)\Big) \\ Y_1(t+1) &= F\Big(Y_1(t), (Y_0(t), Y_2(t)), \xi_1(t+1)\Big) \\ Y_0(t+1) &= F\Big(Y_i(t), Y_{i-1,i+1}(t), \xi_i(t+1)\Big) \end{aligned}$$

- **Note**: $Y_{-1,0,1}(t+1)$ only depends on past $Y_{-1,0,1}[t]$ **and** its law

Comparison of Mean-Field (MF), pairwise (PA) and an approximation to the local equations (MA)



- One can extend the local equations to d-regular trees: the MRF property holds as stated, but one needs to use slightly different symmetries.

- One can extend the local equations to d-regular trees: the MRF property holds as stated, but one needs to use slightly different symmetries.
- One can extend this to random graphs using symmetries in the random graph – in this case, one needs to prove a conditional independence property that also involves the random structure of the random graph ...

Short Summary

- Established limit theorems for empirical measures and marginal dynamics of interacting stochastic processes on sequences of (converging) sparse graphs
  In particular resolves an open problem posed in the literature
- The marginal dynamics yield a new class of stochastic processes worthy of further study
- Provides a more accurate alternative to mean-field limits
- Tip of the icerberg – much work remains to further study equilibrium dynamics (partial results have been obtained) and to fully exploit these results to gain insight into specific applications

Main References for Jump Processes

- **Lacker, R. & Wu**, "Local weak convergence for large networks of interacting processes," *Annals of Applied Probability*, to appear, *Preprint*, arXiv:1904.02585 (2019).
- **Bhamidi, Nam, Nguyen and Sly**, "Survival and extinction of epidemics on random graphs with general degree",*Annals of Probability*, (2021).
- **Ganguly & R.**, "Hydrodynamic limits of non-Markovian interacting particle systems on sparse graphs," *Preprint*, arXiv:2205.01587v2, (2022).
- **Ganguly & R.**, "Interacting jump processes propagate semi-global Markov random fields," *Preprint*, (2022).
- **Ganguly & R.**, "Marginal dynamics of interacting particle systems on regular trees: stationarity properties and Markovian approximations", *In preparation*, (2022).
- **Lacker, R. & Wu**, "Marginal Dynamics of Probabilistic Cellular Automata on Trees," *Preprint* (2022).

Related References for Interacting Diffusions

- **Oliveira, Reis and Stolerman**, "Interacting diffusions on sparse graphs: hydrodynamics from weak limits", *EJP* (2020).
- **Lacker, R. & Wu**, "Locally interacting diffusions as space-time Markov random fields," *Stochastic Processes & their Applications* (2021).
- **Lacker, R. & Wu**, "Marginal dynamics of interacting diffusions on unimodular Galton-Watson trees," *Preprint*, arXiv:2009.11667 (2019).
- **Ichiba, Feng & Fouque**, "Linear-quadratic stochastic differential games on directed chain networks," *Jour. Math. Sci.*, (2021).
- **Ichiba, Feng & J.-P. Fouque**, "Linear-quadratic stochastic differential games on random directed networks," *Jour. Math. Sci.*, (2021).
- **R.**, "Interacting Stochastic Processes on Sparse Random Graphs" *ICM Proceedings, to appear* (2022).

THANK YOU!