Optimal Transport on Quantum Structures
– Workshop –

Book of Abstracts

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Invited talks

Infinite dimensional grassmannians and optimal transport of quantum states

Paolo Antonini
Università del Salento, Italy

We report on a recent work in collaboration with F. Cavalletti where we develop the basic theory of optimal transport for the quantum states of the C^*-algebra of the compact operators on a (separable) Hilbert space. As usual, states are interpreted as the noncommutative replacement of probability measures; via the spectral theorem applied to their density matrices, we associate to states discrete measures on the grassmannian of the finite dimensional subspaces. In this way we can treat them as ordinary probability measures and study the optimal transportation problem.

The metric geometry of the grassmannian, as an infinite dimensional manifold plays a decisive role and part of the talk will be devoted to describe its rich structure. Notably the grassmannian is an Alexandrov space with non negative curvature. Finally we will interpret pure normal states of the tensor product $H \otimes H$ as families of transport maps. This idea leads to the possibility of giving a definition of the Wasserstein cost for such objects.

Applications of OMT in proving isoperimetric and Sobolev inequalities

Zoltán Balogh
University of Bern, Switzerland

OMT is a powerful method in proving sharp isoperimetric and Sobolev inequalities in the general framework of metric spaces satisfying a curvature-dimension condition (CD(K,N) spaces). In this talk we focus on the case of CD(0,N) spaces with an asymptotic volume growth at infinity. We shall establish a sharp isoperimetric inequality in this setting and discuss its consequences. This is a joint work with Alexandru Kristaly.

Common variational principles for Euler, Einstein and Schroedinger equations

Yann Brenier
Universite Paris-Saclay, Orsay, France

In spite of their very different natures, the Euler, Einstein and Schroedinger Equations share very similar variational structures in strong connection with Optimal Transport Theory, involving density fields that are respectively real, matrix and complex-valued. A key point of the analysis is to write all these equations as quadratic system of conservation laws, in particular thanks to the Madelung transform in the case of Schroedinger's equation.

Random matching in $2d$: some recent results

Emanuele Caglioti
Sapienza University of Rome, Italy

I will consider the 2-dimensional random matching problem in two dimensional sets. In a challenging paper, Caracciolo et. al. on the basis of a subtle linearization of the Monge Ampere equation, conjectured that the expected value of the square of the Wasserstein distance, with exponent 2, between two samples of N uniformly distributed points in the unit square is $\log N/2\pi N$ plus corrections.This and other related conjectures has been proved by Ambrosio et al. in a series of challenging papers.

In the talk I will review the results cited above and some related conjectures and proofs also in the case in which the density is non constant and positive in a bounded set. Also, I will discuss the case of densities defined on all the plane.

Modified logarithmic Sobolev inequalities for quantum many-body systems

Angela Capel Cuevas
Universität Tübingen, Germany

Given a thermal quantum dissipative evolution modelled by a quantum Markov semigroup, its mixing time (i.e. time of convergence to its thermal equilibrium) can be bounded using optimal constants associated to a family of non-commutative functional inequalities. In this talk, we will focus on the so-called "modified logarithmic Sobolev inequality" (MLSI), which can be interpreted as the exponential decay rate for the relative entropy between an evolved state and the thermal equilibrium. The existence of an optimal constant for such an inequality is a sufficient condition for a quantum spin system to satisfy "rapid mixing", a property with strong implications in various contexts. For classical spin systems, the problem of estimating MLSI constants, under the assumption of a mixing condition in the Gibbs measure associated to their dynamics, is frequently addressed via a result of quasi-factorization of the entropy in terms of conditional entropies in some sub-algebras. In the past few years, we have extended such a technique to the quantum realm, where we have provided a strategy to prove quantum MLSIs under some decay of correlations on Gibbs states, via results of quasi-factorization of the quantum relative entropy. In this talk, we will present this strategy to prove quantum MLSIs for quantum Markov semigroups modelling thermal dissipative evolutions. We will finish with some specific examples of application of such a strategy to relevant quantum dynamics, with consequences, in particular, in the contexts of quantum memory devices, Gibbs states preparation and dissipative phases of matter.

A mixed-norm estimate of two-particle reduced density matrix of many-body Schrödinger dynamics

Li Chen
University of Mannheim, Germany

We provide a mixed-norm estimate of two-particle reduced density matrix of the solution of N-body Schrödinger equation. Using that we present a new approach to obtain the Vlasov dynamics from the Schrödinger equation through Hartree-Fock dynamics with $\hbar = N^{-1/3}$ as the re-scaled Plank constant. Furthermore, we provide that, in the sense of distribution, the mean-field residue term has higher rate than the semi-classical residue term. This is a joint work with Jinyeop Lee, Yue Li, and Matthew Liew.

The quantum Wasserstein distance of order 1

Giacomo De Palma
University of Bologna, Italy

We propose a generalization of the Wasserstein distance of order 1 to the quantum states of n qudits. The proposal recovers the Hamming distance for the vectors of the canonical basis, and more generally the classical Wasserstein distance for quantum states diagonal in the canonical basis. The proposed distance is invariant with respect to permutations of the qudits and unitary operations acting on one qudit and is additive with respect to the tensor product. We prove a continuity bound for the von Neumann entropy with respect to the proposed distance, which significantly strengthens the best continuity bound with respect to the trace distance. We also propose a generalization of the Lipschitz constant to quantum observables. The notion of quantum Lipschitz constant allows us to compute the proposed distance with a semidefinite program. We prove a quantum version of Marton's Transportation-Cost Inequality (TCI) for product states and a quantum Gaussian concentration inequality for the spectrum of quantum Lipschitz observables. Furthermore, we prove two generalizations of Marton's TCI to Gibbs states of quantum commuting Hamiltonians. The first relies on a quantum generalization of Dobrushin's uniqueness condition, while the second relies on a quantum generalization of Ollivier's coarse Ricci curvature. Finally, we apply our results to prove the equivalence of the canonical and microcanonical ensembles of quantum statistical mechanics.

The quantum 2-Wasserstein (semi-)distances

Michal Eckstein
Jagiellonian University Cracow, Poland

The optimal transport problem, established by Monge and refined by Kantorovich and Wasserstein, has ubiquitous applications in statistics, machine learning, computer vision and early Universe reconstruction. Recently, several approaches towards its quantum version have been proposed. In my talk I will present a direct generalisation of the 2-Wasserstein distance to the quantum realm, based on a specific class of quantum cost matrices. In the simplest case of a projector matrix, the induced quantity is a unitarily invariant semi-distance. Furthermore, it does enjoy the triangle inequality on the space of qubits and numerical evidence strongly suggest that this feature holds in any dimension. More general quantum cost matrices yield a weak distance structure and admit a formulation of the dual optimisation problem. Such quantum analogue of the 2-Wasserstein distance offers a new measure of proximity between quantum states and may prove useful in quantum Generative Adversarial Networks machine learning schemes.

The talk is based on joint works with Rafal Bistron, Sam Cole, Shmuel Friedland and Karol Zyczkowski, arXiv: 2102.07787, 2105.06922, 2204.07405.

Stability and singular limits in plasma physics

Mikaela Iacobelli
ETH Zurich, Switzerland

In this talk we will present two kinetic models that are used to describe the evolution of charged particles in plasmas: the Vlasov-Poisson system and the Vlasov-Poisson system with massless electrons. These systems model respectively the evolution of electrons, and ions in a plasma. We will discuss the well-posedness of these systems, the stability of solutions, and their behaviour under singular limits. Finally, we will introduce a new class of Wasserstein-type distances specifically designed to tackle stability questions for kinetic equations.

Mean-field and semiclassical limit: Wasserstein versus Schatten

Laurent Lafleche
University of Texas at Austin, USA

In this talk, I will review results concerning the problems of the combined mean-field and semiclassical limits from the N-body Scrödinger equation to the Hartree–Fock and Vlasov equations. As quantum analogue of the stability estimates for the Vlasov equation, the methods to get these limits involve either a quantum analogue of the Wasserstein distances introduced by Golse and Paul, or the quantum analogue of the Lebesgue norms defined using scaled Schatten norms.

The different methods do not give the same advantages, leading to different initial data, types of potentials, rates of convergence and time of validity of the estimates. They involve other classical concepts that are translated to the language of quantum theory, such as moments and quantum Sobolev spaces.

Matrix-valued Beckner inequalities

Haojian Li
Baylor University, USA

Beckner inequalities were introduced by Beckner in 1989 for the canonical Gaussian measures on $\mathbb{R}^n$. Beckner inequalities can be viewed an interpolation between logarithmic Sobolev inequalities and Poincare inequalities. I will briefly review classical Beckner inequalities and introduce the matrix-valued Beckner inequalities.

A non-commutative entropic optimal transport approach to quantum composite systems at positive temperature

Lorenzo Portinale
Hausdorff Center for Mathematics Bonn, Germany

The Schrödinger problem consists in finding the most likely evolution of a (random) Brownian particles which evolves from a given initial law to a given target one. It is in fact possible to equivalently describe this problem in terms of entropy regularisation of an optimal transport problem, by perturbing the classical quadratic Wasserstein transport problem with an entropy contribution, weighted by a factor that plays the role of the inverse of the temperature. In this talk, we study a multimarginal, non-commutative analogue of this problem, meaning an entropic regularised optimal transport problem between density matrices on finite dimensional Hilbert spaces. Physically, this describes a composite quantum system at positive temperature, under the knowledge of all its subsystems. Our main contributions are the proof of a duality formula for the multimarginal entropic problem and the introduction and proof of convergence to the optimal states of a non-commutative analogue of the Sinkhorn algorithm. The settings of Bosonic and Fermionic systems are also discussed, with a new variational interpretation of the Pauli's principle.

This is a joint work with Dario Feliciangeli and Augusto Gerolin, arXiv:2106.11217

Quantum extensions of Talagrand, KKL and Freidgut's theorems

Cambyse Rouzé
TU Munich, Germany

In 2008, Montanaro and Osborne [MO] proposed a quantum extension of Boolean functions, namely self-adjoint unitary matrices on $(\mathbb{C}^2)^{\otimes n}$, and extended the celebrated Talagrand L1-L2 variance inequality to this setting. In the classical case, the latter was shown to imply the so-called KKL theorem about the existence of an influential variable for a Boolean function as a simple corollary. However, since the L1 influences and L2 influences do not agree for general quantum Boolean functions, the extension of [MO] does not provide the quantum KKL theorem, in sharp contrast with the classical setting. Therefore a quantum version of the KKL theorem is still missing and was kept as a conjecture in [MO]. In this talk, I will argue that every balanced quantum Boolean function has a geometrically influential variable. I will also derive a quantum analogue of the related Friedgut junta theorem about the approximation of Boolean functions of small total influence by functions of few variables. These results are based on the joint use of recently studied hypercontractivity and gradient estimates. Such generic tools also allow us to derive generalizations of these results in a general von Neumann algebraic setting beyond the case of the quantum hypercube, including examples in infinite dimensions relevant to quantum information theory such as continuous variables quantum systems. If time permits, I will comment on the implications of our results as regards to noncommutative extensions of isoperimetric inequalities and the learnability of quantum observables. This is based on joint work with Melchior Wirth and Haonan Zhang (IST Austria).

Transport in free probability

Dimitri Shlyakhtenko
UCLA, USA

We discuss notions of monotone and optimal transport arising in Voiculescu's free probability theory, related to the Biane-Voiculescu extension of the quadratic Wasserstein distance to tracial non-commutative probability spaces. In particular, we discuss the behavior of Wasserstein distance under small free semicircular perturbations and relate it to free entropy dimension and cohomology.

The Differential Structure of Generators of GNS-Symmetric Quantum Markov Semigroups

Melchior Wirth
ISTA, Austria

Alicki's theorem (for semigroups on matrix algebras) and the results of Cipriani and Sauvageot (for tracially symmetric semigroups) show that Markov generators give rise to a first-order differential calculus. These results are crucial for the construction of the dynamical noncommutative transport distance by Carlen–Maas and the speaker. In this talk I will discuss an extension of these results to GNS-symmetric quantum Markov semigroups on arbitrary von Neumann algebras. Compared to the tracially symmetric case, the derivations satisfy a chain rule with a twist by the modular group, which is encapsulated in the new notion of Tomita bimodules. This could open up the way to extend the dynamical optimal transport methods to GNS-symmetric Markov semigroups on infinite-dimensional von Neumann algebras.

Curvature-dimension conditions for symmetric quantum Markov semigroups

Haonan Zhang
ISTA, Austria

The curvature-dimension condition consists of the lower Ricci curvature bound and upper dimension bound of the Riemannian manifold, which has a number of geometric consequences and is very helpful in proving many functional inequalities. The Bakry-Émery theory and Lott-Sturm-Villani theory allow to extend this notion beyond the Riemannian manifold setting and have seen great progress in the past decades. In this talk, I will first review several notions around lower Ricci curvature bounds in the noncommutative setting and present our work on gradient estimates. Then I will speak about two noncommutative versions of curvature-dimension conditions. Under suitable such curvature-dimension conditions, we prove a family of dimension-dependent functional inequalities, a version of the Bonnet-Myers theorem, and concavity of entropy power in the noncommutative setting. I will also give some examples coming from noncommutative analysis, quantum information and noncommutative probability. This is based on joint work (arXiv:2007.13506, arXiv:2105.08303) with Melchior Wirth (IST Austria).

Monge distance between quantum states

Karol Życzkowski
Jagiellonian University Cracow, Poland
and Center for Theoretical Physics, Poland

To measure the distance between any two quantum states one can consider the Monge distance between the corresponding Husimi distributions (Q-functions). This approach works for pure and mixed states for finite and infinite Hilbert spaces. As the Monge problem is solved with respect to the underlying classical distance the following semiclassical property holds: the distance between any two coherent quantum states is equal to the Euclidean distance between the corresponding points in the classical phase space. Hence this distance is not unitarily invariant and can be applied to analyze a quantum analogue of the Lyapunov exponent.

In the case of infinite dimensional Hilbert space one applies the standard harmonic oscillator coherent states, while for a N–dimensional space we rely on $SU(2)$ spin–coherent states. For several families of states it is possible to evaluate such a distance analytically. In the case of a finite N each pure state can be described in the stellar representation by a constellation of $k = N - 1$ Majorana stars – the zeros of the corresponding Husimi function. The simplified Monge distance can be then computed by solving a simpler, discrete Monge problem for two k-point probability distributions.

An alternative approach of Kantorovich and Wasserstein provides an explicit symmetry between both quantum states analyzed. Furthermore, such a definition does not depend on the selection of the set of coherent states. However, it depends on the choice of the cost matrix, while the optimization is performed over all coupling matrices representing quantum states in N^2–dimensional space with fixed partial traces.

Contributed talks

Quantum transport with Jordan product couplings

Matt Hoogsteder Riera
Universitat Autònoma de Barcelona, Spain

I will talk about our quantum extension of the Kantorovich formulation of classical transport using quantum state couplings based on the Jordan product. This approach stems from the theory of quantum states over time. I will present our preliminary results regarding the computability of the associated transport metric, general properties of this formulation, as well as some open problems that lie ahead.

Approximation of Splines in Wasserstein Spaces

Jorge Justiniano
University of Bonn, Germany

This paper investigates a time discrete variational model for splines in Wasserstein spaces to interpolate probability measures. Cubic splines in Euclidean space are known to minimize of the integrated squared acceleration subject to a set of interpolation constraints. As generalization on the manifold of probability measures the integral over the squared Riemannian acceleration is considered as a spline energy and adding the action functional a regularized spline energy is defined. Both energies are then discretized in time using local Wasserstein-2 distances and the generalized Wasserstein barycenter. The existence of time discrete regularized splines for given interpolation conditions is established and On the subspace of Gaussian distributions, explicit notions for the time discrete and time continuous splines are investigated. The implementation is based on classical optimal transport, entropy regularization and the Sinkhorn algorithm. A variant of the iPALM method is applied for the minimization of the fully discrete functional. A variety of numerical examples demonstrate the robustness of the approach and show striking characteristics of the approach. As a particular application the spline interpolation for synthesized textures is presented.

First-order Conditions for Optimization in the Wasserstein Space

Nicolas Lanzetti
ETH Zurich, Switzerland

We study first-order optimality conditions for constrained optimization in the Wasserstein space, whereby one seeks to minimize a real-valued function over the space of probability measures endowed with the Wasserstein distance. Our analysis combines recent insights on the geometry and the differential structure of the Wasserstein space with more classical calculus of variations. Perhaps surprisingly, we show that simple rationales such as "setting the derivative to zero" and "gradients are aligned at optimality" carry over to the Wasserstein space. We deploy our tools to study and solve optimization problems in the setting of distributionally robust optimization and statistical inference. The generality of our methodology allows us to naturally deal with functionals, such as the mean-variance, the Kullback-Leibler divergence, and the Wasserstein distance, which are traditionally difficult to study in a unified framework.

On the existence of derivations as square roots of generators of state-symmetric quantum Markov semigroups

Matthijs Vernooij
TU Delft, Netherlands

Cipriani and Sauvageot have shown that for any generator L of a tracially symmetric quantum Markov semigroup on a C^*-algebra A there exists a densely defined derivation δ from A to a Hilbert bimodule H such that $L = \delta^* \circ \delta$. Here we show that this construction of a derivation can in general not be generalised to quantum Markov semigroups that are symmetric with respect to a non-tracial state. In particular we show that all derivations to Hilbert bimodules can be assumed to have a concrete form, and then we use this form to show that in the finite-dimensional case the existence of such a derivation is equivalent to the existence of a positive matrix solution of a system of linear equations. We solve this system of linear equations for concrete examples using Mathematica to complete the proof.