TANGENT SETS AND ASSOUAD DIMENSION

ANTTI KÄENMÄKI AND ALEX RUTAR

Abstract. We discuss recent advancements in dimension theory related to tangents. Our main conclusion is that for self-affine sets, the maximal Hausdorff dimension of a tangent is equal to the Assouad dimension.

Contents

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1. Introduction

One of the fundamental geometric objects in analysis is the notion of a *tangent*. Often, tangents will exhibit substantially more regularity than the original object: for example, manifolds or rectifiable sets appear almost linear at almost every point and at sufficiently high resolutions. As we will see, tangents also provide a useful focal point for for discussing and defining purely "local" properties of a given object.

A old observation, and one which we find particularly motivating for these notes, is that the theory of tangents is indeed still very rich even for sets without global regularity. This is the case for many sets in fractal geometry, and we will see that a robust theory of tangents is a useful viewpoint to understand both fractal sets with structure (for instance, some form of dynamical invariance) and without structure.

The primary goal of this survey is to explore the relationship between tangents and geometry, and to see how structure can arise at microscopic resolutions which is not visible at macroscopic scales. We will see how tangents relate to the *Assouad dimension*, which is a coarse measurement of scaling which arose naturally in embedding theory, and also how tangents provide a robust language to quantify inhomogeneity in sets which are otherwise relatively homogeneous (such as overlapping self-similar sets, or self-affine sets).

In Section 2, we cover the necessary preliminaries, Section 3 then introduces the Assouad dimension and its fundamental properties, as well as defines the notion of a *weak tangent*. The main result in this section is that the Assouad dimension of any weak tangent is bounded above by the Assouad dimension of the original set.

Moving on to Section 4, we focus our attention on self-similar and self-affine sets. We review some results on separation conditions for self-similar sets in the real line, with a particular focus on those separation conditions which are relevant for Assouad dimension. We show that a self-similar set in the real line which does not satisfy the *weak separation condition* has a weak tangent which is an interval. In particular, the Assouad dimension of such a self-similar set is 1. We then use this result to construct a planar self-similar set whose Assouad dimension increases under an orthogonal projection.

In Section 5, we show that the maximal Hausdorff dimension of weak tangents is the Assouad dimension. Actually, we show that there exists a weak tangent with Hausdorff content (at the exponent given by the Assouad dimension) at least 1. We use this result to obtain a lower bound for the Assouad dimension of typical orthogonal projections.

Finally, in Section 6, we define the notion of a *tangent* and prove that any set for which the Assouad dimension is in a certain sense topologically stable that the maximal Hausdorff dimension of tangents is equal to the Assouad dimension. In particular, this result applies to self-affine sets. As an example, we also provide a careful study of a particular family of self-affine sets: *Bedford–McMullen carpets.*

2. Hausdorff content and dimension

2.1. **Density result for Hausdorff content.** For $0 < \delta \leqslant \infty$, we define the *$s$-dimensional Hausdorff δ-content* of $X \subset \mathbb{R}^d$ is

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_i \operatorname{diam}(U_i)^s : X \subset \bigcup_i U_i \text{ and } \operatorname{diam}(U_i) \leqslant \delta\right\}.$$

Note that the Hausdorff content can be equivalently defined by using covers consisting only of open sets or closed balls. The Hausdorff content is an outer measure—but we emphasize that is highly non-additive and not a Borel measure. Since the infimum is monotone with respect to inclusion, we have $\mathcal{H}^s_\eta(X) \leqslant \mathcal{H}^s_\delta(X)$ whenever $0 < \delta \leqslant \eta$. The *$s$-dimensional Hausdorff measure* of $X \subset \mathbb{R}^d$ is

$$\mathcal{H}^s(X) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(X) = \sup_{\delta > 0} \mathcal{H}^s_\delta(X).$$

The Hausdorff measure $\mathcal{H}^s$ is a Borel measure such that for every $X \subset \mathbb{R}^d$ there is a Borel set $X' \supset X$ such that $\mathcal{H}^s(X') = \mathcal{H}^s(X)$. It is straightforward to see that $\mathcal{H}^s(X) = 0$ if and only if $\mathcal{H}^s_\infty(X) = 0$.

Let us denote the Lebesgue measure on $\mathbb{R}^d$ by $\mathcal{L}^d$. Recall by the translation invariance and the scaling stability of the Lebesgue measure,

$$\mathcal{L}^d(B(x,r)) = r^d \alpha(d) \tag{2.1}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x,r)$ is the closed ball centered at x with radius r and $\alpha(d) = \mathcal{L}^d(B(0,1))$ is the volume of the unit ball.

Lemma 2.1. *If $0 \leqslant s \leqslant d$, then $\mathcal{H}^s_\infty(B(x,r)) = 2^s r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$.*

Proof. By the definition of the Hausdorff content, we have $\mathcal{H}^s_\infty(B(x,r)) \leqslant 2^s r^s$. Therefore, it suffices to show the other inequality. Let $\{B(x_i, r_i)\}_i$ be a countable cover of $B(x,r)$. Notice that, by (2.1), we have

$$r = \alpha(d)^{-\frac{1}{d}} \mathcal{L}^d(B(x,r))^{\frac{1}{d}}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Therefore, by the fact that $0 \leqslant \frac{s}{d} \leqslant 1$, the subadditivity and monotonicity of the Lebesgue measure imply

$$\begin{aligned}\sum_i (2r_i)^s &= 2^s \alpha(d)^{-\frac{s}{d}} \sum_i \mathcal{L}^d(B(x_i, r_i))^{\frac{s}{d}} \geqslant 2^s \alpha(d)^{-\frac{s}{d}} \Big(\sum_i \mathcal{L}^d(B(x_i, r_i)) \Big)^{\frac{s}{d}} \\ &\geqslant 2^s \alpha(d)^{-\frac{s}{d}} \mathcal{L}^d \Big(\bigcup_i B(x_i, r_i) \Big)^{\frac{s}{d}} \geqslant 2^s \alpha(d)^{-\frac{s}{d}} \mathcal{L}^d(B(x,r)^{\frac{s}{d}} = 2^s r^s.\end{aligned}$$

Since this holds for all countable covers $\{B(x_i, r_i)\}_i$, we get $\mathcal{H}^s_\infty(B(x,r)) \geqslant 2^s r^s$. □

The following theorem is a density result for the Hausdorff content.

Theorem 2.2. *If $X \subset \mathbb{R}^d$ and $\mathcal{H}^s(X) < \infty$, then*

$$1 \leqslant \limsup_{r \downarrow 0} \frac{\mathcal{H}^s_\infty(X \cap B(x,r))}{r^s} \leqslant 2^s$$

for $\mathcal{H}^s$-almost all $x \in X$.

Proof. The upper bound follows from the monotonicity of the Hausdorff content and Lemma 2.1. To see the lower bound, we prove that the set

$$A = \left\{ x \in X : \limsup_{r \downarrow 0} \frac{\mathcal{H}^s_\infty(X \cap B(x,r))}{r^s} < 1 \right\}$$

satisfies $\mathcal{H}^s(A) = 0$. Notice that $A \subset \bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}, \frac{1}{n}}$, where

$$A_{\lambda,\delta} = \{x \in X : \mathcal{H}^s_\infty(X \cap B(x,r)) \leqslant (1-\lambda) r^s \text{ for all } 0 < r \leqslant \delta\}$$

defined for all $0 < \lambda, \delta \leqslant 1$ satisfies $A_{\lambda,\lambda} \subset A_{\lambda,\delta}$ whenever $\delta \leqslant \lambda$. We observe that it suffices to show that

$$\mathcal{H}^s_\delta(A_{\lambda,\delta}) = 0 \tag{2.2}$$

for all $0 < \lambda, \delta \leqslant 1$. Assuming this, monotonicity of the Hausdorff content gives $\mathcal{H}^s_\delta(A_{\frac{1}{n},\frac{1}{n}}) \leqslant \mathcal{H}^s_\delta(A_{\frac{1}{n},\delta}) = 0$ for all $0 < \delta \leqslant \frac{1}{n}$ and, by letting $\delta \downarrow 0$, the subadditivity of the Hausdorff measure implies $\mathcal{H}^s(A) \leqslant \sum_{n\in\mathbb{N}} \mathcal{H}^s(A_{\frac{1}{n},\frac{1}{n}}) = 0$ as required.

To show (2.2), fix $0 < \lambda, \delta \leqslant 1$ and let $\varepsilon > 0$. Let $\{U_i\}_i$ be a cover of $A_{\lambda,\delta}$ such that $A_{\lambda,\delta} \cap U_i \neq \emptyset$ with $0 < \operatorname{diam}(U_i) \leqslant \delta$ for all i and

$$\sum_i \operatorname{diam}(U_i)^s \leqslant \mathcal{H}^s_\delta(A_{\lambda,\delta}) + \varepsilon.$$

Then $\mathcal{H}^s_\delta(A_{\lambda,\delta} \cap U_i) = \mathcal{H}^s_\infty(A_{\lambda,\delta} \cap U_i)$ and $A_{\lambda,\delta} \cap U_i \subset X \cap B(x_i, \operatorname{diam}(U_i))$, where $x_i \in A_{\lambda,\delta} \cap U_i$. By the subadditivity and monotonicity of the Hausdorff content, we thus have

$$\begin{aligned}
\mathcal{H}^s_\delta(A_{\lambda,\delta}) &\leqslant \sum_i \mathcal{H}^s_\delta(A_{\lambda,\delta} \cap U_i) = \sum_i \mathcal{H}^s_\infty(A_{\lambda,\delta} \cap U_i) \\
&\leqslant \sum_i \mathcal{H}^s_\infty(X \cap B(x_i, \operatorname{diam}(U_i))) \leqslant (1-\lambda) \sum_i \operatorname{diam}(U_i)^s \\
&\leqslant (1-\lambda)(\mathcal{H}^s_\delta(A_{\lambda,\delta}) + \varepsilon).
\end{aligned}$$

By letting $\varepsilon \downarrow 0$, we get $\mathcal{H}^s_\delta(A_{\lambda,\delta}) \leqslant (1-\lambda)\mathcal{H}^s_\delta(A_{\lambda,\delta})$. Since $\mathcal{H}^s_\delta(A_{\lambda,\delta}) \leqslant \mathcal{H}^s(A_{\lambda,\delta}) \leqslant \mathcal{H}^s(X) < \infty$ by the monotonicity of the Hausdorff measure, we see that this is possible only when $\mathcal{H}^s_\delta(A_{\lambda,\delta}) = 0$. □

2.2. **Hausdorff measure and Ahlfors regularity.** If $X \subset \mathbb{R}^d$ is a Borel set, then the Hausdorff measure satisfies

$$\mathcal{H}^s(X) = \sup\{\mathcal{H}^s(K) : K \subset X \text{ is compact such that } \mathcal{H}^s(K) < \infty\}; \tag{2.3}$$

see, for example, Mattila [28, Theorem 8.13]. It is easy to see that $\mathcal{H}^0$ is the counting measure and $\mathcal{H}^1$ is the length measure. Relying on the isodiametric inequality, it can be shown that $\mathcal{H}^d = 2^d\alpha(d)^{-1}\mathcal{L}^d$; see Evans and Gariepy [6, Theorem 2.5]. By (2.1) and Lemma 2.1, we thus have $\mathcal{H}^d(B(x,r)) = 2^d r^d = \mathcal{H}^d_\infty(B(x,r))$ for all $x \in \mathbb{R}^d$ and $r > 0$. Furthermore, if $f\colon X \to \mathbb{R}^d$ is a Lipschitz map, i.e. there is

$\lambda > 0$ such that

$$|f(x) - f(y)| \leqslant \lambda |x - y|$$

for all $x, y \in X$, then it follows easily that

$$\mathcal{H}^s(f(X)) \leqslant \lambda^s \mathcal{H}^s(X). \tag{2.4}$$

In particular, if $f\colon X \to \mathbb{R}^d$ is a bi-Lipschitz map, i.e. there are $\eta, \lambda > 0$ such that

$$\eta |x - y| \leqslant |f(x) - f(y)| \leqslant \lambda |x - y|$$

for all $x, y \in X$, then, by applying (2.4) to the inverse $f^{-1}\colon f(X) \to X$, we have

$$\eta^s \mathcal{H}^s(X) \leqslant \mathcal{H}^s(f(X)) \leqslant \lambda^s \mathcal{H}^s(X). \tag{2.5}$$

It follows that the Hausdorff measure $\mathcal{H}^s$ is translation and rotation invariant, and is also stable under scaling:

$$\mathcal{H}^s(X + z) = \mathcal{H}^s(X) \qquad \text{and} \qquad \mathcal{H}^s(\lambda X) = \lambda^s \mathcal{H}^s(X), \tag{2.6}$$

where $X + z = \{x + z : x \in X\}$ and $\lambda X = \{\lambda x : x \in X\}$. We note that (2.4)–(2.6) are valid also for the Hausdorff content. For a detailed treatment of basic properties of the Hausdorff measure, the reader is referred to Evans and Gariepy [6, §2], Falconer [10, §2], and Mattila [28, §4].

The properties of the Hausdorff measure can be studied by finding general measures with certain behavior. We say that a Borel measure μ on $\mathbb{R}^d$ is *Ahlfors s-regular* if there is $C \geqslant 1$ such that

$$C^{-1} r^s \leqslant \mu(B(x, r)) \leqslant C r^s$$

for all $x \in \operatorname{spt}(\mu)$ and $0 < r < \operatorname{diam}(\operatorname{spt}(\mu))$. A compact set $X \subset \mathbb{R}^d$ is *Ahlfors s-regular* if it supports an Ahlfors s-regular measure. A Borel measure μ satisfying $\mu(B(x, r)) \leqslant C r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$ is called an *s-Frostman measure.*

Theorem 2.3. *Let μ be a finite Borel measure on $\mathbb{R}^d$, $X \subset \mathbb{R}^d$ be a Borel set, and $0 < C < \infty$.*

(1) If for every $x \in X$ it holds that

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, r))}{r^s} \leqslant C,$$

then $\mathcal{H}^s(X) \geqslant C^{-1}\mu(X)$.

(2) *If for every* $x \in X$ *it holds that*

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} \geqslant C^{-1},$$

then $\mathcal{H}^s(X) \leqslant 2^s C\mu(X)$.

Proof. (1) Fix $\varepsilon > 0$ and write

$$X_n = \{x \in X : \mu(B(x,r)) \leqslant (C+\varepsilon)r^s \text{ for all } 0 < r \leqslant \tfrac{1}{n}\}$$

for all $n \in \mathbb{N}$. Notice that $X_1 \subset X_2 \subset \cdots$ and, by the assumption, $X = \bigcup_{n\in\mathbb{N}} X_n$. Let $\{U_i\}_i$ be a cover of X such that $\operatorname{diam}(U_i) \leqslant \frac{1}{n}$ for all i. For every i and n with $U_i \cap X_n \neq \emptyset$ we choose $x_{i,n} \in U_i \cap X_n$. Then clearly

$$X_n \subset \bigcup_{i\,:\,U_i\cap X_n\neq\emptyset} U_i \subset \bigcup_{i\,:\,U_i\cap X_n\neq\emptyset} B(x_{i,n}, \operatorname{diam}(U_i))$$

and hence, by the definition of X_n,

$$\mu(X_n) \leqslant \sum_{i\,:\,U_i\cap X_n\neq\emptyset} \mu(B(x_{i,n}, \operatorname{diam}(U_i))) \leqslant (C+\varepsilon)\sum_i \operatorname{diam}(U_i)^s$$

for all $n \in \mathbb{N}$. Since this holds for any $\frac{1}{n}$-cover of X, it follows that $\mu(X_n) \leqslant (C+\varepsilon)\mathcal{H}^s_{\frac{1}{n}}(X) \leqslant (C+\varepsilon)\mathcal{H}^s(X)$. Therefore, $\mu(X) = \lim_{n\to\infty}\mu(X_n) \leqslant (C+\varepsilon)\mathcal{H}^s(X)$ and the claim follows by letting $\varepsilon \downarrow 0$.

(2) Let $\varepsilon > 0$. By Mattila [28, Theorem 1.10(2)], there exists an open set $U \supset X$ such that $\mu(U \setminus X) < \varepsilon$. Fix $\delta > 0$ and define

$$\mathcal{B} = \{B(x,r) \subset U : x \in X,\ 0 < 2r \leqslant \delta, \text{ and } \mu(B(x,r)) \geqslant (C^{-1}-\varepsilon)r^s\}.$$

Notice that, by the assumption, $X \subset \bigcup_{B\in\mathcal{B}} B$ and $\inf\{r > 0 : B(x,r) \in \mathcal{B}\} = 0$ for all $x \in X$. Let $K \subset X$ be a compact set such that $\mathcal{H}^s(K) < \infty$. By Vitali's covering theorem for Radon measures, see Mattila [28, Theorem 2.8], there are pairwise disjoint balls $B_1, B_2, \ldots \in \mathcal{B}$ such that $\mathcal{H}^s(K \setminus \bigcup_{i\in\mathbb{N}} B_i) = 0$. Since the

Hausdorff content is subadditive, we get

$$\begin{aligned}\mathcal{H}^s_\delta(K) &\leqslant \mathcal{H}^s_\delta\biggl(\bigcup_{i\in\mathbb{N}} B_i\biggr) \leqslant \sum_{i\in\mathbb{N}} \operatorname{diam}(B_i)^s \leqslant 2^s(C^{-1}-\varepsilon)^{-1}\sum_{i\in\mathbb{N}}\mu(B_i)\\ &\leqslant 2^s(C^{-1}-\varepsilon)^{-1}\mu(U) \leqslant 2^s(C^{-1}-\varepsilon)^{-1}(\mu(X)+\varepsilon).\end{aligned}$$

By letting $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we see that $\mathcal{H}^s(K) \leqslant 2^s C\mu(X)$. Since this holds for all compact sets $K \subset X$ with $\mathcal{H}^s(K) < \infty$, the claim follows from (2.3). □

For more details on densities, the reader is referred to Mattila [28, §6] and Käenmäki [20]. Theorem 2.3 has numerous consequences. The first is an observation that the Hausdorff measure restricted to the Ahlfors regular set is Ahlfors regular.

Theorem 2.4. *A compact set $X \subset \mathbb{R}^d$ is Ahlfors s-regular if and only if there is $C \geqslant 1$ such that*

$$C^{-1}r^s \leqslant \mathcal{H}^s(X \cap B(x,r)) \leqslant Cr^s$$

for all $x \in X$ and $0 < r < \operatorname{diam}(X)$.

Proof. Since the restriction $\mathcal{H}^s|_X$ is a Borel measure, it suffices to prove that the Ahlfors regularity of X implies the Ahlfors regularity of $\mathcal{H}^s|_X$. Suppose that μ is a Borel measure supported on X and $C \geqslant 1$ such that

$$C^{-1}r^s \leqslant \mu(B(x,r)) \leqslant Cr^s \tag{2.7}$$

for all $x \in X$ and $0 < r < \operatorname{diam}(X)$. By Theorem 2.3 and (2.7), we have

$$C^{-2}r^s \leqslant C^{-1}\mu(B(x,r)) \leqslant \mathcal{H}^s(X \cap B(x,r)) \leqslant 2^s C\mu(B(x,r)) \leqslant 2^s C^2 r^s$$

for all $x \in X$ and $0 < r < \operatorname{diam}(X)$. □

The second consequence is that we can replace the Hausdorff content in Theorem 2.2 by the Hausdorff measure.

Theorem 2.5. *If $X \subset \mathbb{R}^d$ and $\mathcal{H}^s(X) < \infty$, then*

$$1 \leqslant \limsup_{r\downarrow 0} \frac{\mathcal{H}^s(X \cap B(x,r))}{r^s} \leqslant 2^s$$

for $\mathcal{H}^s$-almost all $x \in X$.

Proof. By Theorem 2.2 and the fact that $\mathcal{H}^s_\infty(A) \leqslant \mathcal{H}^s(A)$ for all sets $A \subset \mathbb{R}^d$, it suffices to prove the right-hand side inequality. Suppose to the contrary that there are $\lambda > 2^s$ and a Borel set $A \subset X$ with $\mathcal{H}^s(A) > 0$ such that

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^s|_X(B(x,r))}{r^s} \geqslant \lambda$$

for all $x \in A$, where $\mathcal{H}^s|_X$ is the restriction of $\mathcal{H}^s$ to X. By Theorem 2.3(2), it follows that $\mathcal{H}^s(A) \leqslant 2^s \lambda^{-1} \mathcal{H}^s|_X(A) < \mathcal{H}^s(A)$ which is a contradiction. $\square$

The third consequence is an observation that every set with positive Hausdorff measure supports a Frostman measure.

Theorem 2.6. *A Borel set $X \subset \mathbb{R}^d$ satisfies $\mathcal{H}^s(X) > 0$ if and only if there exists an s-Frostman measure μ such that $\mu(X) > 0$.*

Proof. Let us first assume that there are a Borel measure μ satisfying $\mu(X) > 0$ and a constant $C \geqslant 1$ such that $\mu(B(x,r)) \leqslant Cr^s$ for all $x \in \mathbb{R}^d$ and $r > 0$. Choose $R > 0$ such that $\mu(X \cap B(0,R)) > 0$ and let $\nu = \mu|_{B(0,R)}$ be the restriction of μ to $B(0,R)$. Since $\nu(B(x,r)) \leqslant Cr^s$ for all $x \in \mathbb{R}^d$ and $r > 0$ and, in particular, $\nu(\mathbb{R}^d) \leqslant CR^s < \infty$, Theorem 2.3(1) gives $\mathcal{H}^s(X) \geqslant C^{-1}\nu(X) = C^{-1}\mu(X \cap B(0,R)) > 0$ as required.

To prove the other direction, suppose that $\mathcal{H}^s(X) > 0$. Recalling (2.3), let $K \subset X$ be a compact set such that $0 < \mathcal{H}^s(K) < \infty$. By Theorem 2.5, the set

$$K' = \left\{ x \in K : \limsup_{r \downarrow 0} \frac{\mathcal{H}^s|_K(B(x,r))}{r^s} \leqslant 2^s \right\}$$

satisfies $\mathcal{H}^s(K \setminus K') = 0$. By Egorov's theorem, we find a compact set $K_0 \subset K'$ with $\mathcal{H}^s(K_0) > 0$ and $r_0 > 0$ such that $\mathcal{H}^s(K \cap B(x,r)) \leqslant 2^{1+s} r^s$ for all $x \in K_0$ and $0 < r < r_0$. Writing $\mu = \mathcal{H}^s|_{K_0}$, we thus have $\mu(K_0) > 0$ and

$$\mu(B(x,r)) \leqslant \mathcal{H}^s(K \cap B(x,r)) \leqslant 2^{1+s} r^s$$

for all $x \in \mathbb{R}^d$ and $0 < r < r_0$. Since

$$\frac{\mu(B(x,r))}{r^s} \leqslant \frac{\mathcal{H}^s(K)}{r_0^s}$$

for all $x \in \mathbb{R}^d$ and $r \geqslant r_0$, we have shown that μ is an s-Frostman measure with $\mu(X) \geqslant \mu(K_0) > 0$. $\square$

Theorem 2.3 is also required in proving the variational principle for the Hausdorff dimension, i.e. connecting Hausdorff dimensions of measures and their supports.

2.3. **Hausdorff dimension.** Suppose that $X \subset \mathbb{R}^d$ and $0 \leqslant s < t < \infty$. It is straightforward to see that if $\mathcal{H}^s(X) < \infty$, then $\mathcal{H}^t(X) = 0$, and if $\mathcal{H}^t(X) > 0$, then $\mathcal{H}^s(X) = \infty$. The *Hausdorff dimension* of X is

$$\begin{aligned} \dim_{\mathrm{H}}(X) &= \inf\{s \geqslant 0 : \mathcal{H}^s(X) < \infty\} = \inf\{s \geqslant 0 : \mathcal{H}^s(X) = 0\} \\ &= \sup\{s \geqslant 0 : \mathcal{H}^s(X) = \infty\} = \sup\{s \geqslant 0 : \mathcal{H}^s(X) > 0\}, \end{aligned}$$

where we interpret $\sup \emptyset = 0$. For example, if X is Ahlfors s-regular, then, by Theorem 2.4, $\dim_{\mathrm{H}}(X) = s$. Monotonicity of the Hausdorff measure implies that the Hausdorff dimension is monotone, i.e.

$$\dim_{\mathrm{H}}(X) \leqslant \dim_{\mathrm{H}}(X')$$

whenever $X \subset X' \subset \mathbb{R}^d$. Therefore, by the subadditivity of the Hausdorff measure, the Hausdorff dimension is countably stable, i.e.

$$\dim_{\mathrm{H}}\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \sup_{i\in\mathbb{N}} \dim_{\mathrm{H}}(X_i)$$

for all $X_1, X_2, \ldots \subset \mathbb{R}^d$. As a single point clearly has $\mathcal{H}^0$ measure one, countable stability implies that any countable set has zero Hausdorff dimension. It is easy to see that the Hausdorff dimension gives maximal dimension to open sets, i.e. $\dim_{\mathrm{H}}(U) = d$ for all open sets $U \subset \mathbb{R}^d$. Hence, $\dim_{\mathrm{H}}(\overline{\mathbb{Q}}) = \dim_{\mathrm{H}}(\mathbb{R}) = 1 > 0 = \dim_{\mathrm{H}}(\mathbb{Q})$ and the Hausdorff dimension is not stable under taking closure. Furthermore, if $f\colon X \to \mathbb{R}^d$ is a Lipschitz map, then it follows immediately from (2.4) that

$$\dim_{\mathrm{H}}(f(X)) \leqslant \dim_{\mathrm{H}}(X).$$

We thus see that orthogonal projections cannot increase the Hausdorff dimension. If $f\colon X \to \mathbb{R}^d$ is a bi-Lipschitz map, then (2.5) implies that

$$\dim_{\mathrm{H}}(f(X)) = \dim_{\mathrm{H}}(X).$$

In other words, the Hausdorff dimension is invariant under bi-Lipschitz maps. For a detailed treatment of basic properties of the Hausdorff dimension, the reader is referred to Falconer [10, §2] and Mattila [28, §4].

Suppose that μ is a Borel measure on $\mathbb{R}^d$. The *lower pointwise dimension* of μ at $x \in \mathbb{R}^d$ is

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}$$

and the *upper Hausdorff dimension* of μ is

$$\overline{\dim}_{\mathrm{H}}(\mu) = \operatorname*{ess\,sup}_{x \sim \mu} \underline{\dim}_{\mathrm{loc}}(\mu, x).$$

For example, if μ is Ahlfors s-regular, then $\overline{\dim}_{\mathrm{H}}(\mu) = s$. For more details on pointwise dimensions, the reader is referred to Falconer [9, §10]. The following variational principle is well-known in a form where the maximum is replaced by a supremum. Since the question whether the supremum can be attained does not seem to be so well documented, we modify Falconer, Fraser, and Käenmäki [11, Theorem 3.1] and present the full details in the following.

Theorem 2.7. *If $X \subset \mathbb{R}^d$ is a Borel set, then*

$$\dim_{\mathrm{H}}(X) = \max\{\overline{\dim}_{\mathrm{H}}(\mu) : \mu \text{ is a finite Borel measure on } X\}.$$

Proof. Fix a finite Borel measure μ supported on X and let $s < \overline{\dim}_{\mathrm{H}}(\mu)$. Choose a Borel set $A \subset X$ such that $\mu(A) > 0$ and $\underline{\dim}_{\mathrm{loc}}(\mu, x) > s$ for all $x \in A$. It follows that

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} \leqslant 1$$

for all $x \in A$ and hence, by Theorem 2.3(1), $\mathcal{H}^s(X) \geqslant \mathcal{H}^s(A) \geqslant \mu(A) > 0$ and $\dim_{\mathrm{H}}(X) \geqslant s$. By letting $s \uparrow \overline{\dim}_{\mathrm{H}}(\mu)$, we see that $\dim_{\mathrm{H}}(X) \geqslant \overline{\dim}_{\mathrm{H}}(\mu)$. Therefore, to prove the claim, it suffices to find a finite Borel measure μ supported on X such that $\dim_{\mathrm{H}}(X) \leqslant \overline{\dim}_{\mathrm{H}}(\mu)$.

Write $s_n = \dim_{\mathrm{H}}(X) - \frac{1}{n}$ for all $n \in \mathbb{N}$. Recall that, by (2.3), for each $n \in \mathbb{N}$ there exists a compact set $K_n \subset X$ such that $0 < \mathcal{H}^{s_n}(K_n) < \infty$. Define

$$\mu_n = \frac{\mathcal{H}^{s_n}|_{K_n}}{\mathcal{H}^{s_n}(K_n)} \qquad \text{and} \qquad \mu = \sum_{n \in \mathbb{N}} 2^{-n} \mu_n,$$

and note that μ is a Borel probability measure. Let $s > \overline{\dim}_{\mathrm{H}}(\mu)$ and notice that the set

$$A = \{x \in X : \underline{\dim}_{\mathrm{loc}}(\mu, x) \leqslant \overline{\dim}_{\mathrm{H}}(\mu)\}$$

has full measure, $\mu(A) = 1$. It follows that $\underline{\dim}_{\mathrm{loc}}(\mu, x) < s$ and

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} \geqslant 1$$

for all $x \in A$ and hence, by Theorem 2.3(2), $\mathcal{H}^s(A) \leqslant 2^s \mu(A) < \infty$ and $\dim_{\mathrm{H}}(A) \leqslant s$. By letting $s \downarrow \overline{\dim}_{\mathrm{H}}(\mu)$, we see that $\dim_{\mathrm{H}}(A) \leqslant \overline{\dim}_{\mathrm{H}}(\mu)$. Since $1 = \mu(A) = \sum_{n\in\mathbb{N}} 2^{-n}\mu_n(A)$, we have $\mu_n(A) = 1$ and $\mathcal{H}^{s_n}(K_n \cap A) = \mathcal{H}^{s_n}(K_n)$ for all $n \in \mathbb{N}$. Therefore, $\mathcal{H}^{s_n}(A) \geqslant \mathcal{H}^{s_n}(K_n \cap A) = \mathcal{H}^{s_n}(K_n) > 0$ and $\dim_{\mathrm{H}}(A) \geqslant s_n = \dim_{\mathrm{H}}(X) - \frac{1}{n}$ for all $n \in \mathbb{N}$. It follows that $\dim_{\mathrm{H}}(X) \leqslant \dim_{\mathrm{H}}(A) \leqslant \overline{\dim}_{\mathrm{H}}(\mu)$. $\square$

3. Assoud dimension and weak tangents

3.1. **Assouad dimension.** For a bounded set $A \subset \mathbb{R}^d$, the *r-covering number* of A, i.e.

$$N_r(A) = \min\biggl\{k \in \mathbb{N} : A \subset \bigcup_{i=1}^{k} B(x_i, r) \text{ for some } x_1, \ldots, x_k \in \mathbb{R}^d\biggr\},$$

is the least number of closed balls of radius $r > 0$ needed to cover A. The *Assouad dimension* of $X \subset \mathbb{R}^d$ is

$$\begin{aligned}\dim_{\mathrm{A}}(X) = \inf\biggl\{ s \geqslant 0 &: \text{there is } C \geqslant 1 \text{ such that for every } 0 < r < R \leqslant 1 \\ &\text{and } x \in X \text{ it holds that } N_r(X \cap B(x,R)) \leqslant C\Bigl(\frac{R}{r}\Bigr)^s \biggr\}.\end{aligned}$$

It follows immediately that the Assouad dimension is monotone, i.e.

$$\dim_{\mathrm{A}}(X) \leqslant \dim_{\mathrm{A}}(X')$$

whenever $X \subset X' \subset \mathbb{R}^d$. It is thus easy to see that the Assouad dimension is finitely stable, i.e.

$$\dim_{\mathrm{A}}(X \cup X') = \max\{\dim_{\mathrm{A}}(X), \dim_{\mathrm{A}}(X')\}$$

for all $X, X' \subset \mathbb{R}^d$. It is also straightforward to see that

$$\dim_{\mathrm{A}}(X \times X') \leqslant \dim_{\mathrm{A}}(X) + \dim_{\mathrm{A}}(X') \tag{3.1}$$

for all $X, X' \subset \mathbb{R}^d$. The inequality in (3.1) can be strict; see Robinson [30, §9.2]. The Assouad dimension is stable under taking closure, i.e. $\dim_{\mathrm{A}}(\overline{X}) = \dim_{\mathrm{A}}(X)$ for all $X \subset \mathbb{R}^d$, and it gives maximal dimension to open sets, i.e. $\dim_{\mathrm{A}}(U) = d$ for all open sets $U \subset \mathbb{R}^d$. Hence, $\dim_{\mathrm{A}}(\mathbb{Q}) = \dim_{\mathrm{A}}(\overline{\mathbb{Q}}) = \dim_{\mathrm{A}}(\mathbb{R}) = 1$ and the Assouad dimension is not countably stable since a single point clearly has zero Assouad dimension. On a related note, it is illustrative and straightforward to see that $\dim_{\mathrm{A}}(\{\frac{1}{n} : n \in \mathbb{N}\}) = 1$ and $\dim_{\mathrm{A}}(\{\frac{1}{2^n} : n \in \mathbb{N}\}) = 0$. Finally, the Assouad dimension is invariant under bi-Lipschitz maps, i.e.

$$\dim_{\mathrm{A}}(f(X)) = \dim_{\mathrm{A}}(X)$$

for all $X \subset \mathbb{R}^d$ and bi-Lipschitz maps $f \colon X \to \mathbb{R}^d$. For a detailed treatment of basic properties of the Assouad dimension, the reader is referred to Luukkainen [25, §3] and Fraser [15, §2].

The following lemma shows that the Hausdorff dimension is bounded above by the Assouad dimension.

Lemma 3.1. *If $X \subset \mathbb{R}^d$, then $\dim_{\mathrm{H}}(X) \leqslant \dim_{\mathrm{A}}(X)$.*

Proof. By the countable stability of the Hausdorff dimension, it suffices to show that

$$\dim_{\mathrm{H}}(X \cap B(x,1)) \leqslant \dim_{\mathrm{A}}(X)$$

for all $x \in X$. Let $s > \dim_{\mathrm{A}}(X)$ and notice that, by the definition of the Assouad dimension, there exists $C \geqslant 1$ such that for every $x \in X$ and $0 < r < 1$ we have

$$N_r(X \cap B(x,1)) \leqslant C\Big(\frac{1}{r}\Big)^s.$$

Hence, by the definition of the Hausdorff content,

$$\mathcal{H}^s_r(X \cap B(x,1)) \leqslant N_r(X \cap B(x,1))(2r)^s \leqslant 2^s C.$$

By letting $r \downarrow 0$, we get $\mathcal{H}^s(X \cap B(x,1)) < \infty$ and $\dim_{\mathrm{H}}(X \cap B(x,1)) \leqslant s$. The claim follows by letting $s \downarrow \dim_{\mathrm{A}}(X)$. $\square$

The following result shows that on Ahlfors regular sets the dimensions coincide.

Lemma 3.2. *If $X \subset \mathbb{R}^d$ is Ahlfors s-regular, then* $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{A}}(X) = s$.

Proof. By Theorem 2.4, we have $\dim_{\mathrm{H}}(X) = s$. Therefore, by Lemma 3.1, it is enough to show that $\dim_{\mathrm{A}}(X) \leqslant s$. Fix $x \in X$ and $0 < r < R \leqslant 1$. Let $\{x_i\}_{i\in I} \subset X \cap B(x,R)$ be a maximal collection of points such that the family $\{B(x_i, \frac{r}{2})\}_{i\in I}$ is pairwise disjoint. Note that, by maximality, we have $X \cap B(x,R) \subset \bigcup_{i\in I} B(x_i, r)$. Since $\bigcup_{i\in I} X \cap B(x_i, \frac{r}{2}) \subset X \cap B(x, 2R)$, it follows from Theorem 2.4 that there is $C \geqslant 1$ such that

$$\begin{aligned} C^{-1}\#I(\tfrac{r}{2})^s &\leqslant \sum_{i\in I} \mathcal{H}^s(X \cap B(x_i, \tfrac{r}{2})) = \mathcal{H}^s\Bigl(\bigcup_{i\in I} X \cap B(x_i, \tfrac{r}{2})\Bigr) \\ &\leqslant \mathcal{H}^s(X \cap B(x, 2R)) \leqslant C(2R)^s. \end{aligned}$$

It follows that $N_r(X \cap B(x,R)) \leqslant \#I \leqslant 4^s C^2 (\frac{R}{r})^s$ and $\dim_{\mathrm{A}}(X) \leqslant s$. □

3.2. **Weak tangents.** Let $X \subset \mathbb{R}^d$ be closed and $\mathcal{K}(X) = \{K \subset X \text{ is compact}\}$. The *Hausdorff distance* $d_{\mathrm{H}} \colon \mathcal{K}(X) \times \mathcal{K}(X) \to [0,\infty)$ is defined by setting

$$d_{\mathrm{H}}(A,B) = \max\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\}$$

for all $A, B \in \mathcal{K}(X)$. It is easy to see that $\mathcal{K}(X)$ equipped with the Hausdorff distance is a complete metric space; consult e.g. Edgar [5, §2.5]. Furthermore, the topology generated by the Hausdorff distance is the *Vietoris topology* whose basis consists of sets of the form

$$\langle U_1, \dots, U_n \rangle = \Bigl\{ K \in \mathcal{K}(X) : K \subset \bigcup_{i=1}^n U_i \text{ and } U_i \cap K \neq \emptyset \text{ for all } i \Bigr\},$$

where $U_1, \dots, U_n$ are non-empty open subsets of X. The following result is proved by Mattila and Mauldin [29, Theorem 2.1].

Lemma 3.3. *If $X \subset \mathbb{R}^d$ is closed, then the function $\mathcal{H}^s_\infty \colon \mathcal{K}(X) \to [0,\infty)$ is upper semicontinuous.*

Proof. Let $K \in \mathcal{K}(X)$ and notice that $\mathcal{H}^s_\infty(K) \leqslant \operatorname{diam}(K)^s < \infty$. Fix $c \in [0,\infty)$. By the definition of the Hausdorff content and compactness of K, we have $\mathcal{H}^s_\infty(K) <$

c if and only if there are open sets $U_1, \dots, U_n$ intersecting K such that

$$K \subset \bigcup_{i=1}^{n} U_i \qquad \text{and} \qquad \sum_{i=1}^{n} \operatorname{diam}(U_i)^s < c.$$

Since the set $\langle U_1, \dots, U_n \rangle$, where $\sum_{i=1}^{n} \operatorname{diam}(U_i)^s < c$, is open in the Vietoris topology, we conclude that $\{K \in \mathcal{K}(X) : \mathcal{H}^s_\infty(K) < c\}$ is open and the function $\mathcal{H}^s_\infty \colon \mathcal{K}(X) \to [0, \infty)$ is upper semicontinuous. □

For each $x \in X$ and $r > 0$ the *magnification* at x by r is the homothety $M_{x,r} \colon \mathbb{R}^d \to \mathbb{R}^d$ for which

$$M_{x,r}(z) = \frac{z - x}{r}$$

for all $z \in \mathbb{R}^d$. A set $T \subset \mathbb{R}^d$ is a *weak tangent* of X if there are sequences $(x_n)_{n \in \mathbb{N}}$ of points in X and $(r_n)_{n \in \mathbb{N}}$ of positive reals such that $\lim_{n \to \infty} r_n = 0$ and

$$M_{x_n, r_n}(X) \cap B(0,1) \to T$$

in Hausdorff distance. We denote the collection of all weak tangents of X by $\operatorname{Tan}(X)$. The following lemma, proved by Käenmäki and Rossi [22, Lemma 3.11], shows that weak tangents of weak tangents are weak tangents.

Lemma 3.4. *If $X \subset \mathbb{R}^d$ is closed, then* $\operatorname{Tan}(\operatorname{Tan}(X)) \subset \operatorname{Tan}(X)$.

Proof. Fix $T \in \operatorname{Tan}(X)$ and $T' \in \operatorname{Tan}(T)$. Let $(M_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ be sequences of homotheties such that

$$M_n(X) \cap B(0,1) \to T \qquad \text{and} \qquad L_n(T) \cap B(0,1) \to T'$$

in Hausdorff distance. Let $\varepsilon > 0$ and choose N such that

$$d_{\mathrm{H}}(T', L_N(T) \cap B(0,1)) < \frac{\varepsilon}{2}.$$

Write $L_N(x) = \lambda_N x + t_N$ and choose P such that

$$d_{\mathrm{H}}(T, M_P(X) \cap B(0,1)) < \frac{\varepsilon}{2\lambda_N}.$$

It follows that

$$d_{\mathrm{H}}(T', L_N \circ M_P(X) \cap B(0,1)) < \varepsilon.$$

The claim follows by letting $\varepsilon \downarrow 0$. □

For positive Hausdorff measure sets, the following lemma, proved by Käenmäki and Rutar [23, Lemma 2.6], shows the existence of a weak tangent set having Hausdorff content bounded from below.

Lemma 3.5. *If $X \subset \mathbb{R}^d$ is closed and $\mathcal{H}^s(X) > 0$, then there exists $T \in \operatorname{Tan}(X)$ such that $\mathcal{H}^s_\infty(T) \geqslant 1$.*

Proof. Recalling (2.3), let $K \subset X$ be a compact set such that $0 < \mathcal{H}^s(K) < \infty$. By Theorem 2.2, we have

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^s_\infty(K \cap B(x,r))}{r^s} \geqslant 1$$

for $\mathcal{H}^s$-almost all $x \in K$. Fix such a point $x \in K$. Noticing that $M_{x,r}(K) \cap B(0,1) = r^{-1}(K - x) \cap B(0,1) = r^{-1}((K \cap B(x,r)) - x)$ and recalling (2.6), we can rewrite the above inequality as

$$\limsup_{r \downarrow 0} \mathcal{H}^s_\infty(M_{x,r}(K) \cap B(0,1)) \geqslant 1.$$

Since $\mathcal{H}^s_\infty \colon \mathcal{K}(K) \to [0, \infty)$ is upper semicontinuous by Lemma 3.3, there are $T' \in \operatorname{Tan}(K)$ and a sequence $(r_n)_{n \in \mathbb{N}}$ of positive reals such that $\lim_{n \to \infty} r_n = 0$,

$$M_{x,r_n}(K) \cap B(0,1) \to T'$$

in Hausdorff distance, and $\mathcal{H}^s_\infty(T') \geqslant 1$. Since $M_{x,r_n}(K) \cap B(0,1) \subset M_{x,r_n}(X) \cap B(0,1)$ for all $n \in \mathbb{N}$ and, by possibly going into a subsequence, there is $T \in \operatorname{Tan}(X)$ such that

$$M_{x,r_n}(X) \cap B(0,1) \to T$$

in Hausdorff distance, we have $T' \subset T$ by compactness. Hence, by the monotonicity of the Hausdorff content, $\mathcal{H}^s_\infty(T) \geqslant \mathcal{H}^s_\infty(T') \geqslant 1$. □

Our goal is to generalize the previous lemma to show the existence of such a weak tangent set when the parameter s is the Assouad dimension. This goal will be achieved in Theorem 5.7. To that end, we compare the Assouad dimensions of a set and its weak tangent sets.

Let $\mathbb{Q} \cap [0,1] = \{q_i : i \in \mathbb{N}\}$ be an enumeration of rationals in the unit interval and $X_n = \{q_i : i \in \{1, \ldots, n\}\}$ be a finite set in $[0,1]$. The Assouad dimension is finitely stable and we have $\dim_{\mathrm{A}}(X_n) = 0$ for all $n \in \mathbb{N}$. Since $X_n \to [0,1]$ in Hausdorff distance and $\dim_{\mathrm{A}}([0,1]) = 1$, we see that the Assouad dimension is not necessarily continuous with respect to the Hausdorff distance[1]. The following lemma is proved by Mackay and Tyson [27, Proposition 6.1.5].

Lemma 3.6. *If* $X \subset \mathbb{R}^d$ *is closed, then* $\dim_{\mathrm{A}}(T) \leqslant \dim_{\mathrm{A}}(X)$ *for all* $T \in \mathrm{Tan}(X)$.

Proof. Fix $T \in \mathrm{Tan}(X)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X and $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive reals such that $\lim_{n\to\infty} r_n = 0$ and

$$M_{x_n,r_n}(X) \cap B(0,1) \to T$$

in Hausdorff distance. Without loss of generality, we may assume that $r_n < 1$ for all $n \in \mathbb{N}$. Let $s > \dim_{\mathrm{A}}(X)$ be arbitrary and, by definition of the Assouad dimension, get a constant $C > 0$ so that

$$N_r(B(x,R) \cap X) \leqslant C\Big(\frac{R}{r}\Big)^s$$

for all $x \in X$ and $0 < r \leqslant R < 1$.

Now let $y \in T$ and $0 < r \leqslant R < 1$ be arbitrary. By definition of T, let $n \in \mathbb{N}$ be such that

$$d_{\mathrm{H}}(M_{x_n,r_n}(X) \cap B(0,1), T) \leqslant r.$$

Now, consider the ball $M^{-1}_{x_n,r_n}(B(y,R))$ which contains some point $x \in X$, so in particular $M^{-1}_{x_n,r_n}(B(y,R)) \subset B(x, 2Rr_n)$. By choice of C,

$$N_{rr_n}(B(x,2Rr_n) \cap X) \leqslant C2^d\Big(\frac{R}{r}\Big)^s.$$

Therefore,

$$N_r(M_{x_n,r_n}(X) \cap B(0,1) \cap B(y,R)) \leqslant C2^d\Big(\frac{R}{r}\Big)^s.$$

But each element of T is distance at most r from some element in in $M_{x_n,r_n}(X) \cap B(0,1)$, and moreover each ball $B(x,2r)$ can be covered by D_d balls of radius r,

[1]This observation also disproves [24, Lemma 2.16].

where D_d is some constant depending only on d. Thus

$$\begin{aligned} N_r(T \cap B(y,R)) &\leqslant D_d N_r(M_{x_n,r_n}(X) \cap B(0,1) \cap B(y,R)) \\ &\leqslant D_d 2^d C \Big(\frac{R}{r}\Big)^s. \end{aligned}$$

Since $y \in T$ and $0 < r \leqslant R < 1$ were arbitrary, we conclude that $\dim_{\mathrm{A}}(T) \leqslant s$. Since $s > \dim_{\mathrm{A}}(X)$ was arbitrary, we conclude that $\dim_{\mathrm{A}}(T) \leqslant \dim_{\mathrm{A}}(X)$, as claimed. □

We conclude this section with an immediate corollary.

Corollary 3.7. *If $X \subset \mathbb{R}^d$ is closed, then*

$$\sup\{\dim_{\mathrm{H}}(T) : T \in \mathrm{Tan}(X)\} \leqslant \dim_{\mathrm{A}}(X).$$

Proof. The claim follows directly from Lemmas 3.1 and 3.6. □

4. Self-similar and self-affine sets

4.1. **Self-affine sets and affinity dimension.** Let $\Phi = (\varphi_1, \ldots, \varphi_N)$ be a tuple of affine maps $\varphi_i \colon \mathbb{R}^d \to \mathbb{R}^d$, $\varphi_i(x) = A_i x + v_i$, where $\mathsf{A} = (A_1, \ldots, A_N) \in \mathrm{GL}_d(\mathbb{R})^N$ is a tuple of contractive invertible matrices and $(v_1, \ldots, v_N) \in (\mathbb{R}^d)^N$ is a tuple of translation vectors. By the classical result of Hutchinson [19, §3.1], there exists a unique non-empty compact set $X \subset \mathbb{R}^d$ such that

$$X = \bigcup_{i=1}^{N} \varphi_i(X). \tag{4.1}$$

The set X is called the *self-affine set* associated to Φ. To avoid triviality, we assume that X has at least two points. If the tuple A consists only of a constant multiple of orthogonal matrices, then the maps φ_i are similarities and the self-affine set is called *self-similar*. We use the convention that whenever we speak about a self-affine set, then it is automatically accompanied with a tuple of affine maps which defines it.

We recall that the *singular values* of $A \in \mathrm{GL}_d(\mathbb{R})$ are defined to be the non-negative square roots of the eigenvalues of the positive-semidefinite matrix $A^\top A$ and are denoted $\alpha_1(A), \ldots, \alpha_d(A)$ in non-increasing order. The identities $\alpha_1(A) = \|A\|$,

$\alpha_d(A) = \|A^{-1}\|^{-1}$, and $\prod_{i=1}^d \alpha_i(A) = |\det A|$ are standard. For each $s \geqslant 0$ we define the *singular value function* by setting

$$\varphi^s(A) = \begin{cases} \alpha_1(A) \cdots \alpha_{\lfloor s \rfloor}(A) \alpha_{\lceil s \rceil}(A)^{s - \lfloor s \rfloor}, & \text{if } 0 \leqslant s \leqslant d, \\ |\det(A)|^{\frac{s}{2}}, & \text{if } s > d. \end{cases}$$

The value $\varphi^s(A)$ represents a measurement of the s-dimensional volume of the ellipse $A(B(0,1))$. Note that $\alpha_d(A)^s \leqslant \varphi^s(A) = \|A\|^s$ for all $0 \leqslant s \leqslant 1$ and $\alpha_d(A)^s \leqslant \varphi^s(A) \leqslant \|A\|^s$ for all $s > 1$. By Falconer [7, Lemma 2.1] (see also Käenmäki and Morris [20, §3.4]), the singular value function is sub-multiplicative meaning that $\varphi^s(AB) \leqslant \varphi^s(A)\varphi^s(B)$ for all $A, B \in \mathrm{GL}_d(\mathbb{R})$.

Let $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ be the collection of all infinite words obtained from the letters $\{1, \ldots, N\}$. If $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$, then we define $\sigma \mathtt{i} = \sigma(\mathtt{i}) = i_2 i_3 \cdots$ and $\mathtt{i}|_n = i_1 \cdots i_n$ for all $n \in \mathbb{N}$. The empty word $\mathtt{i}|_0$ is denoted by $\varnothing$. Define $\Sigma_n = \{\mathtt{i}|_n : \mathtt{i} \in \Sigma\}$ for all $n \in \mathbb{N}$ and $\Sigma_* = \bigcup_{n \in \mathbb{N}} \Sigma_n \cup \{\varnothing\}$. Thus Σ_* is the collection of all finite words. The length of a word $\mathtt{j}$ is denoted by $|\mathtt{j}|$ and the concatenation of a finite word $\mathtt{i}$ and $\mathtt{j}$ is denoted by $\mathtt{ij}$. If $\mathtt{i} \in \Sigma_*$, then by $\mathtt{i}^k$ we mean the word $\mathtt{ii} \cdots \mathtt{i}$ where $\mathtt{i}$ is repeated k times. We write

$$\begin{aligned} \varphi_{\mathtt{i}} &= \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}, \\ A_{\mathtt{i}} &= A_{i_1} \cdots A_{i_n} \end{aligned}$$

for all $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$ and $n \in \mathbb{N}$.

For each $\mathsf{A} = (A_1, \ldots, A_N) \in \mathrm{GL}_d(\mathbb{R})^N$ and $s \geqslant 0$ we define the *singular value pressure* by setting

$$P(\mathsf{A}, s) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\mathtt{i} \in \Sigma_n} \varphi^s(A_{\mathtt{i}}).$$

By the sub-multiplicativity of the singular value function, $(\log \sum_{\mathtt{i} \in \Sigma_n} \varphi^s(A_{\mathtt{i}}))_{n \in \mathbb{N}}$ is a sub-additive sequence and hence, the limit above exists by Fekete's lemma. It is also easy to see that the pressure $P(\mathsf{A}, s)$ is continuous and strictly decreasing as a function of s with $P(\mathsf{A}, 0) \geqslant 0$ and $\lim_{s \to \infty} P(\mathsf{A}, s) = -\infty$. We may thus define the *affinity dimension* by setting $\dim_{\mathrm{aff}}(\mathsf{A})$ to be the unique $s \geqslant 0$ for which $P(\mathsf{A}, s) = 0$. If X is a self-affine set, then by $\dim_{\mathrm{aff}}(X)$ we mean the affinity dimension $\dim_{\mathrm{aff}}(\mathsf{A})$. Relying on (4.1), the self-affine set X can naturally be covered by the sets $\varphi_{\mathtt{i}}(B)$, where B is a ball containing X. Observe that in the planar case, each ellipse $\varphi_{\mathtt{i}}(B)$ can be covered by one ball of radius $\alpha_1(A_{\mathtt{i}})\operatorname{diam}(B)$ or by $\alpha_1(A_{\mathtt{i}})/\alpha_2(A_{\mathtt{i}})$ many balls of radius $\alpha_2(A_{\mathtt{i}})\operatorname{diam}(B)$. This gives a motivation to study the limiting behavior of the sums $\sum_{\mathtt{i}\in\Sigma_n}\varphi^s(A_{\mathtt{i}})$ and indeed, it is straightforward to see that

$$\dim_{\mathrm{H}}(X) \leqslant \min\{d, \dim_{\mathrm{aff}}(X)\}; \tag{4.2}$$

see Falconer [7, Theorem 5.4].

The *canonical projection* $\pi\colon \Sigma \to X$ is defined by setting

$$\pi\mathtt{i} = \pi(\mathtt{i}) = \sum_{n=1}^{\infty} A_{\mathtt{i}|_{n-1}} v_{i_n}$$

for all $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$. It is easy to see that the image of Σ is the self-affine set, $\pi(\Sigma) = X$. Separation conditions allow simple interplay between Σ and X. We say that X satisfies the *strong separation condition* if $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ whenever $i \neq j$. The strong separation condition is characterized by the requirement that the canonical projection is one-to-one. We say that X satisfies the *open set condition* if there exists an open set $U \subset \mathbb{R}^2$ such that $\varphi_i(U) \subset U$ for all $i \in \{1, \ldots, N\}$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ whenever $i \neq j$. If such a set U also intersects X, then we say that X satisfies the *strong open set condition*. Observe that the strong separation condition implies the strong open set condition.

4.2. **Self-similar sets and separation conditions.** Let us next survey some known results on separation conditions for self-similar sets. Fix a self-similar set $X \subset \mathbb{R}^d$ and let $\mathsf{A} = (r_1 O_1, \ldots, r_N O_N)$, where $0 < r_i < 1$ and $O_i \in O(d)$ for all $i \in \{1, \ldots, N\}$, be the associated tuple of matrices. In this case, the affinity dimension is called *similarity dimension* and we denote it by $\dim_{\mathrm{sim}}(X)$. Notice that $\dim_{\mathrm{sim}}(X)$ is the unique $s \geqslant 0$ for which $\sum_{i=1}^{N} r_i^s = 1$. Let us endow the group of all affine maps with the topology of pointwise convergence and define

$$\begin{aligned}\Sigma(x,r) = \{\mathtt{i} \in \Sigma_* : \operatorname{diam}(\varphi_{\mathtt{i}}(X)) \leqslant r < \operatorname{diam}(\varphi_{\mathtt{i}^-}(X)) \\ \text{and } \varphi_{\mathtt{i}}(X) \cap B(x,r) \neq \emptyset\}\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Recall that $\mathcal{H}^s(X) < \infty$ for $s = \dim_{\mathrm{H}}(X)$ by Falconer [8, Theorem 4]. The following theorem characterizes $\mathcal{H}^s(X) > 0$ for $s = \dim_{\mathrm{sim}}(X)$.

Theorem 4.1. *If $X \subset \mathbb{R}^d$ is a self-similar set and $s = \dim_{\mathrm{sim}}(X)$, then the following seven conditions are equivalent:*

(1) *X satisfies the open set condition,*
(2) *X satisfies the strong open set condition,*
(3) $\sup\{\#\Sigma(x,r) : x \in X$ *and* $r > 0\} < \infty$,
(4) *the identity is not in the closure of* $\{\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Sigma_*$ *such that* $\mathtt{i} \neq \mathtt{j}\}$,
(5) *there is $\eta > 0$ such that $|\varphi_{\mathtt{i}} - \varphi_{\mathtt{j}}| \geqslant \eta \operatorname{diam}(\varphi_{\mathtt{i}}(X))$ for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$ with $\mathtt{i} \neq \mathtt{j}$,*
(6) $\mathcal{H}^s(X) > 0$,
(7) *X is Ahlfors s-regular,*

Proof. Notice that (2) ⇒ (1) is a triviality and (7) ⇒ (6) follows from Theorem 2.4. Hutchinson [19, §5.3] proved the implications (1) ⇒ (3) ⇒ (7), Bandt and Graf [2] showed that (6) ⇔ (4) ⇔ (5), and finally, Schief [32, Theorem 2.1] verified the remaining implication (6) ⇒ (2). □

We say that a self-similar set $X \subset \mathbb{R}^d$ satisfies the *weak separation condition* if

$$\sup\{\#\Phi(x,r) : x \in X \text{ and } r > 0\} < \infty,$$

where

$$\Phi(x,r) = \{\varphi_{\mathtt{i}} : \operatorname{diam}(\varphi_{\mathtt{i}}(X)) \leqslant r < \operatorname{diam}(\varphi_{\mathtt{i}^-}(X)) \\ \text{and } \varphi_{\mathtt{i}}(X) \cap B(x,r) \neq \emptyset\}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. By Theorem 4.1, the open set condition is valid if and only if the weak separation condition holds and $\varphi_{\mathtt{i}} \neq \varphi_{\mathtt{j}}$ for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$ with $\mathtt{i} \neq \mathtt{j}$.

Theorem 4.2. *If $X \subset \mathbb{R}^d$ is a self-similar set, then the following three conditions are equivalent:*

(1) *X satisfies the weak separation condition,*
(2) *the identity is not a limit point of* $\{\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Sigma_*$ *such that* $\mathtt{i} \neq \mathtt{j}\}$,
(3) *there is $\eta > 0$ such that $|\varphi_{\mathtt{i}} - \varphi_{\mathtt{j}}| \geqslant \eta \operatorname{diam}(\varphi_{\mathtt{i}}(X))$ for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$ with $\varphi_{\mathtt{i}} \neq \varphi_{\mathtt{j}}$.*

Furthermore, if $s = \dim_{\mathrm{H}}(X)$*, then the following three conditions follow from the above conditions:*

(4) $\mathcal{H}^s(X) > 0$,
(5) X *is Ahlfors* s*-regular,*
(6) $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{A}}(X)$.

Proof. It follows from Zerner [33, Theorem 1] that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Note that [33, Corollary after Proposition 2] verifies the implication (1) $\Rightarrow$ (4) and, by [12, Corollary 3.1], we have (4) $\Leftrightarrow$ (5). The implication (5) $\Rightarrow$ (6) follows immediately from Lemma 3.2. □

The weak separation condition is intimately connected to the behavior of weak tangets. The following result is by Fraser, Henderson, Olson, and Robinson [16, Theorem 3.1]. Angelevska, Käenmäki, and Troscheit [1, Theorem 4.1] generalized the argument for a more general setting which proof Rutar [31, Theorem 3.4] then modified with simplifications for the self-similar case.

Theorem 4.3. *If a self-similar set* $X \subset \mathbb{R}$ *does not satisfy the weak separation condition, then there exists* $T \in \mathrm{Tan}(X)$ *such that* $\dim_{\mathrm{H}}(T) = 1$.

Proof. Let $(\varphi_1, \ldots, \varphi_N)$ be the associated tuple of similarities $\varphi_i \colon \mathbb{R} \to \mathbb{R}$, $\varphi_i(x) = r_i x + v_i$, where $0 < |r_i| < 1$ and $v_i \in \mathbb{R}$. Without loss of generality, we may assume that $\varphi_1(0) = 0$ and $r_1 > 0$. Since X does not satisfy the weak separation condition, the identity is a limit point of $\{\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Sigma_* \text{ such that } \mathtt{i} \neq \mathtt{j}\}$ by Theorem 4.2. In other words, for each $\varepsilon > 0$ there exist $\mathtt{i} \neq \mathtt{j}$, $0 \leqslant \delta < \varepsilon$, and $|\gamma - 1| < \varepsilon$ such that $(\delta, \gamma) \neq (0, 1)$ and

$$\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}}(x) = \gamma x + \delta \tag{4.3}$$

for all $x \in X$. By appending at most two letters to $\mathtt{i}$ and $\mathtt{j}$ if necessary, we may assume that $r_{\mathtt{i}}, r_{\mathtt{j}}, \delta > 0$.

Fix $m \in \mathbb{N}$ and let $\mathtt{h}_1 = \varnothing$ and $\varepsilon_1 = 1$. For each $\ell \in \{1, \ldots, m\}$ use (4.3) to choose $\mathtt{i}_\ell, \mathtt{j}_\ell \in \Sigma_*$ and $k_\ell \in \mathbb{N}_0$ recursively so that with

$$\begin{aligned}\mathtt{h}_\ell &= \mathtt{j}_{\ell-1} 1^{k_{\ell-1}} \cdots \mathtt{j}_1 1^{k_1},\\ \varepsilon_\ell &= r_1^{k_1+\cdots+k_{\ell-1}} r_{\mathtt{i}_1} \cdots r_{\mathtt{i}_{\ell-1}}\end{aligned}$$

it holds that:

(1) $\varphi_{\mathtt{i}_\ell}^{-1} \circ \varphi_{\mathtt{j}_\ell}(x) = \gamma_\ell x + \delta_\ell$ for all $x \in X$, where $(\delta_\ell, \gamma_\ell) \neq (0,1)$ and
$$0 \leqslant \delta_\ell < \frac{\varepsilon_\ell}{m} \qquad \text{and} \qquad |\gamma_\ell - 1| < \frac{r_1 \varepsilon_\ell}{2m|\varphi_{\mathtt{h}_\ell}(0)|},$$

(2) k_ℓ satisfies
$$\frac{r_1}{m} < \frac{r_1^{-k_\ell}\delta_\ell}{\varepsilon_\ell} \leqslant \frac{1}{m}.$$

Set

$$\begin{aligned}\mathtt{k}_\ell &= \mathtt{i}_m 1^{k_m} \cdots \mathtt{i}_\ell 1^{k_\ell},\\ \varrho &= r_1^{k_1+\cdots+k_m} r_{\mathtt{i}_1} \cdots r_{\mathtt{i}_m}\end{aligned}$$

and note that, by construction,

$$\varphi_{1^{k_\ell}}^{-1} \circ \varphi_{\mathtt{i}_\ell}^{-1} \circ \varphi_{\mathtt{j}_\ell} \circ \varphi_{1^{k_\ell}}(x) = \gamma_\ell x + r_1^{-k_\ell}\delta_\ell$$

for all $x \in X$. Therefore,

$$\begin{aligned}\varphi_{\mathtt{k}_{\ell+1}\mathtt{h}_{\ell+1}}(0) - \varphi_{\mathtt{k}_\ell \mathtt{h}_\ell}(0) &= \varphi_{\mathtt{k}_\ell} \circ \varphi_{1^{k_\ell}}^{-1} \circ \varphi_{\mathtt{i}_\ell}^{-1} \circ \varphi_{\mathtt{j}_\ell} \circ \varphi_{1^{k_\ell}} \circ \varphi_{\mathtt{h}_\ell}(0) - \varphi_{\mathtt{k}_\ell \mathtt{h}_\ell}(0)\\ &= r_1^{k_\ell+\cdots+k_m} r_{\mathtt{i}_\ell} \cdots r_{\mathtt{i}_m}(r_1^{-k_\ell}\delta_\ell + \varphi_{\mathtt{h}_\ell}(0)(\gamma_\ell - 1))\\ &= \varrho \Delta_\ell\end{aligned}$$

for some Δ_ℓ satisfying $\frac{r_1}{2} \leqslant m\Delta_\ell \leqslant 1 + \frac{r_1}{2}$ by the choice of k_ℓ. In particular,

$$\{0, \Delta_1, \ldots, \Delta_1 + \cdots + \Delta_{m-1}\} \subset \frac{X - \varphi_{\mathtt{k}_1\mathtt{h}_1}(0)}{\varrho} = M_{\varphi_{\mathtt{k}_1\mathtt{h}_1}(0),\varrho}(X).$$

By letting $m \to \infty$, we conclude that $[0,1] \in \mathrm{Tan}(X)$ which finishes the proof. $\square$

As a direct corollary, we see that in the real line all six conditions of Theorem 4.2 are equivalent.

Corollary 4.4. *If $X \subset \mathbb{R}$ is a self-similar set such that $s = \dim_{\mathrm{H}}(X) < 1$, then the following four conditions are equivalent:*

(1) X satisfies the weak separation condition,
(2) $\mathcal{H}^s(X) > 0$,
(3) X is Ahlfors s-regular,
(4) $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{A}}(X)$.

Proof. By Theorem 4.2 and its proof, it suffices to show that (4) $\Rightarrow$ (1). If X does not satisfy the weak separation condition, then Theorem 4.3 guarantees the existence of $T \in \mathrm{Tan}(X)$ for which $\dim_{\mathrm{H}}(T) = 1$. As a consequence, the assumption together with Lemmas 3.1 and 3.6 show that $\dim_{\mathrm{H}}(X) < 1 = \dim_{\mathrm{H}}(T) \leqslant \dim_{\mathrm{A}}(T) \leqslant \dim_{\mathrm{A}}(X)$ which contradicts with (4). $\square$

We can also use Theorem 4.3 to show that, perhaps a bit surprisingly, Lipschitz maps can increase the Assouad dimension.

Example 4.5. We follow Fraser [13, §3.1] and construct a planar self-similar set whose Assouad dimension increases under an orthogonal projection. Let $0 < \alpha, \beta, \gamma < 1$ and choose $\varphi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$\begin{aligned}
\varphi_1(x, y) &= \alpha(x, y),\\
\varphi_2(x, y) &= \beta(x, y) + (0, 1 - \beta),\\
\varphi_3(x, y) &= \gamma(x, y) + (1 - \gamma, 0).
\end{aligned}$$

Denote the self-similar set associated to $(\varphi_1, \varphi_2, \varphi_3)$ by X and let $\mathrm{proj}_V \colon \mathbb{R}^2 \to V$, $\mathrm{proj}_V(x, y) = x$, be the orthogonal projection onto the x-axis V, which we identify with $\mathbb{R}$. It is easy to see that $\mathrm{proj}_V(X)$ is a self-similar set associated to (ψ_1, ψ_2, ψ_3), where the maps $\psi_i \colon \mathbb{R} \to \mathbb{R}$ are such that

$$\begin{aligned}
\psi_1(x) &= \alpha x,\\
\psi_2(x) &= \beta x,\\
\psi_3(x) &= \gamma x + (1 - \gamma).
\end{aligned}$$

We may now clearly choose $0 < \alpha, \beta, \gamma < 1$ so that X satisfies the open set condition and $\dim_{\mathrm{H}}(X) < 1$ but $\mathrm{proj}_V(X)$ does not satisfy the weak separation condition. For example, Fraser [13, §3.1] calculated that the choices $\alpha = 2^{-\sqrt{3}}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{10}$ work. Theorem 4.3 guarantees the existence of $T \in \mathrm{Tan}(\mathrm{proj}_V(X))$ for which $\dim_{\mathrm{H}}(T) = 1$. Lemmas 3.1 and 3.6 show that $1 = \dim_{\mathrm{H}}(T) \leqslant \dim_{\mathrm{A}}(T) \leqslant \dim_{\mathrm{A}}(\mathrm{proj}_V(X)) \leqslant 1$. Therefore, by Theorem 4.1 and Lemma 3.2, we have

$$\dim_{\mathrm{A}}(\mathrm{proj}_V(X)) = 1 > \dim_{\mathrm{H}}(X) = \dim_{\mathrm{A}}(X)$$

as claimed.

5. Maximal weak tangent

5.1. **Dyadic cubes.** An interval $I \subset [0,1)$ is called *dyadic* if it is of the form

$$I = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$$

for some integers $j, n \in \mathbb{N}_0$. If $I_1, \ldots, I_d$ are dyadic intervals of the same length, then the product

$$Q = I_1 \times \cdots \times I_d \subset [0,1)^d$$

is a *dyadic cube*. The collection of all dyadic cubes of side length 2^{-n} is denoted by $\mathcal{Q}_n$. We also write $\mathcal{Q} = \bigcup_{n \in \mathbb{N}_0} \mathcal{Q}_n$. It is straightforward to see that if Q and Q' are dyadic cubes such that $Q \cap Q' \neq \emptyset$, then they are contained in each other, i.e. $Q \subset Q'$ or vice versa. Therefore, if $Q \in \mathcal{Q}_n$ is a dyadic cube, then there is a unique dyadic cube, called the *parent* of Q, in $\mathcal{Q}_{n-1}$ which contains Q. Similarly, the 2^d dyadic cubes of $\mathcal{Q}_{n+1}$ contained in Q are called the *children* of Q. In particular, the dyadic cubes $\mathcal{Q}$ can be obtained by applying an iterated function system $(\varphi_1, \ldots, \varphi_{2^d})$ satisfying the open set condition, where $\varphi_1, \ldots, \varphi_{2^d}$ are the unique homotheties taking $[0,1)^d$ surjectively to $Q_1, \ldots, Q_{2^d} \in \mathcal{Q}_1$, respectively.

In this section, we study the Assouad dimension of sets in $[0,1)^d$. This is not a restriction since the Assouad dimension is invariant under bi-Lipschitz maps and any bounded set can be scaled and translated into $[0,1)^d$. For a set $X \subset \mathbb{R}^d$, the *dyadic n-covering number* of X, i.e.

$$D_n(X) = \#\{Q' \in \mathcal{Q}_n : X \cap Q' \neq \emptyset\},$$

is the least number of dyadic cubes of side length 2^{-n} needed to cover $X \cap [0,1)^d$.

Lemma 5.1. *If* $X \subset [0,1)^d$ *and* $0 < t < \dim_{\mathrm{A}}(X)$*, then there are* $m, n \in \mathbb{N}_0$ *and* $Q \in \mathcal{Q}_m$ *such that*

$$\# D_n(X \cap Q) \geqslant 2^{(n-m)t}.$$

Proof. By the definition of the Assouad dimension there are $0 < r < R \leqslant 1$ and $x \in X$ such that

$$N_r(X \cap B(x,R)) > 2^{d+2t}\sqrt{d}^t \Big(\frac{R}{r}\Big)^t.$$

Let $m \in \mathbb{N}_0$ be such that $2^{-m-1} < 2R \leqslant 2^{-m}$. Notice that the closed ball $B(x,R)$ is contained in an union of at most 2^d dyadic cubes in $\mathcal{Q}_m$. Therefore, by the pigeonhole principle, there exists $Q \in \mathcal{Q}_m$ such that

$$N_r(X \cap Q) > 2^{2t}\sqrt{d}^t \Big(\frac{R}{r}\Big)^t > \sqrt{d}^t \Big(\frac{2^{-m}}{r}\Big)^t. \tag{5.1}$$

Let $n \in \mathbb{N}_0$ be such that $\sqrt{d}2^{-n-1} \leqslant r < \sqrt{d}2^{-n}$. Write $k = \# D_n(X \cap Q)$ and let $Q_1', \ldots, Q_k' \in \mathcal{Q}_n$ be such that $Q_i' \subset Q$ and $Q_i' \cap X \neq \emptyset$. Since each Q_i' is contained in a closed ball B_i of radius r, we see that

$$X \cap Q \subset \bigcup_{i=1}^k Q_i' \subset \bigcup_{i=1}^k B_i$$

and $N_r(X \cap Q) \leqslant k$. Therefore, by (5.1),

$$\# D_n(X \cap Q) = k \geqslant \sqrt{d}^t \Big(\frac{2^{-m}}{r}\Big)^t > \Big(\frac{2^{-m}}{2^{-n}}\Big)^t$$

as claimed. □

For each $Q \in \mathcal{Q}$ we let $M_Q \colon \mathbb{R}^d \to \mathbb{R}^d$ be the unique homothety sending Q surjectively to $[0,1)^d$. We define the *maximal relative dyadic n-covering number* of $X \subset [0,1)^d$ to be

$$D_n^*(X) = \max_{Q \in \mathcal{Q}_m} D_n(M_Q(X)) = \max_{\substack{Q \in \mathcal{Q}_m \\ m \in \mathbb{N}_0}} D_{m+n}(X \cap Q).$$

Lemma 5.1 shows that for each $0 < t < \dim_{\mathrm{A}}(X)$ there is $n \in \mathbb{N}$ such that $D_n^*(X) \geqslant 2^{nt}$. We will strenghten this to hold for all large enough n. Furstenberg [18, Lemma 5.1] observed that the sequence $(D_n^*(A))_{n \in \mathbb{N}}$ is submultiplicative.

Lemma 5.2. *If* $X \subset [0,1)^d$*, then* $D_{n+k}^*(X) \leqslant D_n^*(X) D_k^*(X)$ *for all* $n, k \in \mathbb{N}$.

Proof. Let $Q \in \mathcal{Q}$ be such that $D^*_{n+k}(X) = D_{n+k}(M_Q(X))$. If $Q' \in \mathcal{Q}_n$ satisfies $M_Q(X) \cap Q' \neq \emptyset$, then $D_{n+k}(M_Q(X) \cap Q') = D_k(M_{Q'} \circ M_Q(X)) \leqslant D^*_k(X)$ and

$$\begin{aligned} D^*_{n+k}(X) &\leqslant \sum_{Q' \in \mathcal{Q}_n : M_Q(X) \cap Q' \neq \emptyset} D_{n+k}(M_Q(X) \cap Q') \\ &\leqslant D_n(M_Q(X)) D^*_k(X) \leqslant D^*_n(X) D^*_k(X) \end{aligned}$$

as claimed. □

Relying on Fekete's lemma for subadditive sequences, we may now write

$$\Delta(X) = \lim_{n\to\infty} \frac{\log D^*_n(X)}{\log 2^n} \tag{5.2}$$

for all sets $X \subset [0,1)^d$. Note that for each $0 < t < \Delta(X)$ there is $n_0 \in \mathbb{N}$ such that $D^*_n(X) \geqslant 2^{nt}$ for all $n \geqslant n_0$. The following lemma, proved by Käenmäki and Rossi [22, Proposition 3.13], shows that the Assouad dimension is bounded above by Δ.

Lemma 5.3. *If* $X \subset [0,1)^d$, *then* $\dim_{\mathrm{A}}(X) \leqslant \Delta(X)$.

Proof. Let $s > \Delta(X)$ and choose n_0 such that $D^*_n(X) < 2^{ns}$ for all $n \geqslant n_0$. Fix $0 < r < R \leqslant 1$ and $x \in X$. Let $m \in \mathbb{N}_0$ be such that $2^{-m-1} < 2R \leqslant 2^{-m}$ and notice that the closed ball $B(x,R)$ is contained in an union of at most 2^d dyadic cubes in $\mathcal{Q}_m$. If $r < \sqrt{d}2^{-n_0+1}R$, then we choose $n \geqslant n_0$ such that $\sqrt{d}2^{-m-n-1} \leqslant r < \sqrt{d}2^{-m-n}$. Let $Q \in \mathcal{Q}_m$ be such that $Q \cap B(x,R) \neq \emptyset$ and notice that $D_n(M_Q(X)) = D_{m+n}(X \cap Q) \leqslant D^*_n(X)$. Write $k = D_n(M_Q(X))$ whence $k < 2^{ns}$. Let $Q'_1, \ldots, Q'_k \in \mathcal{Q}_n$ be such that $M_Q(X) \cap [0,1)^d \subset \bigcup_{i=1}^k Q'_i$ and denote the center of each Q'_i by $x_i \in \mathbb{R}^d$. Since $\bigcup_{i=1}^k Q'_i \subset \bigcup_{i=1}^k B(x_i, 2^m r)$, we have

$$N_r(X \cap Q) = N_{2^m r}(M_Q(X) \cap [0,1)^d) \leqslant k < 2^{ns}.$$

Therefore,

$$\begin{aligned} N_r(X \cap B(x,R)) &\leqslant \sum_{Q \in \mathcal{Q}_m \,:\, Q \cap B(x,R) \neq \emptyset} N_r(X \cap Q) \leqslant 2^d N_r(X \cap Q) \\ &< 2^d 2^{ns} = 2^{d+2s} \Big(\frac{2^{-m-2}}{2^{-m-n}} \Big)^s < 2^{d+2s} \sqrt{d}^s \Big(\frac{R}{r} \Big)^s. \end{aligned}$$

If $\sqrt{d}2^{-n_0+1}R \leqslant r < R$, then the previous estimate applied to $\varrho = 2^{-n_0}R$ satisfying $2^{-n_0}r < \varrho < r$ gives

$$\begin{aligned} N_r(X \cap B(x,R)) &\leqslant N_\varrho(X \cap B(x,R)) \\ &< 2^{d+2s}\sqrt{d}^s\left(\frac{R}{\varrho}\right)^s < 2^{d+(n_0+2)s}\sqrt{d}^s\left(\frac{R}{r}\right)^s \end{aligned}$$

and, consequently, $\dim_{\mathrm{A}}(X) \leqslant s$. The claim follows by letting $s \downarrow \Delta(X)$. □

5.2. **Discrete Frostman measure.** If a collection $\{w_Q\}_{Q\in\mathcal{Q}_n}$ of non-negative real numbers satisfies $\sum_{Q\in\mathcal{Q}_n} w_Q = 1$, then the Borel probability measure

$$\mu = \sum_{Q\in\mathcal{Q}_n} w_Q \frac{\mathcal{L}^d|_Q}{\mathcal{L}^d(Q)}$$

is the $\mathcal{Q}_n$-*discrete measure* with respect to weights $\{w_Q\}_{Q\in\mathcal{Q}_n}$. Suppose that μ is a Borel probability measure on $[0,1)^d$ and $Q \in \mathcal{Q}_m$ is such that $\mu(Q) > 0$ for some $m \in \mathbb{N}_0$. Recall that M_Q is the unique homothety sending Q surjectively to $[0,1)^d$, $\mu|_Q$ is the restriction of μ to Q, and $(M_Q)_*\mu$ is the push-forward of μ under M_Q. The Borel probability measure

$$\mu^Q = \frac{(M_Q)_*(\mu|_Q)}{\mu(Q)}$$

is the *magnification* of μ with respect to Q. The following lemma shows that a magnified discrete measure is a discrete measure.

Lemma 5.4. *If μ is a $\mathcal{Q}_n$-discrete measure with respect to weights $\{w_Q\}_{Q\in\mathcal{Q}_n}$ and $Q \in \mathcal{Q}_m$ is such that $\mu(Q) > 0$ where $m \in \{0,\ldots,n-1\}$, then μ^Q is a $\mathcal{Q}_{n-m}$-discrete measure with respect to weights $\{\mu(Q)^{-1}w_{M_Q^{-1}(Q')}\}_{Q'\in\mathcal{Q}_{n-m}}$.*

Proof. Notice first that

$$\begin{aligned} \mu^Q = \frac{(M_Q)_*(\mu|_Q)}{\mu(Q)} &= \mu(Q)^{-1} \sum_{Q'\in\mathcal{Q}_n\,:\,Q'\subset Q} w_{Q'} \frac{\mathcal{L}^d|_{M_Q(Q')}}{\mathcal{L}^d(M_Q(Q'))} \\ &= \sum_{Q'\in\mathcal{Q}_{n-m}} \mu(Q)^{-1} w_{M_Q^{-1}(Q')} \frac{\mathcal{L}^d|_{Q'}}{\mathcal{L}^d(Q')}. \end{aligned}$$

Since

$$\begin{aligned}\sum_{Q' \in \mathcal{Q}_{n-m}} \mu(Q)^{-1} w_{M_Q^{-1}(Q')} &= \mu(Q)^{-1} \sum_{Q' \in \mathcal{Q}_n \,:\, Q' \subset Q} w_{Q'} \\ &= \mu(Q)^{-1} \sum_{Q' \in \mathcal{Q}_n} w_{Q'} \frac{\mathcal{L}^d|_{Q'}(Q)}{\mathcal{L}^d(Q')} = 1,\end{aligned}$$

the Borel probability measure μ^Q is a $\mathcal{Q}_{n-m}$-discrete measure with respect to weights $\{\mu(Q)^{-1} w_{M_Q^{-1}(Q')}\}_{Q' \in \mathcal{Q}_{n-m}}$. □

The maximal relative dyadic covering number introduces us a discrete measure supported on a neighborhood of a magnification of the set.

Lemma 5.5. *For every $n \in \mathbb{N}$ there exist $Q \in \mathcal{Q}$ and a $\mathcal{Q}_n$-discrete measure with respect to weights $\{w_{Q'}\}_{Q' \in \mathcal{Q}_n}$, where*

$$w_{Q'} = \begin{cases} D_n^*(X)^{-1}, & \textit{if } M_Q(X) \cap Q' \neq \emptyset, \\ 0, & \textit{if } M_Q(X) \cap Q' = \emptyset \end{cases}$$

and $M_Q \colon \mathbb{R}^d \to \mathbb{R}^d$ is the unique homothety sending Q surjectively to $[0,1)^d$.

Proof. Fix $n \in \mathbb{N}$ and let $Q \in \mathcal{Q}$ be such that $D_n^*(X) = D_n(M_Q(X))$. Since

$$\sum_{Q' \in \mathcal{Q}_n} w_{Q'} = \sum_{Q' \in \mathcal{Q}_n \,:\, M_Q(X) \cap Q' \neq \emptyset} D_n(M_Q(X))^{-1} = 1,$$

the claim follows. □

A Borel probability measure μ on $[0,1)^d$ is a $\mathcal{Q}_n$-*discrete s-Frostman measure* if

$$\mu(Q) \leqslant \operatorname{diam}(Q)^s$$

for all $Q \in \mathcal{Q}_\ell$ and $\ell \in \{0, \ldots, n\}$.

Proposition 5.6. *If $X \subset [0,1)^d$, $0 < s < \Delta(X)$, and $n \in \mathbb{N}$, then there exists a $\mathcal{Q}_n$-discrete s-Frostman measure μ, i.e.*

$$\mu(Q') \leqslant \operatorname{diam}(Q')^s$$

for all $Q' \in \mathcal{Q}_\ell$ and $\ell \in \{0,\dots,n\}$ such that there is a dyadic cube $Q \in \mathcal{Q}_N$ with $N \geqslant n$ such that μ is supported on the closed $\sqrt{d}2^{-N}$-neighborhood of $M_Q(X)$, where $M_Q\colon \mathbb{R}^d \to \mathbb{R}^d$ is the unique homothety sending Q surjectively to $[0,1)^d$.

Proof. Fix $n \in \mathbb{N}$ and let $s < t < \min\{s+1, \Delta(X)\}$. Recalling (5.2), let $k_0 \in \mathbb{N}$ be such that

$$\#D_k^*(X) \geqslant 2^{kt} \tag{5.3}$$

for all $k \geqslant k_0$. Choose an integer $k \geqslant \max\{k_0, \frac{nd}{t-s}\} \geqslant n+1$. Recalling Lemma 5.5, let $Q \in \mathcal{Q}$ be such that μ_0 is a $\mathcal{Q}_k$-discrete measure with respect to weights $\{D_k^*(X)^{-1}\}_{Q' \in \mathcal{Q}_k \,:\, M_Q(X) \cap Q' \neq \emptyset}$. If there are $N \in \{n,\dots,k-1\}$ and $Q_0' \in \mathcal{Q}_N$ such that $\mu_0^{Q_0'}$ is a $\mathcal{Q}_n$-discrete s-Frostman measure, then the proof is finished. Otherwise, since μ_0 is a probability measure and $\#\mathcal{Q}_n = 2^{nd}$, we choose $Q_0 \in \mathcal{Q}_n$ such that $\mu_0(Q_0) \geqslant 2^{-nd}$ and notice that there are $\ell_1 \in \{0,\dots,n\}$ and $Q_1 \in \mathcal{Q}_{\ell_1}$ such that, writing $\mu_1 = \mu_0^{Q_0}$, we have

$$\mu_1(Q_1) = \mu_0^{Q_0}(Q_1) > \operatorname{diam}(Q_1)^s = \sqrt{d}^s 2^{-\ell_1 s}. \tag{5.4}$$

Writing $Q_1' = M_{Q_0}^{-1}(Q_1)$, we see that $Q_1' \in \mathcal{Q}_{n+\ell_1}$ such that $Q_1' \subset Q_0$ and

$$\begin{aligned}
\mu_1(Q_1) = \frac{\mu_0(Q_1')}{\mu_0(Q_0)} &= \frac{1}{\mu_0(Q_0)D_k^*(X)} \sum_{Q' \in \mathcal{Q}_k \,:\, M_Q(X) \cap Q' \neq \emptyset} \frac{\mathcal{L}^d|_{Q'}(Q_1')}{\mathcal{L}^d(Q')} \\
&\leqslant 2^{nd} D_k^*(X)^{-1} \cdot \#\{Q' \in \mathcal{Q}_k : Q' \subset Q_1' \text{ and } M_Q(X) \cap Q' \neq \emptyset\} \\
&= 2^{nd} D_k^*(X)^{-1} D_{k-n-\ell_1}(M_{Q_1'} \circ M_Q(X)) \\
&\leqslant 2^{nd} D_k^*(X)^{-1} D_{k-n-\ell_1}^*(X).
\end{aligned} \tag{5.5}$$

The estimates (5.3) and (5.4) thus give

$$D_{k-n-\ell_1}^*(X) \geqslant 2^{-nd} D_k^*(X) \mu_1(Q_1) > 2^{-nd} 2^{kt} \sqrt{d}^s 2^{-\ell_1 s}. \tag{5.6}$$

We may now repeat the above procedure with the iterated bound (5.6) in place of (5.3). Indeed, let μ_2 be the $\mathcal{Q}_{k-n-\ell_1}$-discrete measure with respect to weights $\{D_{k-n-\ell_1}^*(X)^{-1}\}_{Q' \in \mathcal{Q}_{k-n-\ell_1} \,:\, M_Q(X) \cap Q' \neq \emptyset}$ for some $Q \in \mathcal{Q}$ given by Lemma 5.5. Again, if μ_2 is a $\mathcal{Q}_n$-discrete s-Frostman measure, we are done. Otherwise, there are $\ell_2 \in \{0,\dots,n\}$ and $Q_2 \in \mathcal{Q}_{\ell_2}$ such that

$$\mu_2(Q_2) > \operatorname{diam}(Q_2)^s = \sqrt{d}^s 2^{-\ell_2 s}. \tag{5.7}$$

Repeating the reasoning done in (5.5), we see that

$$\begin{aligned}\mu_2(Q_2) &= D^*_{k-n-\ell_1}(X)^{-1} \sum_{Q' \in \mathcal{Q}_{k-n-\ell_1} \,:\, M_Q(X) \cap Q' \neq \emptyset} \frac{\mathcal{L}^d|_{Q'}(Q_2)}{\mathcal{L}^d(Q')} \\ &= D^*_{k-n-\ell_1}(X)^{-1} D_{k-n-(\ell_1+\ell_2)}(M_{Q_2} \circ M_Q(X)) \\ &\leqslant D^*_{k-n-\ell_1}(X)^{-1} D^*_{k-n-(\ell_1+\ell_2)}(X)\end{aligned}$$

and hence, by (5.6) and (5.7),

$$D^*_{k-n-(\ell_1+\ell_2)}(X) \geqslant D^*_{k-n-\ell_1}(X)\mu_2(Q_2) > 2^{-nd} 2^{kt} \sqrt{d}^{2s} 2^{-(\ell_1+\ell_2)s}.$$

Continuing inductively, we see that after m steps either the $\mathcal{Q}_{k-n-(\ell_1+\cdots+\ell_{m-1})}$-discrete measure μ_m given by Lemma 5.5 is a $\mathcal{Q}_n$-discrete s-Frostman measure or there is $\ell_m \in \{0, \ldots, n\}$ such that

$$D^*_{k-n-(\ell_1+\cdots+\ell_m)}(X) > 2^{-nd} 2^{kt} \sqrt{d}^{ms} 2^{-(\ell_1+\cdots+\ell_m)s}.$$

If m is such that $k - n < \ell_1 + \cdots + \ell_m \leqslant k$, then

$$2^{k(t-s)-nd} \leqslant 2^{-nd} 2^{kt} \sqrt{d}^{ms} 2^{-ks} < D^*_{k-n-(\ell_1+\cdots+\ell_m)}(X) \leqslant 1$$

and $k(t-s) - nd < 1$ which is contradicting with the choice of k. Thus μ_m is a $\mathcal{Q}_n$-discrete s-Frostman measure. $\square$

Furstenberg [18, Theorem 5.1] proved that if $X \subset [0,1)^d$ is closed, then there exists $T \in \mathrm{Tan}(X)$ such that $\Delta(X) = \dim_{\mathrm{H}}(T)$. Recalling Corollary 3.7 and Lemma 5.3, this implies $\dim_{\mathrm{A}}(X) = \dim_{\mathrm{H}}(T)$ and the Assouad dimension is thus characterized by weak tangents. The result for the Assouad dimension was first explicitly observed by Käenmäki, Ojala, and Rossi [21, Proposition 5.7].

Theorem 5.7. *If $X \subset [0,1)^d$ is closed, then $\dim_{\mathrm{A}}(X) = \Delta(X)$ and there exists $T \in \mathrm{Tan}(X)$ such that $\mathcal{H}^s_\infty(T) \geqslant 1$ where $s = \dim_{\mathrm{A}}(X)$.*

Proof. We may assume that $\Delta(X) > 0$ as otherwise, by Lemma 5.3, $\dim_{\mathrm{A}}(X) = \Delta(X) = 0$ and the existence of the claimed tangent set is trivial. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers strictly smaller than $\Delta(X)$ such that $\lim_{n\to\infty} s_n = \Delta(X)$. For every $n \in \mathbb{N}$ and $0 < s_n < \Delta(X)$, recalling Proposition 5.6, let $Q_n \in \mathcal{Q}_N$ with $N \geqslant n$ and μ_n be a $\mathcal{Q}_n$-discrete s_n-Frostman measure supported on the closed $\sqrt{d}2^{-N}$-neighborhood of $M_{Q_n}(X)$. In other words,

$$\mu_n(Q) \leqslant \operatorname{diam}(Q)^{s_n} = \sqrt{d}^{s_n} 2^{-\ell s_n} \tag{5.8}$$

for all $Q \in \mathcal{Q}_\ell$ and $\ell \in \{0, \ldots, n\}$. Going into a subsequence, if necessary, we choose $T \in \operatorname{Tan}(X)$ and a Borel probability measure μ such that

$$M_{Q_n}(X) \cap [0,1)^d \to T$$

in Hausdorff distance and

$$\mu_n \to \mu$$

in weak* topology; see Mattila [28, Definition 1.21]. Observe that, by compactness, μ is supported on T. Write $s = \Delta(X)$, let $0 < r < 1$, and choose $\ell \in \mathbb{N}$ such that $2^{-\ell-1} \leqslant 2r < 2^{-\ell}$. Fix $x \in T$ and notice that the open ball $B^o(x,r)$ can intersect at most 2^d many dyadic cubes $Q \in \mathcal{Q}_\ell$. Therefore, by the Portmanteau theorem, see Mattila [28, Theorem 1.24], and (5.8),

$$\begin{aligned} \mu(B^o(x,r)) &\leqslant \liminf_{n\to\infty} \mu_n(B^o(x,r)) \leqslant 2^d \liminf_{n\to\infty} \sqrt{d}^{s_n} 2^{-\ell s_n} \\ &\leqslant 2^d \liminf_{n\to\infty} \sqrt{d}^{s_n} (4r)^{s_n} = 2^d \sqrt{d}^{s} 4^s r^s. \end{aligned}$$

By Theorem 2.3(1), we have $\mathcal{H}^s(T) \geqslant 2^{-d}\sqrt{d}^{-s}4^{-s}\mu(T) = 2^{-d}\sqrt{d}^{-s}4^{-s} > 0$ and hence, $\dim_{\mathrm{H}}(T) \geqslant s = \Delta(X)$. Recalling Corollary 3.7, we see that $\dim_{\mathrm{A}}(X) \geqslant \Delta(X)$ which, together with Lemma 5.3, gives the first claim. Furthermore, by Lemma 3.5, there is $T' \in \operatorname{Tan}(T)$ such that $\mathcal{H}^s_\infty(T') \geqslant 1$. Since Lemma 3.4 ensures that $T' \in \operatorname{Tan}(X)$, we have finished the proof. □

If $X \subset [0,1)^d$ is closed and $s = \dim_{\mathrm{A}}(X)$, it would be interesting to know if there exists an Ahlfors s-regular weak tangent set $T \in \operatorname{Tan}(X)$. Regardless, Theorem 5.7 gives us an immediate corollary.

Corollary 5.8. *If $X \subset [0,1)^d$ is closed, then*

$$\dim_{\mathrm{A}}(X) = \max\{\dim_{\mathrm{H}}(T) : T \in \operatorname{Tan}(X)\}.$$

Proof. The claim follows directly from Corollary 3.7 and Theorem 5.7. □

The above observation introduces a way to calculate the Assouad dimension of a set by considering its weak tangents. We will apply this approach and study what can in general be said about the Assouad dimension of orthogonal projections. Recall that, by Example 4.5, orthogonal projections can increase the Assouad dimension. Let $G(d,k)$ be the collection of all k-dimensional linear subspaces in $\mathbb{R}^d$. It is a compact smooth manifold of dimension $k(d-k)$ and inherits a Haar measure $\gamma_{d,k}$ from the orthogonal group $O(d)$. If $V \in G(d,k)$, then we denote the orthogonal projection onto V by $\mathrm{proj}_V \colon \mathbb{R}^d \to V$. The following result is due to Fraser [14, Theorem 2.9].

Theorem 5.9. *If $X \subset [0,1)^d$ and $k \in \{1,\ldots,d-1\}$, then*

$$\dim_{\mathrm{A}}(\mathrm{proj}_V(X)) \geqslant \min\{k, \dim_{\mathrm{A}}(X)\}$$

for $\gamma_{d,k}$-almost all $V \in G(d,k)$.

Proof. Since $\mathrm{proj}_V(\overline{X}) \subset \overline{\mathrm{proj}_V(X)}$ and the Assouad dimension is stable under taking closure, we may assume that X is closed. By Corollary 5.8, there exists $T \in \mathrm{Tan}(X)$ such that $\dim_{\mathrm{H}}(T) = \dim_{\mathrm{A}}(X)$. Let $(M_n)_{n\in\mathbb{N}}$ be the sequence of homotheties for which

$$M_n(X) \cap B(0,1) \to T \tag{5.9}$$

in Hausdorff distance. Write $M_n(x) = r_n x + v_n$, where $r_n > 0$ and $v_n \in \mathbb{R}^d$. Since M_n is a homothety, the map $L_{V,n} = \mathrm{proj}_V \circ M_n \circ \mathrm{proj}_V^{-1} \colon V \to V$ is well-defined for all $V \in G(d,k)$. Note that

$$L_{V,n}(x) = \mathrm{proj}_V(r_n \,\mathrm{proj}_V^{-1}(x) + v_n) = r_n x + \mathrm{proj}_V(v_n)$$

for all $x \in V$ and hence, $L_{V,n}$ is a homothety. Therefore, by (5.9),

$$\begin{aligned} L_{V,n}(\mathrm{proj}_V(X)) \cap \mathrm{proj}_V(B(0,1)) &= \mathrm{proj}_V(M_n(X)) \cap \mathrm{proj}_V(B(0,1)) \\ &\supset \mathrm{proj}_V(M_n(X) \cap B(0,1)) \to \mathrm{proj}_V(T) \end{aligned}$$

in Hausdorff distance. By going into a subsequence, if necessary, we find $T' \in \mathrm{Tan}(\mathrm{proj}_V(X))$ such that

$$L_{V,n}(\mathrm{proj}_V(X)) \cap \mathrm{proj}_V(B(0,1)) \to T' \supset \mathrm{proj}_V(T).$$

By Lemma 3.6, monotonicity of the Assouad dimension, and Lemma 3.1, we have

$$\dim_{\mathrm{A}}(\mathrm{proj}_V(X)) \geqslant \dim_{\mathrm{A}}(T') \geqslant \dim_{\mathrm{A}}(\mathrm{proj}_V(T)) \geqslant \dim_{\mathrm{H}}(\mathrm{proj}_V(T)). \tag{5.10}$$

Furthermore, by Marstrand's projection theorem, see Mattila [28, Corollaries 9.4 and 9.8], and the choice of $T \in \mathrm{Tan}(X)$, we have

$$\dim_{\mathrm{H}}(\mathrm{proj}_V(T)) = \min\{k, \dim_{\mathrm{H}}(T)\} = \min\{k, \dim_{\mathrm{A}}(X)\} \tag{5.11}$$

for $\gamma_{d,k}$-almost all $V \in G(d,k)$. The proof follows by combining (5.10) and (5.11). □

Fraser and Käenmäki [17, Theorem 2.1] showed that for every upper semi-continuous function $f\colon G(2,1) \to [0,1]$ there exists a compact set $X \subset \mathbb{R}^2$ with $\dim_{\mathrm{A}}(X) = 0$ such that $\dim_{\mathrm{A}}(\mathrm{proj}_V(X)) = f(V)$ for all $V \in G(2,1)$. The result demonstrates that $\dim_{\mathrm{A}}(\mathrm{proj}_V(X))$ can take on any countable number of distinct values with positive measure and also, can avoid all values almost surely.

6. Tangents on self-affine sets

6.1. **Tangents and maximality on self-affine sets.** A set $T \subset \mathbb{R}^d$ is a *tangent* of $X \subset \mathbb{R}^d$ at $x \in \mathbb{R}^d$ if there is a sequence $(r_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty} r_n = 0$ and

$$M_{x,r_n}(X) \cap B(0,1) \to T$$

in Hausdorff distance. We denote the collection of tangents of X at x by $\mathrm{Tan}(X,x)$. Note that every tangent is a weak tangent, $\mathrm{Tan}(X,x) \subset \mathrm{Tan}(X)$. We remark that Lemma 3.5 shows the existence of a tangent with Hausdorff content bounded from below. The following example of Le Donne and Rajala [24, Example 2.20] shows that, in general, the Assouad dimension cannot be characterized by tangents.

Example 6.1. Let

$$X = \{0\} \cup \bigcup_{k=1}^{\infty} \bigcup_{\ell=0}^{k} \{2^{-k} + \ell 4^{-k}\} \subset [0,1].$$

It is straightforward to see that $\dim_{\mathrm{A}}(X) = 1$ but $\dim_{\mathrm{A}}(T) = 0$ for all $T \in \mathrm{Tan}(X,x)$ and $x \in X$.

By Corollary 5.8, the Assouad dimension gets realized on a weak tangent. This is particularly important detail in the study of self-affine sets as such sets often undergo a metamorphosis in approaching the weak tangent. A common technique in studying self-affine sets is to relate the underlying geometry to symbolic properties associated with the space of all infinite words. Therefore, if the Assouad dimension of a self-affine set was realized on a tangent, then upper bounding the Assouad dimension would be easier since one may fix in advance an infinite word for the point. The following theorem, proved by Käenmäki and Rutar [23, Theorem 2.12], guarantees that this is indeed the case.

Theorem 6.2. *If $X \subset \mathbb{R}^d$ is a self-affine set and $s = \dim_{\mathrm{A}}(X)$, then there exist $x \in X$ and $T \in \mathrm{Tan}(X, x)$ such that $\mathcal{H}^s_\infty(T) \geqslant 2^{-s}$.*

Proof. Let $(\varphi_1, \ldots, \varphi_N)$ be the associated tuple of affine maps $\varphi_i \colon \mathbb{R}^d \to \mathbb{R}^d$, $\varphi_i(x) = A_i x + v_i$, where $(A_1, \ldots, A_N) \in \mathrm{GL}_d(\mathbb{R})^N$ and $(v_1, \ldots, v_N) \in (\mathbb{R}^d)^N$. Fix $x_1 \in X$ and $0 < r_1 \leqslant 1$. Since X is a self-affine set, there is an affine map f_1 such that $f_1(X) \subset X \cap B(x_1, r_1)$. Indeed, if $\mathtt{i} \in \Sigma$ is such that $\pi(\mathtt{i}) = x_1$, then we may choose $f_1 = \varphi_{\mathtt{i}|_n}$ where n is the smallest integer such that $\alpha_1(A_{\mathtt{i}|_n}) \operatorname{diam}(X) < r_1$. Since $\dim_{\mathrm{A}}(f_1(X)) = s$, Theorem 5.7 guarantees the existence of $T_1 \in \mathrm{Tan}(f_1(X))$ such that $\mathcal{H}^s_\infty(T_1) \geqslant 1$. Thus there exists a homothety $M_1 \colon \mathbb{R}^d \to \mathbb{R}^d$, $M_1(x) = \lambda_1 x + v_1$, such that $0 \in M_1(X)$, it is expanding by $\lambda_1 \geqslant 1$, and

$$d_{\mathrm{H}}(T_1, M_1(f_1(X)) \cap B(0,1)) \leqslant 1.$$

Choose next $x_2 \in X$ and $0 < r_2 \leqslant \frac{1}{2}$ such that $B(x_2, r_2) \subset M_1^{-1}(B^o(x_1, r_1))$, and repeat the above construction. Iterating, we obtain a sequence $(f_n)_{n\in\mathbb{N}}$ of affine maps, a sequence $(T_n)_{n\in\mathbb{N}} \in (\mathrm{Tan}(f_n(X)))^{\mathbb{N}}$ of compact sets, and a sequence $(M_n)_{n\in\mathbb{N}}$ of homotheties, each expanding by $\lambda_n \geqslant n$, such that

(1) $f_n(X) \subset X$,
(2) $\mathcal{H}^s_\infty(T_n) \geqslant 1$,
(3) $M_{n+1}^{-1}(B(0,1)) \subset M_n^{-1}(B(0,1))$,
(4) $d_{\mathrm{H}}(T_n, M_n(f_n(X)) \cap B(0,1)) \leqslant \frac{1}{n}$.

Let $x = \lim_{n\to\infty} M_n^{-1}(0)$ and note that, by (3), $x \in M_n^{-1}(B(0,1))$ for all $n \in \mathbb{N}$. Recalling (4), let L_n be a homothety such that it is contracting by $\frac{1}{2}$ and

$$d_{\mathrm{H}}\left(L_n(T_n), \frac{\lambda_n(f_n(X)-x)}{2} \cap B(0,1)\right) \leqslant \frac{1}{n}. \tag{6.1}$$

Observe that, by (2.6) and (2), we have $\mathcal{H}^s_\infty(L_n(T_n)) \geqslant 2^{-s}$. Passing to a subsequence if necessary, we may set

$$T_0 = \lim_{n\to\infty} \frac{\lambda_n(f_n(X)-x)}{2} \cap B(0,1) \qquad \text{and} \qquad T = \lim_{n\to\infty} \frac{\lambda_n(X-x)}{2} \cap B(0,1).$$

By (6.1) and (1), we have $\lim_{n\to\infty} L_n(T_n) = T_0 \subset T \in \operatorname{Tan}(X,x)$. Since the Hausdorff content is upper semicontinuous by Lemma 3.3, we conclude that

$$\mathcal{H}^s_\infty(T) \geqslant \mathcal{H}^s_\infty(T_0) \geqslant \limsup_{n\to\infty} \mathcal{H}^s_\infty(L_n(T_n)) \geqslant 2^{-s}$$

as required. □

It is an immediate corollary of Theorem 6.2 that the Assouad dimension of a self-affine set gets realized on a tangent at some point. It is worthwhile to emphasize that the result does not assume any separation condition.

Corollary 6.3. *If $X \subset \mathbb{R}^d$ is a self-affine set, then*

$$\dim_{\mathrm{A}}(X) = \max\{\dim_{\mathrm{H}}(T) : x \in X \text{ and } T \in \operatorname{Tan}(X,x)\}.$$

Proof. The claim follows directly from Corollary 3.7 and Theorem 6.2. □

6.2. **Assouad dimension of Bedford-McMullen carpets.** We will consider a particular class of self-affine sets. Let $q > p \geqslant 2$ and $N \in \{2,\dots,pq\}$ be integers. Write $A = \operatorname{diag}(\frac{1}{p},\frac{1}{q}) \in \mathrm{GL}_2(\mathbb{R})$ and choose $I \subset \{0,\dots,p-1\} \times \{0,\dots,q-1\}$ to be a set of N elements. The *Bedford-McMullen carpet* is the self-affine set $X \subset [0,1]^2$ associated to a tuple $(\varphi_1,\dots,\varphi_N)$ of affine maps which all have the same linear part A and the translation part is from the set $\{(\frac{j}{p},\frac{k}{q}) \in [0,1]^2 : (j,k) \in I\}$. Write $n_j = \#\{k : (j,k) \in I\}$ to denote the number of sets $\varphi_i([0,1)^2)$ the vertical line $\{(\frac{j}{p}, y) : y \in \mathbb{R}\}$ intersects. We use the convention that whenever we speak about a Bedford-McMullen carpet, then it is automatically accompanied with this notation.

Our goal is to determine the Assouad dimension of a Bedford-McMullen carpet X. The result is due to Mackay [26, Theorem 1.1]. Our approach below relies on the fact that the Assouad dimension of a self-affine set gets realized on a tangent and the proof is a modification of Bárány, Käenmäki, and Yu [3, proof of Theorem 3.2]. To that end, we begin with an observation that a vertical slice of any tangent set can be affinely embedded into X.

Lemma 6.4. *If $X \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition and $V \in G(2,1)$ is the y-axis, then for every $x \in X$ and $T \in \mathrm{Tan}(X,x)$ there exist $z \in X$ such that*

$$\dim_{\mathrm{A}}(T \cap V) \leqslant \dim_{\mathrm{A}}(\mathrm{proj}_V(T)) \leqslant \dim_{\mathrm{A}}(X \cap (V+z)).$$

Proof. Let $\mathtt{i} \in \Sigma$ be such that $\pi\mathtt{i} = x$ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive reals such that $\lim_{n\to\infty} r_n = 0$ and

$$M_{x,r_n}(X) \cap B(0,1) \to T$$

in Hausdorff distance. For each $n \in \mathbb{N}$ choose $k_n \in \mathbb{N}$ such that $q^{-k_n-2} \leqslant r_n < q^{-k_n-1}$. Notice that $X \cap B(x,r_n) \subset \varphi_{\mathtt{i}|_{k_n}}(X)$. By compactness, going into a subsequence if necessary, there is $x \in X$ such that $\pi\sigma^{-k_n}\mathtt{i} \to z$. Hence,

$$\varphi_{\mathtt{i}|_{k_n}}^{-1} \circ M_{x,r_n}^{-1} \to \lambda\,\mathrm{proj}_V + z$$

uniformly on $B(0,1)$, where $\lambda \in [q^{-2}, q^{-1}]$, and thus,

$$\varphi_{\mathtt{i}|_{k_n}}^{-1}(X \cap B(x,r_n)) = \varphi_{\mathtt{i}|_{k_n}}^{-1} \circ M_{x,r_n}^{-1}(M_{x,r_n}(X) \cap B(0,1)) \to \lambda\,\mathrm{proj}_V(T) + z$$

in Hausdorff distance. Since $\varphi_{\mathtt{i}|_{k_n}}^{-1}(X \cap B(x,r_n)) \subset X$, we get by compactness that $\lambda\,\mathrm{proj}_V(T) + z \subset X$. As trivially $T \cap V \subset \mathrm{proj}_V(T)$, the claim follows by recalling that the Assouad dimension is monotone and invariant under bi-Lipschitz maps. $\square$

By considering maximal tangent sets, this observation easily converts to an upper bound for the Assouad dimension by means of the projection onto the x-axis and the maximal vertical slice.

Proposition 6.5. *If $X \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition and $V \in G(2,1)$ is the y-axis, then for every $x \in X$ and $T \in \mathrm{Tan}(X, x)$ it holds that*

$$\dim_{\mathrm{A}}(T) \leqslant \dim_{\mathrm{A}}(\mathrm{proj}_{V^\perp}(X)) + \max_{x \in \mathbb{R}^2} \dim_{\mathrm{A}}(X \cap (V + x)).$$

Proof. Since trivially $T \subset \mathrm{proj}_{V^\perp}(T) \times \mathrm{proj}_V(T)$, we get from (3.1) that

$$\dim_{\mathrm{A}}(T) \leqslant \dim_{\mathrm{A}}(\mathrm{proj}_{V^\perp}(T)) + \dim_{\mathrm{A}}(\mathrm{proj}_V(T)).$$

By Lemma 6.4, there exists $z \in X$ such that

$$\dim_{\mathrm{A}}(\mathrm{proj}_V(T)) \leqslant \dim_{\mathrm{A}}(X \cap (V + z))$$

and the proof is finished. □

The construction of the Bedford-McMullen carpet has a lot of regularity and therefore it is expected that the dimensions of the projection and the maximal slice play a role also in the lower bound. In the following lemma, we express these quantities by means of the data given to define the carpet.

Proposition 6.6. *If $X \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition and $V \in G(2,1)$ is the y-axis, then*

$$\dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp}(X)) = \dim_{\mathrm{A}}(\mathrm{proj}_{V^\perp}(X)) = \frac{\log \#\{j \in \{1, \ldots, p\} : n_j \neq 0\}}{\log p}$$

and

$$\max_{x \in \mathbb{R}^2} \dim_{\mathrm{H}}(X \cap (V + x)) = \max_{x \in \mathbb{R}^2} \dim_{\mathrm{A}}(X \cap (V + x)) = \max_{j \in \{1, \ldots, p\}} \frac{\log n_j}{\log q}.$$

Proof. We identify both the y-axis V and the x-axis $V^\perp$ with $\mathbb{R}$. It is easy to see that $\mathrm{proj}_{V^\perp}(X) \subset [0,1]$ is the self-similar set associated to $\#\{j \in \{1, \ldots, p\} : n_j \neq 0\}$ many homotheties $\psi_j \colon [0,1] \to [0,1]$, $\psi_j(x) = \frac{1}{p}x + \frac{j}{p}$, where $n_j \neq 0$. Furthermore, as $\mathrm{proj}_{V^\perp}(X)$ satisfies the strong separation condition, Theorems 4.1 and 4.2 give

$$\begin{aligned} \dim_{\mathrm{A}}(\mathrm{proj}_{V^\perp}(X)) &= \dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp}(X)) = \dim_{\mathrm{sim}}(\mathrm{proj}_{V^\perp}(X)) \\ &= \frac{\log \#\{j \in \{1, \ldots, p\} : n_j \neq 0\}}{\log p} \end{aligned}$$

as required.

To prove the second claim, notice that it suffices to maximize the dimensions of $X\cap(V+(x,0))$ over $x\in[0,1]$. Let $j_0\in\{1,\dots,p\}$ be such that $n_{j_0}=\max_{j\in\{1,\dots,p\}} n_j$. Define $z=\lim_{n\to\infty}\psi_{j_0^n}(0)=\sum_{k=1}^{\infty}\frac{j_0}{p^k}\in[0,1]$ and notice that $X\cap(V+(z,0))$ is the self-similar set associated to n_{j_0} many homotheties $\gamma_k\colon[0,1]\to[0,1]$, $\gamma_k(y)=\frac{1}{q}y+\frac{k}{q}$, where $(j_0,k)\in I$. Furthermore, as $X\cap(V+(z,0))$ satisfies the strong separation condition, Theorems 4.1 and 4.2 imply that

$$\begin{aligned}\dim_{\mathrm{A}}(X\cap(V+(z,0)))&=\dim_{\mathrm{H}}(X\cap(V+(z,0)))\\&=\dim_{\mathrm{sim}}(X\cap(V+(z,0)))=\frac{\log n_{j_0}}{\log q}.\end{aligned}$$

By the definition of z, it is evident that the dimension of $X\cap(V+(x,0))$ reaches its maximal value at $x=z$. □

To find the lower bound for the Assouad dimension, we bound the dimensions of the projection and the maximal slice from above by means of tangent sets.

Lemma 6.7. *If $X\subset\mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition and $V\in G(2,1)$ is the y-axis, then for every $x\in X$ and $T\in\mathrm{Tan}(X,x)$ it holds that*

$$\dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp}(X))\leqslant\dim_{\mathrm{H}}(T\cap(V^\perp+y))$$

for all $y\in T\cap V\cap B(0,\frac{1}{2})$.

Proof. Let $\mathtt{i}\in\Sigma$ be such that $\pi\mathtt{i}=x=(x_1,x_2)$ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive reals such that $\lim_{n\to\infty}r_n=0$ and

$$M_{x,r_n}(X)\cap B(0,1)\to T$$

in Hausdorff distance. For each $n\in\mathbb{N}$ choose $k_n\in\mathbb{N}$ such that $p^{-k_n+1}\leqslant r_n<p^{-k_n+2}$. Notice that $\varphi_{\mathtt{i}|_{k_n}}(X)\subset B(0,1)$ and

$$\frac{r_n}{p^{-k_n}}\,\mathrm{proj}_{V^\perp}(M_{x,r_n}(\varphi_{\mathtt{i}|_{k_n}}(X)))+(x_1,0)=\mathrm{proj}_{V^\perp}(X).$$

Since $M_{x,r_n}(\varphi_{\mathtt{i}|_{k_n}}(X)) \subset \mathbb{R} \times [-\frac{q^{-k_n}}{r_n}, \frac{q^{-k_n}}{r_n}]$ and $\frac{q^{-k_n}}{r_n} \to 0$, going into a subsequence if necessary, we see that

$$M_{x,r_n}(\varphi_{\mathtt{i}|_{k_n}}(X)) \to \lambda(\operatorname{proj}_{V^\perp}(X) - (x_1, 0))$$

in Hausdorff distance, where $\lambda \in [p^{-2}, p^{-1}]$. Recalling that $M_{x,r_n}(\varphi_{\mathtt{i}|_{k_n}}(X)) \subset M_{x,r_n}(X) \cap B(0,1)$, we get by compactness that

$$\lambda(\operatorname{proj}_{V^\perp}(X) - (x_1, 0)) \subset T \cap V^\perp.$$

By the construction of the Bedford-McMullen carpet, it is evident that an affine copy of $\operatorname{proj}_{V^\perp}(X)$ is contained also in $T \cap (V^\perp + y)$ for all $y \in T \cap V \cap B(0, \frac{1}{2})$. The claim follows by recalling that the Hausdorff dimension is monotone and invariant under bi-Lipschitz maps. □

Recalling Lemma 6.4, the second lemma shows the existence of a tangent set having maximal vertical slice.

Lemma 6.8. *If $X \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition and $V \in G(2,1)$ is the y-axis, then there exist $z \in X$ and $T \in \operatorname{Tan}(X, z)$ such that*

$$\max_{x \in \mathbb{R}^2} \dim_{\mathrm{H}}(X \cap (V + x)) \leqslant \dim_{\mathrm{H}}(T \cap V).$$

Proof. Recall from the proof of Proposition 6.6 that the maximum in the claim is attained at $\dim_{\mathrm{H}}(X \cap (V + x))$, where $X \cap (V + x)$ is a self-similar set. Therefore, by Corollary 6.3 and Proposition 6.6, there are $z \in X \cap (V + x)$ and $T \in \operatorname{Tan}(X \cap (V + x), z)$ such that $\dim_{\mathrm{H}}(T) = \dim_{\mathrm{A}}(X \cap (V + x)) = \dim_{\mathrm{H}}(X \cap (V + x))$. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\lim_{n \to \infty} r_n = 0$ and

$$M_{z,r_n}(X \cap (V + x)) \cap B(0,1) \to T$$

in Hausdorff distance. Note that, as M_{z,r_n} is a homothety, we have $M_{z,r_n}(V + x) = V$. By going into a subsequence, if necessary, we find $T' \in \operatorname{Tan}(X, z)$ such that

$$M_{z,r_n}(X) \cap B(0,1) \to T'$$

in Hausdorff distance. By compactness, we have $T \subset T' \cap V$ and thus,

$$\dim_{\mathrm{H}}(X \cap (V + x)) = \dim_{\mathrm{H}}(T) \leqslant \dim_{\mathrm{H}}(T' \cap V)$$

as required. $\square$

The above two lemmas now convert to a lower bound for the Assouad dimension by means of the projection onto the x-axis and the maximal vertical slice.

Proposition 6.9. *If $X \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition and $V \in G(2,1)$ is the y-axis, then there exist $z \in X$ and $T \in \mathrm{Tan}(X,z)$ such that*

$$\dim_{\mathrm{H}}(T) \geqslant \dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp}(X)) + \max_{x \in \mathbb{R}^2} \dim_{\mathrm{H}}(X \cap (V + x)).$$

Proof. By Lemma 6.8, there exist $z \in X$ and $T \in \mathrm{Tan}(X,z)$ such that

$$\max_{x \in \mathbb{R}^2} \dim_{\mathrm{H}}(X \cap (V + x)) \leqslant \dim_{\mathrm{H}}(T \cap V) \tag{6.2}$$

and, by Lemma 6.7, it holds that

$$\dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp}(X)) \leqslant \dim_{\mathrm{H}}(T \cap (V^\perp + y)) \tag{6.3}$$

for all $y \in T \cap V \cap B(0, \frac{1}{2})$. Relying on Proposition 6.6 and (6.2), choose $0 < s < \dim_{\mathrm{H}}(T \cap V)$ and, by Theorem 2.6, let μ be an s-Frostman measure on $T \cap V \cap B(0, \frac{1}{2})$. By Marstrand's slicing theorem, see Bishop and Peres [4, Theorem 3.3.1], we have

$$\dim_{\mathrm{H}}(T \cap (V^\perp + y)) \leqslant \dim_{\mathrm{H}}(T) - s \tag{6.4}$$

for μ-almost all $y \in T \cap V \cap B(0, \frac{1}{2})$. By letting $s \uparrow \dim_{\mathrm{H}}(T \cap V)$, we get from (6.3), (6.4), and (6.2) that

$$\begin{aligned} \dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp}(X)) &\leqslant \dim_{\mathrm{H}}(T) - \dim_{\mathrm{H}}(T \cap V) \\ &\leqslant \dim_{\mathrm{H}}(T) - \max_{x \in \mathbb{R}^2} \dim_{\mathrm{H}}(X \cap (V + x)) \end{aligned}$$

as required. $\square$

As a corollary, we are able to determine the Assouad dimension of a Bedford-McMullen carpet.

Corollary 6.10. *If $X \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying the strong separation condition, then*

$$\dim_{\mathrm{A}}(X) = \frac{\log \#\{j \in \{1, \ldots, p\} : n_j \neq 0\}}{\log p} + \max_{j \in \{1,\ldots,p\}} \frac{\log n_j}{\log q}.$$

Proof. By Corollary 6.3, the claim follows directly from Propositions 6.5, 6.6 and 6.9. □

Recall from Theorem 4.2 that a self-similar set X satisfying the weak separation condition has $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{A}}(X)$. By (4.2), we have $\dim_{\mathrm{H}}(X) \leqslant \dim_{\mathrm{aff}}(X)$ for all self-affine sets X. In the following example, we show that a Bedford-McMullen carpet X can have $\dim_{\mathrm{aff}}(X) < \dim_{\mathrm{A}}(X)$ even if the strong separation condition is satisfied.

Example 6.11. Write $\mathsf{A} = (A, \ldots, A) \in \mathrm{GL}_2(\mathbb{R})^N$, where $A = \mathrm{diag}(\frac{1}{p}, \frac{1}{q})$, and note that $P(\mathsf{A}, s) = \log(N\varphi^s(A))$. Hence,

$$\dim_{\mathrm{aff}}(X) = \begin{cases} \frac{\log N}{\log p}, & \text{if } N \in \{2, \ldots, p\}, \\ 1 + \frac{\log N/p}{\log q}, & \text{if } N \in \{p+1, \ldots, pq\}. \end{cases} \tag{6.5}$$

Let $X \subset [0,1]^2$ be the Bedford-McMullen carpet associated to the following choices: Fix $q = 5$, $p = 4$, and $N = 5$, and choose the translation vectors such that $n_1 = 3$, $n_2 = 0$, $n_3 = 1 = n_4$, and that the strong separation condition is satisfied. Then, by (6.5) and Corollary 6.10, we have

$$\dim_{\mathrm{aff}}(X) = 2 - \frac{\log 4}{\log 5} < \frac{\log 3}{\log 4} + \frac{\log 3}{\log 5} = \dim_{\mathrm{A}}(X).$$

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