Hyperbolic networks

2022 spring

Outline

- **Introduction**
- **Hyperbolic network models**
- **Hyperbolic embedding of networks**
- **Communities in hyperbolic networks**

Introduction

PRELIMINARIES

What are hyperbolic networks?

- **Model networks** (graphs) generated by placing nodes in hyperbolic spaces.
- **Real networks** embedded into a hyperbolic space.

- Random Hyperbolic Graph (RHG) or $\mathbb{S}^1/\mathbb{H}^2$ model:

  D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, M. Boguñá: Hyperbolic geometry of complex networks. *Phys. Rev. E.* **82**, 036106 (2010).

- Popularity Similarity Optimisation (PSO) model:

  F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov: Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012).

- HyperMap for embedding into hyperbolic space:

  F. Papadopoulos, C. Psomas, D. Krioukov:
  Network Mapping by Replaying Hyperbolic Growth. *IEEE/ACM Transactions on Networking.* **23**, 198–211 (2015).

- Coalescent embeddings:

  A. Muscoloni, J. M. Thomas, S. Ciucci, G. Bianconi, C. V. Cannistraci: Machine learning meets complex networks via coalescent embedding in the hyperbolic space. *Nat. Commun.* **8**, 1615 (2017).

Why is it a good idea to place the nodes of a network into hyperbolic spaces?

Network models

What is the goal/motivation of a network model?

Network models

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- Generate interesting graphs...

Network models

What is the goal/motivation of a network model?

- Generate interesting graphs...
- **Reproduce statistical properties of the networks representing real systems**.

Watts-Strogatz model:

- Regular ring network with random rewiring.
- Can generate small-world and highly clustered networks.

D. Watts and S. H. Strogatz, *Nature* **393**,409 (1998)

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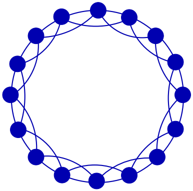

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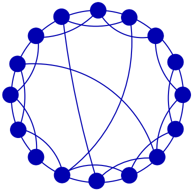
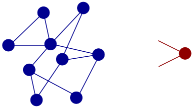


D. Watts and S. H. Strogatz, *Nature* **393**,409 (1998)

Barabási-Albert model

- Network growth with preferential attachment.
- Generates scale-free networks where $p(k) \propto k^{-3}$.



A.-L. Barabási, R. Albert, *Science* **286**, 509–512 (1999).

Network models

What features are we after?

Network models

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Most networks representing real complex systems are in most cases:

- **Small-world**
- **Highly clustered**
- **Inhomogeneous** in terms of the degree (scale-free).

Network models

Can we have all of these in a simple model?

Network models

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Holme-Kim model:

- B-A model with extra triad formation steps
- Can generate scale-free networks with a tunable clustering coefficient.

P. Holme and B. J. Kim, *Phys. Rev. E* **65**, 026107 (2002).

Network models

What about random geometric graphs?

Random geometric graphs:

- Place nodes (uniformly) at random in a (Euclidean) space,
- and connect them according to the distance.

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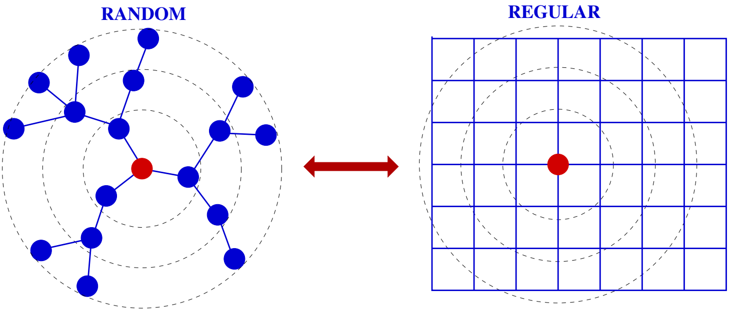
But can we have small-world, highly clustered and scale-free networks in this approach?

Small-world vs Regular



$$N(\ell) \approx \langle k \rangle^{\ell}$$
$$\langle k \rangle^{\ell} \approx N$$
$$\langle \ell \rangle \approx \frac{\ln N}{\ln \langle k \rangle}$$

$$N(\ell) \sim \ell^2$$
$$\langle \ell \rangle^2 \sim N$$
$$\langle \ell \rangle \sim N^{1/2}$$

Small-world vs Euclidean

- The number of nodes in concentric shells around a given node grows exponentially in a small-world network.

$$\updownarrow$$

- The volume of a sphere displays only a polynomial growth in Euclidean spaces.

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- The volume of a sphere displays only a polynomial growth in Euclidean spaces.

$\rightarrow$ We cannot have large Euclidean random geometric graphs that are also small-world!

- **However, the volume of spheres grows exponentially in hyperbolic spaces, thus, they are more suited for hosting small world networks!**

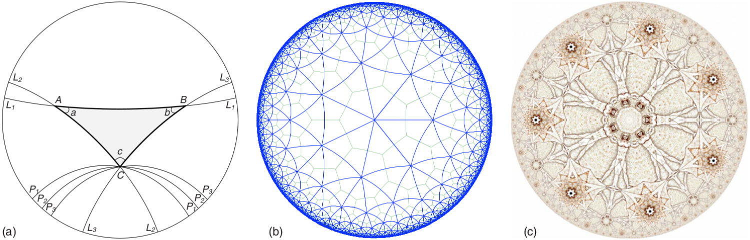
HYPERBOLIC GEOMETRY

Hyperbolic geometry

- A hyperbolic space is a metric space with constant negative curvature K, usually characterised by $\zeta = \sqrt{-K}$.

Hyperbolic geometry

- A hyperbolic space is a metric space with constant negative curvature K, usually characterised by $\zeta = \sqrt{-K}$.
- Poincaré disk model of 2d hyperbolic space:



(Figure from Krioukov et al.,*Phys. Rev. E.* **82**, 036106 (2010))

Hyperbolic geometry

- Comparing different geometries:

Property	Euclidean	Spherical	Hyperbolic
Curvature K	0	>0	<0
Parallel lines	1	0	∞
Triangles are	Normal	Thick	Thin
Shape of triangles			
Sum of angles in triangles	π	$>\pi$	$<\pi$
Circle length	$2\pi r$	$2\pi \sin \zeta r$	$2\pi \sinh \zeta r$
Disk area	$2\pi r^2/2$	$2\pi(1-\cos \zeta r)$	$2\pi(\cosh \zeta r - 1)$

(Table from Krioukov et al.,*Phys. Rev. E.* **82**, 036106 (2010))

Hyperbolic geometry

- Hyperbolic geometry on YouTube:
  - From CodeParade:
    Non-Euclidean Geometry Explained - Hyperbolica Devlog #1
  - From Henry Segerman:
    Illuminating hyperbolic geometry
  - From Numberphile:
    Playing Sports in Hyperbolic Space - Numberphile

Native disk representation

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- The circle perimeter and area are

$$L(r) = 2\pi \sinh(\zeta r),$$
$$A(r) = 2\pi \left(\cosh(\zeta r) - 1\right),$$

both grow as $e^{\zeta r}$ as a function of r.

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  both grow as $e^{\zeta r}$ as a function of r.
- The hyperbolic law of cosines for the hyperbolic distance x between two points (r, θ) and (r', θ'):

$$\cosh(\zeta x) = \cosh(\zeta r)\cosh(\zeta r') - \sinh(\zeta r)\sinh(\zeta r')\cos(\Delta\theta),$$

  where $\Delta\theta = \pi - |\pi - |\theta - \theta'||$ is the angular difference.

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  where $\Delta\theta = \pi - |\pi - |\theta - \theta'||$ is the angular difference.
- For sufficiently large ζr, $\zeta r'$ and $\Delta\theta > 2\sqrt{e^{-2\zeta r} + e^{-2\zeta r'}}$ the distance can be approximated as

$$x \simeq r + r' + \frac{2}{\zeta}\ln\left(\sin\left(\frac{\Delta\theta}{2}\right)\right) \approx r + r' + \frac{2}{\zeta}\ln\left(\frac{\Delta\theta}{2}\right).$$

Hyperbolic network models

POPULARITY SIMILARITY OPTIMISATION MODEL

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Popularity and similarty during network growth

Plausible effects governing the connection process in growing networks representing real complex systems:

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Popularity and similarty during network growth

Plausible effects governing the connection process in growing networks representing real complex systems:

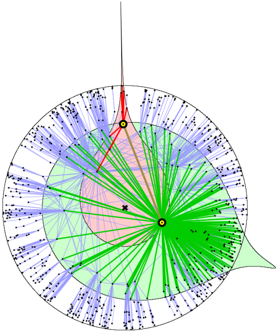
- **Similarity** between the entities represented by the nodes is enhancing the pairwise connection probability.
- **Popularity** (degree) of an entity can enhance the probability for connecting to any other node in general.

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov: Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Popularity and similarity in the native disk

Analogy of these two properties in the native disk:

- **Similarity**: the angle and the angular separation $\Delta\theta$ can provide a simple model of similarity.
- **Popularity**: the radial distance from the disk center can model the popularity. (Smaller radius corresponds to larger popularity).



(Figure from Krioukov et al., *Phys. Rev. E.* **82**, 036106 (2010)

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov: Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Concept of the PSO model

- A **growing network model** where we add a new node at each iteration to the native disk:

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Concept of the PSO model

- A **growing network model** where we add a new node at each iteration to the native disk:
  - the angular coordinates are chosen uniformly at random,
  - the radial coordinates are chosen such that the node density is uniform.
- The node pairs are connected according to a probability depending on the **hyperbolic distance**.

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
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Concept of the PSO model

How should we set the radial coordinates?

- We know that the disk area is increasing exponentially with the radius...

→ the radial coordinate of the new nodes should increase logarithmically with the node index (or birth time):

$$r_t = \ln(t)$$

→ The hyperbolic distance between nodes s and t becomes approximately

$$x_{st} \simeq r_s + r_t + \frac{2}{\zeta}\ln\left(\frac{\theta_{st}}{2}\right).$$

If we set $\zeta = 2$,

$$e^{x_{st}} \simeq \underbrace{s\cdot t}_{\text{pop.}}\cdot\underbrace{\frac{\theta_{st}}{2}}_{\text{sim.}}$$

the distance (exponentiated) is basically the logarithm of the product between popularity and similarity.

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Concept of the PSO model

How should we connect the node pairs?

→ A basic idea:

- always connect to the closest m nodes.
- connect to all nodes within some radius R.
- (with appropriate choice of R the two can be made equivalent)

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov: Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Model 0
Definition

The PSO model (Model 0)

- The curvature $K <$ parametrised by $\zeta = \sqrt{-K}$ is set to $\zeta = 2$, making the formula for the hyp. distance even simpler.
- The only free parameters are the number of nodes N and the average degree parametrised by $m = \langle k \rangle /2$.
- The network is grown according to the following rules:
  - At iteration t, the new node obtains a radial coordinate $r_t = \ln t$, and an angular coordinate $\theta_t \in [0, 2\pi]$ uniformly at random.
  - If $t < m$, it connects to all previous nodes, otherwise it connects to the closest m nodes.

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Model 0
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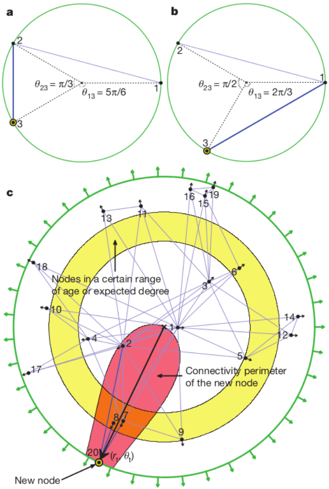
Illustration from the original paper:



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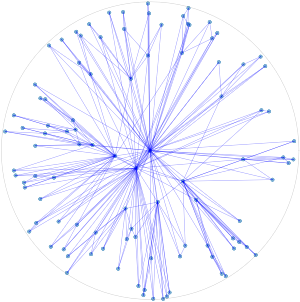
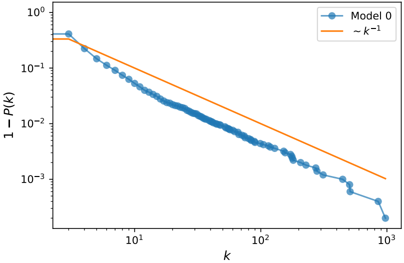
Model 0
Illustration

Network with 100 nodes:



Degree distribution (complementary cumulative):



Average clustering coeff.: 0.85

→ The model generates scale-free networks with $\gamma \approx 2$!

→ The clustering coefficient is also high!

Model 0

It would be nice

- if we could control the degree decay exponent γ...
- if we could control the average clustering coefficient $\langle C\rangle$...

Model 1

Controlling the degree distribution: **popularity fading**.

- The degree is determined by the radial coordinate, with nodes closer to the origin gaining more connections.

→ We could modify the network generation process by slowly pulling the old nodes outwards to decrease their popularity...

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Model 1
Definition

The PSO model (Model 1)

- The curvature $K <$ parametrised by $\zeta = \sqrt{-K}$ is set to $\zeta = 2$, making the formula for the hyp. distance even simpler.
- Free parameters are N, $m = \langle k \rangle /2$ and β, controlling the popularity fading.
- The network is grown according to the following rules:
  - At iteration t, the new node obtains a radial coordinate $r_t(t) = \ln t$, and an angular coordinate $\theta_t \in [0, 2\pi]$ uniformly at random.
  - **Popularity fading**: The radial coordinate of all existing nodes is updated as
  $$r_s(t) = \beta r_s(s) + (1 - \beta) r_t(t)$$
  (At $\beta = 1$ we recover Model 0).
  - If $t < m$, the new node connects to all previous nodes, otherwise it connects to the closest m nodes.

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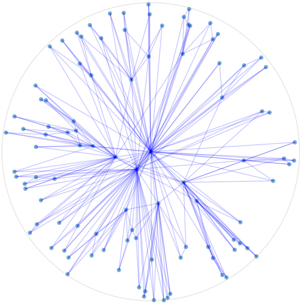
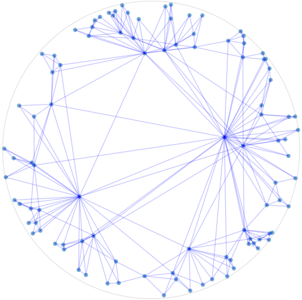
Model 0 vs Model 1

Network generated with Model 0:
$(N = 100,\ m = 3,\ \beta = 1)$

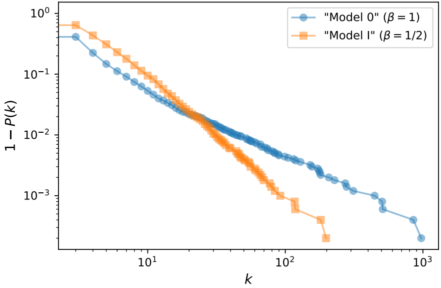


Network generated with Model 1:
$(N = 100,\ m = 3,\ \beta = \frac{1}{2})$



Model 0 vs Model 1

The degree distributions:



Analytic calculation of the degree distribution

Main steps:

- Convert

  "connect to m closest nodes"

  into

  "connect to all nodes within a cutoff radius R"
  (by appropriate choice of R).

- Using that, show that the linking probability between a new node t and an old node s is equivalent to that in a generalised B-A model.

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Analytic calculation of the degree distribution

- What is the **expected number of nodes within a radius** R from the node appearing at t?
- The prob. that s is closer than R is

$$P(x_{st} < R) \simeq P\left(r_s + r_t + \frac{2}{\zeta}\ln(\theta_{st}/2) < R\right) = P\left(\theta_{st} < 2e^{-\frac{\zeta}{2}(r_s+r_t-R)}\right).$$

Since we have set $\zeta = 2$, and θst is uniform in $[0, \pi]$

$$P(x_{st} < R) \simeq P\left(\theta_{st} < 2e^{-(r_s+r_t-R)}\right) = \frac{2}{\pi}e^{-(r_s+r_t-R)}$$

- By summing over all existing nodes we gain

$$\bar{N}(R) = \sum_{i=1}^{t} P(x_{it} < R) \simeq \int_1^t P(x_{it} < R)di = \frac{2}{\pi}e^{-(r_t-R)}\int_1^t e^{-r_i(t)}di.$$

- The integral can be expressed as:

$$\int_1^t e^{-r_i(t)}di. = \begin{cases} \frac{e^{-(1-\beta)r_t}}{1-\beta}\left[e^{(1-\beta)r_t} - 1\right] = \frac{1}{1-\beta}\left[1 - e^{-(1-\beta)r_t}\right]. & \text{if } \beta < 1 \\ \ln(t) = r_t & \text{if } \beta = 1. \end{cases}$$

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Analytic calculation of the degree distribution

- The expected number of nodes within a radius R from the node t:

$$\bar{N}(R) = \begin{cases} \frac{2}{\pi} e^{-(r_t - R)} \frac{1}{1-\beta} \left[1 - e^{-(1-\beta) r_t}\right], & \text{if } \beta < 1 \\ \frac{2}{\pi} e^{-(r_t - R)} r_t & \text{if } \beta = 1. \end{cases}$$

- By setting $\bar{N} = m$, we can define a t-dependent cutoff radius, for which the expected number of older nodes within is m as

$$R_t = \begin{cases} r_t - \ln\left[\frac{2}{\pi} \frac{\left[1 - e^{-(1-\beta) r_t}\right]}{m(1-\beta)}\right], & \text{if } \beta < 1 \\ r_t - \ln\left[\frac{2}{\pi} \frac{r_t}{m}\right] & \text{if } \beta = 1. \end{cases}$$

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Analytic calculation of the degree distribution

- Let's focus now on the probability that t is connecting to s:

$$\Pi(s,t) = P(x_{st} < R_t) = \frac{2}{\pi} e^{-(r_s(t)+r_t(t)-R_t)} = \begin{cases} \dfrac{e^{-r_s(t)}m}{\frac{1}{1-\beta}\left[1-e^{-(1-\beta)r_t}\right]}, & \text{if } \beta < 1 \\ \dfrac{e^{-r_s(t)}m}{r_t} & \text{if } \beta = 1. \end{cases}$$

By realising that the denominator is $\int_1^t e^{-r_i} di$,

$$\Pi(s,t) = m\frac{e^{-r_s(t)}}{\int_1^t e^{-r_i(t)}di} = m\frac{e^{-\beta r_s(s)-(1-\beta)r_t(t)}}{\int_1^t e^{-\beta r_i(i)-(1-\beta)r_t(t)}di} = m\frac{e^{-\beta r_s(s)}}{\int_1^t e^{-\beta r_i(i)}di},$$

or equivalently

$$\Pi(s,t) = m\frac{s^{-\beta}}{\int_1^t i^{-\beta}di} = m\frac{\left(\frac{s}{t}\right)^{-\beta}}{\int_1^t \left(\frac{i}{t}\right)^{-\beta}di}.$$

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Analytic calculation of the degree distribution

- Dorogovtsev, Mendes and Samukhin generalised the B-A model where a new node bringing m new links is choosing s as

$$P(s) \propto k_s(t) - m + A,$$

where A is a further model-parameter.

- The connection probability is

$$\Pi(s,t) = m\frac{k_s(t) - m + A}{t(m+A)}.$$

The degree of node introduced at $t = s$ can be written as

$$\bar{k}_s(t) = m + A\left[\left(\frac{s}{t}\right)^{-\beta} - 1\right],$$

where β is an exponent $\beta \in (0,1)$ depending on the model parameters, and

$$\beta = \frac{1}{1-\gamma} \quad \leftrightarrow \quad \gamma = 1 + \frac{1}{\beta}.$$

N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, *Phys. Rev. Lett.* **85**, 4633 (2000)

Analytic calculation of the degree distribution

- By replacing $k_s(t)$ by its expected value

$$\bar{\Pi}(s,t) = m\frac{\bar{k}_s(t) - m + A}{t(m+A)} = m\frac{A\left(\frac{s}{t}\right)^{-\beta}}{\int_1^t (k_i(t) - m + A)di} = m\frac{A\left(\frac{s}{t}\right)^{-\beta}}{A\int_1^t \left(\frac{i}{t}\right)^{-\beta} di} =$$

$$m\frac{\left(\frac{s}{t}\right)^{-\beta}}{\int_1^t \left(\frac{i}{t}\right)^{-\beta} di}$$

- This is exactly the same as the connection prob. in Model 1!

→ **Model 1 is generating scale-free networks where γ is controlled by β as**

$$\gamma = 1 + \frac{1}{\beta}.$$

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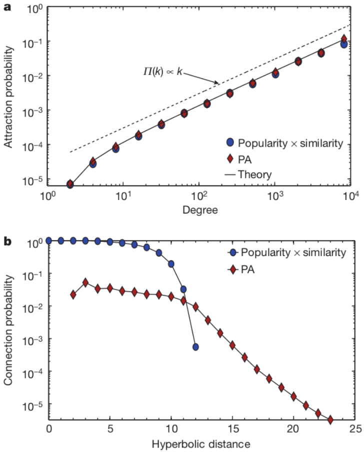
Analytic calculation of the degree distribution

Comparing PSO and preferential attachment in the original paper:



F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Analytic calculation of the degree distribution

- Let us turn back to the expected degree of node s:

$$\bar{k}_s(t) \sim \left(\frac{s}{t}\right)^{-\beta} = e^{-\beta(r_s(s)-r_s(t))}$$

Using that $r_s(t) = \beta r_s(s) + (1-\beta)r_t(t)$ we can write $\beta r_s(s) = r_s(t) + (\beta-1)r_t(t)$, hence

$$\bar{k}_s(t) \sim e^{-(r_s(t)-r_t(t))}.$$

- Thus, the expected node degree is determined by the radial coordinate, or equivalently, by the birth time of the node, and the expected degree is decreasing as a function of r.

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How to control also the clustering coefficient?

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Model 2

Concept

How to control also the clustering coefficient?

- The large C comes from the relatively "strict" connection rule, where we connect to everybody within R_t and to no one farther away...

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How to control also the clustering coefficient?

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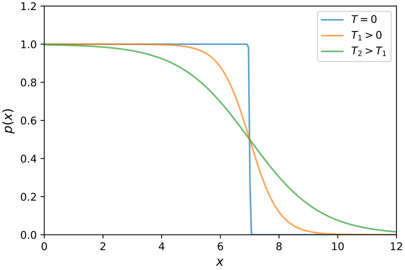
How to control also the clustering coefficient?

- The large C comes from the relatively "strict" connection rule, where we connect to everybody within R_t and to no one farther away...
- Softening this rule can decrease C.
- A natural idea is to use

$$p(x_{st}) = \frac{1}{1 + e^{\frac{x_{st} - R_t}{T}}}$$



F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov: Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

The PSO model (Model 2)

- The curvature $K <$ parametrised by $\zeta = \sqrt{-K}$ is set to $\zeta = 2$.
- Parameters: N, $m = \langle k \rangle /2$, β, and T, controlling $\langle C \rangle$.
- The network is grown according to the following rules:
  - At iteration t, the new node obtains $r_t(t) = \ln t$, and $\theta_t \in [0, 2\pi]$ uniformly at random.
  - The radial coordinate of all existing nodes is updated as

$$r_s(t) = \beta r_s(s) + (1 - \beta) r_t(t)$$

  - If $t < m$, the new node connects to all previous nodes.
  - Otherwise repeat until m links are realised:
    - Choose a node s uniformly at random.
    - Connect to this node according to

$$p(x_{st}) = \frac{1}{1 + e^{\frac{1}{T}(x_{st} - R_t)}}$$

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Model 2
Degree distribution

What happens to the degree distribution with this modification?

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What happens to the degree distribution with this modification?

- Let's write the distance dependent connection prob. as

$$p(x_{st}) = \frac{1}{1 + e^{\frac{1}{T}(r_s + r_t + \ln(\theta_{st}/2) - R_t)}} = \frac{1}{1 + \left(X(s,t)\frac{\theta_{st}}{2}\right)^{\frac{1}{T}}},$$

where we introduced $X(s,t) = e^{r_s + r_t - R_t}$.

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  where we introduced $X(s,t)=e^{r_s+r_t-R_t}$.

- Since θ_{st} is uniformly random in $[0,\pi]$, and nodes are chosen at random, the prob. that t connects to s in one round is

$$P(s,t) = \frac{1}{t}\frac{1}{\pi}\int_0^{\pi}\frac{1}{1+\left(X(s,t)\frac{\theta_{st}}{2}\right)^{\frac{1}{T}}}d\theta_{st}.$$

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$$P(s,t)=\frac{1}{t}\frac{1}{\pi}\int_0^{\pi}\frac{1}{1+\left(X(s,t)\frac{\theta_{st}}{2}\right)^{\frac{1}{T}}}d\theta_{st}.$$

- If $T<1$, and assuming that $X(s,t)>>1$, by change of variables the integral can be approximated as

$$P(s,t)\approx\frac{2T}{t\sin(\pi T)}\frac{1}{X(s,t)}.$$

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Model 2

Degree distribution

- The probability that node t is connecting to s overall can be written as

$$\Pi(s,t) = m\frac{P(s,t)}{\int_1^r P(i,t)di} = m\frac{X(s,t)^{-1}}{\int_1^t X(i,t)^{-1}di} = m\frac{e^{-r_s(t)}}{\int_1^t e^{-r_i(t)}di}.$$

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- This is the same as in Model 1!

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→ **Thus, the degree distribution is not affected by changing from Model 1 to Model 2, and γ is still controlled (only) by β!**

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What about the cutoff radius R_t?

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We have calculated the same integral before, thus,

$$\bar{N}(R_t) = \begin{cases} \frac{2T}{\sin(\pi T)} e^{-(r_t-R_t)} \frac{1}{1-\beta} \left[1 - e^{-(1-\beta)r_t}\right]. & \text{if } \beta < 1 \\ \frac{2T}{\sin(\pi T)} e^{-(r_t-R_t)} r_t & \text{if } \beta = 1 \end{cases}$$

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- Based on that

$$R_t = \begin{cases} r_t - \ln\left[\frac{2T}{\sin(\pi T)} \frac{\left[1-e^{-(1-\beta)r_t}\right]}{m(1-\beta)}\right], & \text{if } \beta < 1 \\ r_t - \ln\left[\frac{2T}{\sin(\pi T)} \frac{r_t}{m}\right] & \text{if } \beta = 1. \end{cases}$$

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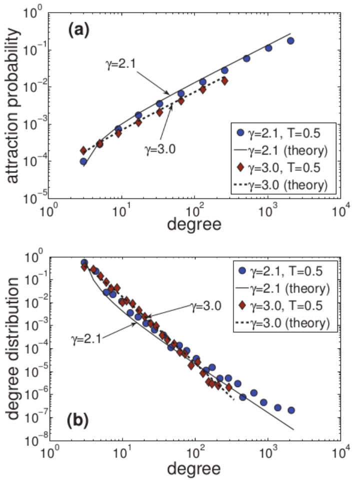

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  - At low T nodes connect almost only to the closest other notes.
  - $\rightarrow$ Due to the triangle inequality a lot of triangles are formed.
  - At high T nodes can connect to nodes further away as well.
  - $\rightarrow$ The number of triangles (and consequently, $\bar{C}$) is reduced.

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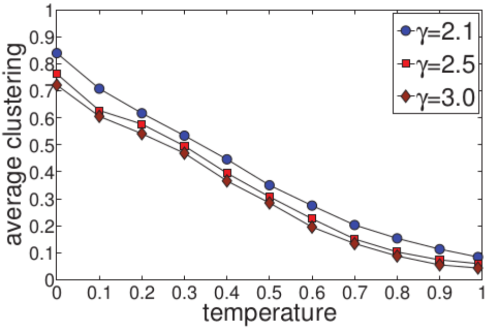
Model 2
Simulations

From the original paper:



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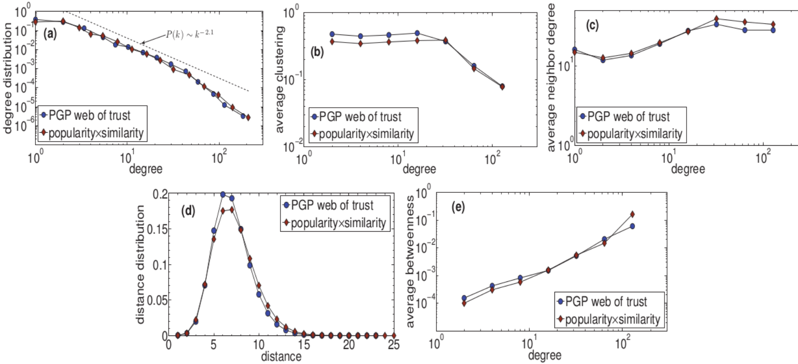
Model 2
Comparing with a real network

From the original paper:



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$$R_t = \begin{cases} r_t - \frac{2}{\zeta} \ln \left[\frac{2T}{\sin(\pi T)} \frac{\left[1 - e^{-\frac{\zeta}{2}(1-\beta) r_t}\right]}{m(1-\beta)} \right], & \text{if } \beta < 1 \\ r_t - \frac{2}{\zeta} \ln \left[\frac{2T}{\sin(\pi T)} \frac{\zeta r_t}{m} \right] & \text{if } \beta = 1. \end{cases}$$

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- **With these modifications the same results hold.**

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Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

PSO model

The PSO model (canonical form)

- Parameters: $\zeta = \sqrt{-K}$, $m = \langle k \rangle /2$, $\beta \in (0,1]$, and $T \in [0,1)$.
- The network is grown according to the following rules:
  - At time step t, the new node appears at $r_t(t) = \frac{2}{\zeta}\ln t$, and $\theta_t \in [0, 2\pi]$
  - The radial coordinate of all existing nodes is updated as
    $$r_s(t) = \beta r_s(s) + (1-\beta) r_t(t)$$
  - If $t < m$, the new node connects to all previous nodes.
  - Otherwise repeat until m links are realised:
    - Choose a node s uniformly at random.
    - Connect to this node according to $p(x_{st}) = \frac{1}{1+e^{\frac{\zeta}{2T}(x_{st}-R_t)}}$, where

$$R_t = \begin{cases} r_t - \frac{2}{\zeta}\ln\left[\frac{2T}{\sin(\pi T)}\frac{\left[1-e^{-\frac{\zeta}{2}(1-\beta)r_t}\right]}{m(1-\beta)}\right], & \text{if } \beta < 1 \\ r_t - \frac{2}{\zeta}\ln\left[\frac{2T}{\sin(\pi T)}\frac{\zeta r_t}{m}\right] & \text{if } \beta = 1. \end{cases}$$

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The PSO model

The main properties of the generated network:

- **The degree distribution is scale-free.**
- **High clustering coefficient.**
- **The degree of the nodes is determined by their radial coordinate.**

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The PSO model

Can we also define a version for $T > 1$?

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The PSO model

Can we also define a version for $T > 1$?

- The former calculation of the degree distribution does not hold.

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The PSO model

Can we also define a version for $T > 1$?

- The former calculation of the degree distribution does not hold.
- In order to retain the same β dependency of the degree distribution, the initial radial coordinate of the nodes has to be changed to $r_t = \frac{2T}{\zeta}\ln t$ instead of $r_t = \frac{2}{\zeta}\ln t$.

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Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

EXTENDED POPULARITY SIMILARITY OPTIMISATION MODEL

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth.
IEEE/ACM Transactions on Networking. **23**, 198–211 (2015).

The E-PSO model
Motivation

- In real complex networks new connections may appear also between already existing nodes as well...

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- E.g., Internet, World Wide Web, online social media, etc.

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Motivation

- In real complex networks new connections may appear also between already existing nodes as well...
- E.g., Internet, World Wide Web, online social media, etc.
- Let's extend the model with this feature!

The E-PSO model

Extension to the PSO-model:

- Grow the network according the PSO model...

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Extension to the PSO-model:

- Grow the network according the PSO model...
- However, at each time step, after connecting the new node with m links to the already existing nodes, we also distribute L extra internal links between the old nodes:

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- Random $i, j < t$ pairs of nodes are selected at random, and are linked according to

$$p(x_{ij}) = \frac{1}{1 + e^{\frac{\zeta}{2T}(x_{ij} - R_t)}}.$$

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- The above step is repeated until L internal connections are realised.

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The E-PSO model

- The average degree becomes

$$\langle k \rangle = 2(m + L).$$

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

The E-PSO model

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→ The degree decay exponent is still $\gamma = 1 + \frac{1}{\beta}$ in the asymptotic limit.

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
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The E-PSO model

What is the difference then?

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth.
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The E-PSO model

What is the difference then?

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F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth.
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The E-PSO model

What is the difference then?

- The extra internal links can decrease further the average distance.
- The densification of sub-graphs spanning between nodes with $k > k_{\text{min}}$ as a function of k_{min} observed in some real networks can be reproduced this way.

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth.
IEEE/ACM Transactions on Networking. **23**, 198–211 (2015).

The E-PSO model

How about distributing the extra "internal" links straight away together with the new links coming with the new node?

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→ The number of new links m on the new nodes is now not uniform, instead depends on t.

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The E-PSO model

How about distributing the extra "internal" links straight away together with the new links coming with the new node?

- → The number of new links m on the new nodes is now not uniform, instead depends on t.
- Still, this allows a formulation of the model that will be very beneficial when trying to embed real networks into the hyperbolic space.

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth. *IEEE/ACM Transactions on Networking.* **23**, 198–211 (2015).

The E-PSO model

- The average number of links created between node s and all previous nodes up to a certain time t if we have also extra internal link formation:

$$\bar{m}_s(t) \simeq m + L\frac{2(1-\beta)}{(1-t^{-(1-\beta)})^2(2\beta-1)}\left[\left(\frac{t}{s}\right)^{2\beta-1} - 1\right]\left(1 - s^{-(1-\beta)}\right).$$

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- At the end of the network generation process $t = N$, thus, for node s the total number of links to previous nodes is

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→ We could replace m in the PSO model by the $\bar{m}_s$ above!

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth. *IEEE/ACM Transactions on Networking.* **23**, 198–211 (2015).

The E-PSO model

The E-PSO model

- In the Extended PSO model we have the following parameters: $\zeta=\sqrt{-K}$, m, L $\beta\in(0,1]$, and $T\in[0,1)$.
- The network is grown according to the rules of the PSO model.
- However, at time step t, the number of new links with which we connect the new node to the already existing part is

$$m_t \simeq m + L\frac{2(1-\beta)}{(1-N^{-(1-\beta)})^2(2\beta-1)}\left[\left(\frac{N}{t}\right)^{2\beta-1}-1\right]\left(1-t^{-(1-\beta)}\right).$$

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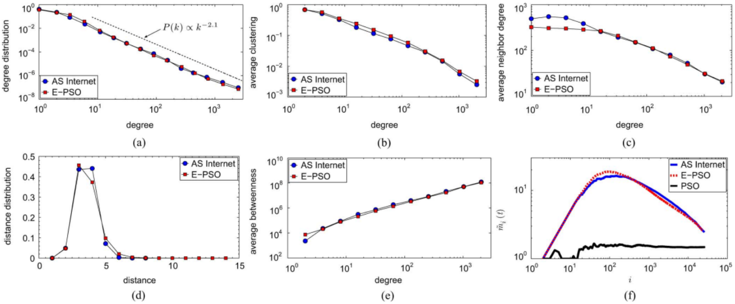
The E-PSO model

Comparing the Internet on the level of Autonomous Systems with the E-PSO model in the original paper:



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The E-PSO model

What about generalising also for link deletion?

→ In the model where we add extra links between old nodes, the basic idea would be something like this:

- Grow the network according the E-PSO model.
- However at each time step, after distributing L_+ extra internal links between the old nodes, also delete L_- links between the old nodes...

B. Kovács, G. Palla: Optimisation of the coalescent hyperbolic embedding of complex networks. *Sci. Rep.* **11**, 8350 (2021).

The E-PSO model

OK, but how to choose the links to be deleted?

- At $T = 0$, the natural choice is to delete the links that connect the node pairs i, j with the largest x_{ij}.

→ At $T > 0$ we can extend this by declaring that for any existing link between old nodes i, j:

  - the probability to survive the link removal is $p(x_{ij}) = \frac{1}{1+e^{\frac{\zeta}{2T}(x_{ij}-R_t)}}$,
  - and the probability to be removed is $1 - p(x_{ij}) = \frac{1}{1+e^{-\frac{\zeta}{2T}(x_{ij}-R_t)}}$.

- With this definition, when $L_+ = L_-$, we recover a network equivalent to a graph generated by the original PSO model (without any insertion or deletion of internal links).

B. Kovács, G. Palla: Optimisation of the coalescent hyperbolic embedding of complex networks. *Sci. Rep.* **11**, 8350 (2021).

The E-PSO model

Again, we can turn this into a model where the extra internal link addition or link deletion is carried out already at the birth of the new node:

- Let's redefine L as the net number of added and removed internal links, $L = L_+ - L_-$.
- The expected number of connections from node s to previous nodes at the end of the network generation process:

$$\bar{m}_s \simeq m + L\frac{2(1-\beta)}{(1-N^{-(1-\beta)})^2(2\beta-1)}\left[\left(\frac{N}{s}\right)^{2\beta-1} - 1\right]\left(1 - s^{-(1-\beta)}\right).$$

- This looks identical to m_s in the previous version, however an important difference is that now L can also be negative.

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Sci. Rep. **11**, 8350 (2021).

The E-PSO model

The E-PSO model'

- In the Extended PSO model' we have the following parameters: $\zeta = \sqrt{-K}$, m, L $\beta \in (0, 1]$, and $T \in [0, 1)$. The L can be now also negative.
- The network is grown according to the rules of the PSO model.
- However, at time step t, the number of new links with which we connect the new node to the already existing part is

$$m_t \simeq m + L\frac{2(1-\beta)}{(1-N^{-(1-\beta)})^2(2\beta-1)}\left[\left(\frac{N}{t}\right)^{2\beta-1} - 1\right]\left(1-t^{-(1-\beta)}\right).$$

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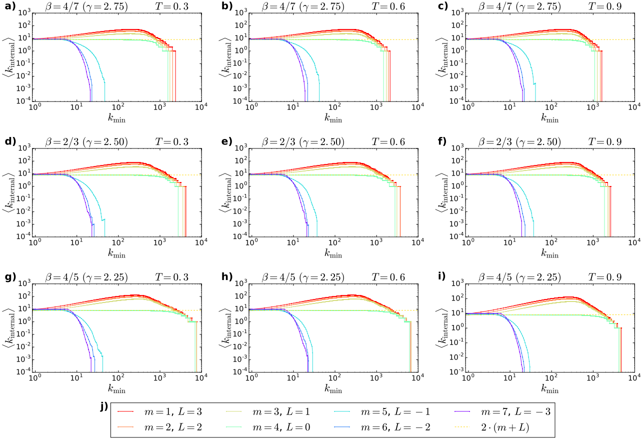
The E-PSO model

Average internal degree of subgraphs spanning between nodes with $k > k_{\min}$ as a function of $k_{\min}$ for E-PSO' networks:



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NONUNIFORM POPULARITY OPTIMISATION MODEL

A. Muscoloni, C. V. Cannistraci: A nonuniform popularity-similarity optimization (nPSO) model to efficiently generate realistic complex networks with communities. *New. J Phys.* **20**, 052002 (2018).

nPSO model

How about non-uniform angular coordinates?

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How about non-uniform angular coordinates?

- In the region of higher node density we can also expect a higher link density (due to "locality").

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nPSO model

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→ In the vicinity of the peaks communities are going to be formed in the resulting network!

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nPSO model

What sort of distributions should we use for θ?

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nPSO model

What sort of distributions should we use for θ?

- Gaussian mixture: superposition of Gaussian distributions.
- Gamma mixture: superposition of Gamma distributions.
- Gaussian and Gamma mixture: superposition of Gaussian and Gamma distributions.
- etc.

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nPSO model

Superposing distributions:

- Suppose we aim for n communities.
- We can define the $\mu_{1...n} \in [0, 2\pi)$ community centers (expected values),
- and also the $\sigma_{1...n} > 0$ community spreads (standard deviations).
- Furthermore, $p_{1...n}$ with $\sum_i p_i = 1$ define the relative community sizes in terms of the number of community members.
- The mixture is

$$\rho(\theta) = \sum_{c=1}^{n} p_c \rho(\theta \mid \mu_c, \sigma_c)$$

- When the sampled θ falls beyond $[0, 2\pi)$, it is shifted back using the modulo operator.

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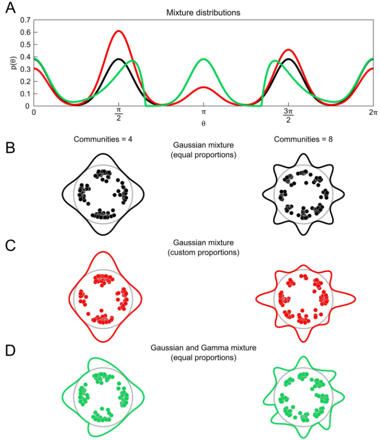
Examples for non-uniform θ distributions:



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nPSO model

nPSO model

- Parameters:
  - The usual PSO parameters: N, m, β, T,
  - The parameters characterising the angular distribution: n, $\{\mu_c\}$, $\{\sigma_c\}$, $\{p_c\}$.
- Grow the network according to the PSO model.
- However, the angular coordinate of the new node is sampled from the non-uniform mixture distribution instead of the uniform distribution over $[0, 2\pi]$.

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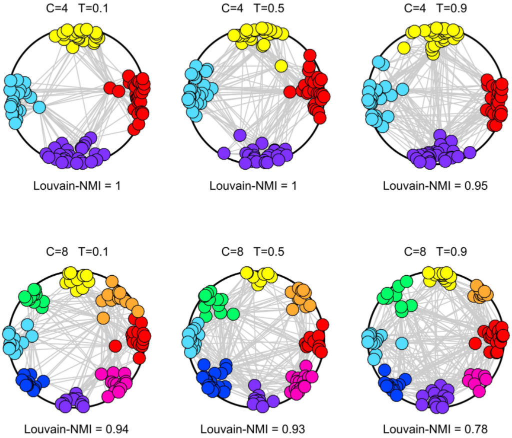

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RANDOM HYPERBOLIC GRAPH MODEL

D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, M. Boguñá:
Hyperbolic geometry of complex networks. *Phys. Rev. E.* **82**, 036106 (2010).

Static network models

A large class of network models are static:

- Erdős-Rényi model,
- Configuration model,
- Static scale-free model,
- Stochastic block model,
- Hidden variable model,
- etc.

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- etc.

$\rightarrow$ **What about a static hyperbolic model?**

Concept of the $\mathbb{H}^2$ model:

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Concept of the $\mathbb{H}^2$ model:

- Place N nodes uniformly at random inside a circle of radius R in the native disk.

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Concept of the $\mathbb{H}^2$ model:

- Place N nodes uniformly at random inside a circle of radius R in the native disk.
- Connect the node pairs according to a probability decaying with the hyperbolic distance.

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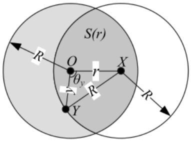
→ Simplest idea is to connect with all other nodes closer than R:



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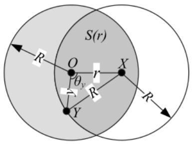

- The $\langle k \rangle$ can be controlled by the choice of R, and the resulting network is scale-free and highly clustered.

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The $\mathbb{H}^2$ model
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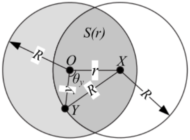

- The $\langle k \rangle$ can be controlled by the choice of R, and the resulting network is scale-free and highly clustered.
- To allow control also over the degree decay exponent γ, the radial coordinates have to be turned slightly non-uniform (similarly to popularity fading in PSO).

D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, M. Boguñá:
Hyperbolic geometry of complex networks. *Phys. Rev. E.* **82**, 036106 (2010).

The $\mathbb{H}^2$ model

- The $\mathbb{H}^2$ model is also equivalent to the $\mathbb{S}^1$ model...

The $\mathbb{S}^1$ model

- The $\mathbb{S}^1$ approach aims at modelling a network with one of the simplest possible underlying metric structure, a circle.
- It is also a **hidden variable model**, where the connection probability is affected by "hidden" variables associated to the nodes.

M. Á. Serrano D. Krioukov, M. Boguñá: Self-Similarity of Complex Networks and Hidden Metric Spaces. *Phys. Rev. Lett.* **100**, 078701 (2008).

The $\mathbb{S}^1$ model

The $\mathbb{S}^1$ model

- Parameters: N, the hidden variable distribution $\rho(\kappa)$, a connection function $p(\chi)$, and μ, controlling the average degree.
- Place the nodes uniformly at random on a circle of radius $\frac{N}{2\pi}$.
- Assign hidden variables drawn from $\rho(\kappa)$. Let us focus on the case where
$$\rho(\kappa) = \frac{(\gamma - 1)\kappa^{-\gamma}}{\kappa_0^{1-\gamma}}.$$
- Connect node pairs at θ, θ' separated by an arc distance of $d = N\Delta\theta/2\pi$ with probability
$$p(\chi) \quad \text{where } \chi = \frac{d}{\mu\kappa\kappa'}.$$
($p(\chi)$ can be any integrable function).

M. Á. Serrano D. Krioukov, M. Boguñá: Self-Similarity of Complex Networks and Hidden Metric Spaces. *Phys. Rev. Lett.* **100**, 078701 (2008).

The $\mathbb{S}^1$ model

- With appropriate choice of κ_0, the expected degree of a node with hidden variable κ becomes $\bar{k}(\kappa) = \kappa$.

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The $\mathbb{S}^1$ model

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- How to map this to $\mathbb{H}^2$?
- In $\mathbb{H}^2$ the degree is controlled by r. The mapping

$$r_t = \hat{R} - 2\ln\left(\frac{\kappa_t}{\kappa_0}\right) \quad \leftrightarrow \quad \kappa_t = \kappa_0 e^{\frac{\hat{R}-r_t}{2}}$$

with $\hat{R} = 2\ln\left(\frac{N}{\mu\pi\kappa_0^2}\right)$ is connecting equivalent models where

$$p(\chi) = p\left(\frac{d}{\mu\kappa_s\kappa_t}\right) = p\left(\frac{N\theta_{st}}{2\pi\mu\kappa_s\kappa_t}\right) = p\left(\frac{N\theta_{st}}{2\pi\mu\kappa_0^2}e^{\frac{r_s+r_t-2\hat{R}}{2}}\right) =$$

$$p\left(\frac{N\theta_{st}}{2\pi\mu\kappa_0^2}e^{\frac{r_s+r_t-\hat{R}}{2}}\frac{\mu\pi\kappa_0^2}{N}\right) = p\left(e^{\frac{r_s+r_t+\ln(\theta_{st}/2)-\hat{R}}{2}}\right) = p\left(e^{\frac{x_{st}-\hat{R}}{2}}\right)$$

D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, M. Boguñá:
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The $\mathbb{S}^1/\mathbb{H}^2$ model

Concept of the $\mathbb{S}^1/\mathbb{H}^2$ model:

- Place N nodes uniformly at random on a circle (the $\mathbb{S}^1$ space), and assign a hidden variable to each node that controls its attractiveness.
- Connect node pairs according to a probability that depends both on the hidden variables and the angular separation.
- To obtain a hyperbolic network, convert the hidden variables into radial coordinates in the native disk, and your nodes are now placed in the $\mathbb{H}^2$ space.

G. García-Pérez, A. Allard, M. Á. Serrano, M. Boguñá:
Mercator: uncovering faithful hyperbolic embeddings of complex networks. *New J. Phys.* **21**, 123033 (2019).

The $\mathbb{S}^1/\mathbb{H}^2$ model

The $\mathbb{S}^1/\mathbb{H}^2$ model

- Parameters: N, $\langle k \rangle$, the degree decay exponent γ of the target degree distribution, and $\alpha > 1$, controlling the average clustering coefficient.
- Assign each node i an angular coordinate of $\theta_i \in [0, 2\pi)$ uniformly at random, and a hidden variable κ_i sampled from $\rho(\kappa) = (\gamma - 1) \cdot \frac{\kappa^{-\gamma}}{\kappa_0^{1-\gamma}}$, where $\kappa_0 = \frac{\gamma-2}{\gamma-1} \cdot \langle k \rangle$.
- Each pair of nodes $i - j$ is connected with probability

$$p_{ij} = \frac{1}{1 + \left(\frac{N \cdot \Delta\theta_{ij}}{2\pi \cdot \mu \cdot \kappa_i \cdot \kappa_j} \right)^{\alpha}},$$

  where $\Delta\theta_{ij} = \pi - |\pi - |\theta_i - \theta_j||$ is the angular distance between the nodes, and $\mu = \frac{\alpha}{2\pi \langle k \rangle} \cdot \sin\left(\frac{\pi}{\alpha}\right)$.
- Convert the hidden variables into a radial coordinates in the native disk (at $K = -1$) as $r_i = \hat{R} - 2\ln\left(\frac{\kappa_i}{\kappa_0}\right)$, where $\hat{R} = 2\ln\left(\frac{N}{\mu\pi\kappa_0^2}\right)$.

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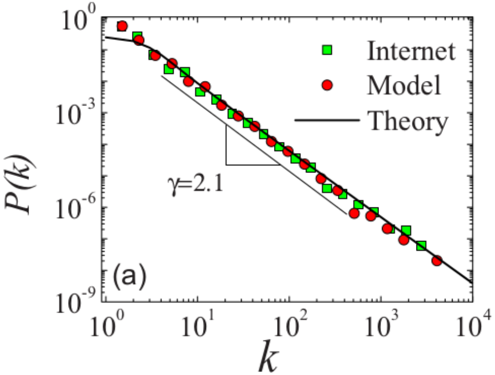
The $\mathbb{S}^1/\mathbb{H}^2$ model

Simulation results from the original paper:



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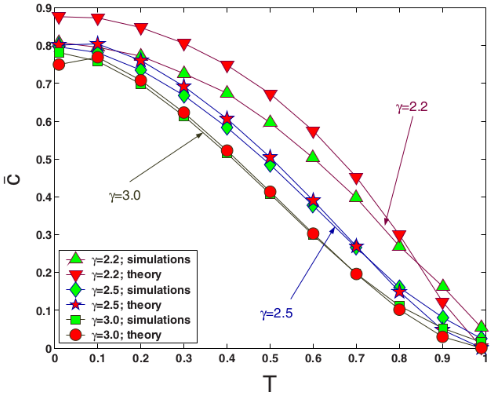

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Hyperbolic embedding

WHY EMBED A NETWORK?

Why embed a network?

- Embedding a network into a hyperbolic space can be considered as the "inverse" problem of modelling:
  - instead of drawing links based on coordinates
  - we try to guess coordinates based on links.

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→ Interesting problem on its own.

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  - instead of drawing links based on coordinates
  - we try to guess coordinates based on links.

→ Interesting problem on its own.

- Practical benefits:
  - can be used for greedy routing.
  - can be used for missing link prediction.
  - can provide input for machine learning tasks.
  - can define a clearly organised intuitive layout!

Greedy routing

If the network is embedded into a metric space, a navigation protocol using only local information can be defined:

Greedy routing

If the network is embedded into a metric space, a navigation protocol using only local information can be defined:

- Based on the target coordinate and the coordinates of the neighbours, the current node will forward to the neighbour that is the closest to the target.

Greedy routing

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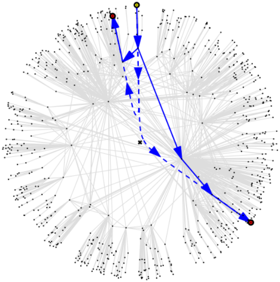
Greedy routing can be extremely efficient in random graphs generated by hyperbolic models!

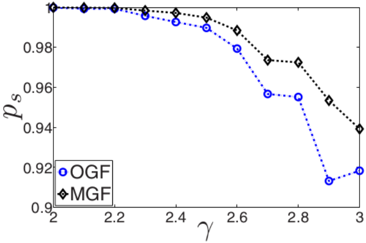
Greedy routing

In RHG networks, shortest paths, greedy routing paths and geodesic lines are usually very close to each other:

Illustration:



Fraction of successful greedy routing paths:



D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, M. Boguñá:
Hyperbolic geometry of complex networks. *Phys. Rev. E.* **82**, 036106 (2010).

How to embed a network?

How to embed a network?

- **Likelihood optimisation**
  (with respect to an assumed hyperbolic model).

- **Dimension reduction.**

- **Mixing the above two ideas.**

LIKELIHOOD OPTIMISATION

Likelihood optimisation

Let's assume a network model $\mathcal{M}$ in general, with parameter set $\{\sigma\}$, specifying the connection probability between node pairs in some way

$$P(A_{ij}=1)=p_{\mathcal{M}}(i,j\mid\{\sigma\}),\qquad P(A_{ij}=0)=1-p_{\mathcal{M}}(i,j\mid\{\sigma\}).$$

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$$\mathcal{L}(\mathbf{A})=P(\mathbf{A}\mid\{\sigma\})=\prod_{i<j}\left[p_{\mathcal{M}}(i,j\mid\{\sigma\})\right]^{A_{ij}}\left[1-p_{\mathcal{M}}(i,j\mid\{\sigma\})\right]^{1-A_{ij}}$$

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$$\ln\mathcal{L}(\mathbf{A})=\sum_{i<j}A_{ij}\ln\left[p_{\mathcal{M}}(i,j\mid\{\sigma\})\right]+\sum_{i<j}(1-A_{ij})\ln\left[1-p_{\mathcal{M}}(i,j\mid\{\sigma\})\right].$$

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Here $\{\sigma\}$ are fixed, and A_{ij} can vary if e.g., we generate more samples using $\mathcal{M}$.

Likelihood optimisation

- What if we observe a given network, and would like to find the best fitting $\{\sigma\}$?

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  - Since connection probabilities cannot exceed 1, $\ln \mathcal{L}(\mathbf{A}) < 0$.
  - Equivalently to maximising $\ln \mathcal{L}(\mathbf{A})$ we can minimise the logarithmic loss

$$LL = -\ln \mathcal{L}(\mathbf{A}) = \\ -\sum_{i<j} A_{ij} \ln \left[p_{\mathcal{M}}(i,j \mid \{\sigma\}) \right] - \sum_{i<j} (1 - A_{ij}) \ln \left[1 - p_{\mathcal{M}}(i,j \mid \{\sigma\}) \right].$$

Likelihood optimisation

Bayesian inference:
According to Bayes' theorem, the conditional probability that the observed $\mathbf{A}$ was generated using $\{\sigma\}$ is

$$\underbrace{P(\{\sigma\} \mid \mathbf{A}; \lambda)}_{\text{posterior}} = \frac{\overbrace{P(\mathbf{A} \mid \{\sigma\})}^{\text{likelihood}} \overbrace{P(\{\sigma\} \mid \lambda)}^{\text{prior}}}{\underbrace{P(\mathbf{A} \mid \lambda)}_{\text{evidence}}},$$

where

- **Prior:** The distribution of the model parameters, controlled by hyperparameter λ.
- **Evidence:** Also called as marginal likelihood:

  $$P(\mathbf{A} \mid \lambda) = \int P(\mathbf{A} \mid \{\sigma\}) P(\{\sigma\} \mid \lambda) d\sigma_1 ... d\sigma_n.$$

  Does not depend on $\{\sigma\}$, thus, **can be also treated as a constant**.
- **Posterior**: the distribution of $\{\sigma\}$ we are interested in, depending on both the observed data and the prior.

Likelihood optimisation

- **Uninformed prior**: If we have no prior belief regarding the values of $\{\sigma\}$ we can assume a uniform distribution over all possible values.

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$$P(\{\sigma\} \mid \mathbf{A}) = \frac{P(\mathbf{A} \mid \{\sigma\}) \overbrace{P(\{\sigma\})}^{\text{constant}}}{\underbrace{P(\mathbf{A})}_{\text{constant}}} \propto P(\mathbf{A} \mid \{\sigma\})$$

the posterior distribution becomes simply proportional to the likelihood.

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  the posterior distribution becomes simply proportional to the likelihood.
- How to sample from the posterior distribution?

→ Using **Markov-Chain Monte Carlo (MCMC)** methods:

  - the sampled σ form a Markov-Chain, where the next σ is chosen from candidates in the vicinity of the present value,
  - and the acceptance probabilities are set such that in the long run, the distribution of the sampled σ follows the posterior.

Likelihood optimisation

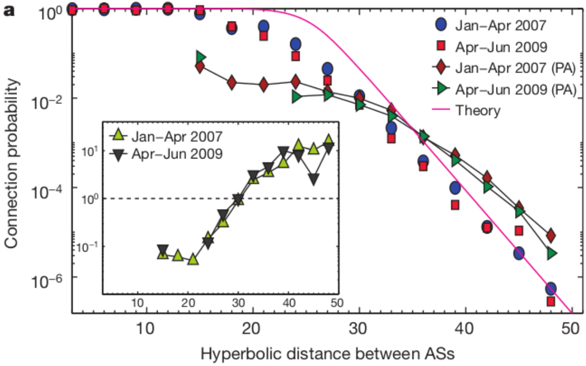
Likelihood optimisation for the PSO-model:

- The m, β and T parameters can be estimated based on overall network properties such as $\langle k \rangle$, $\langle C \rangle$ and γ.
- The radial coordinates can be set by matching the actual degree of the node to the expected degree at r, using that $\bar{k}_s(t) \sim e^{r_t - r_s(t)}$.
- The angular coordinates are optimised using MCMC.

F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, D. Krioukov:
Popularity versus similarity in growing networks. *Nature* **489**, 53 (2012)

Likelihood optimisation

Likelihood optimisation in the original PSO paper:



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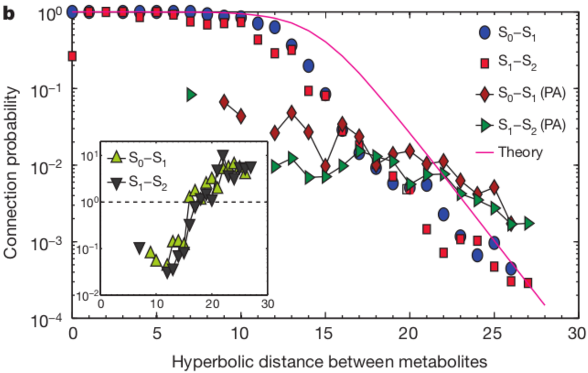

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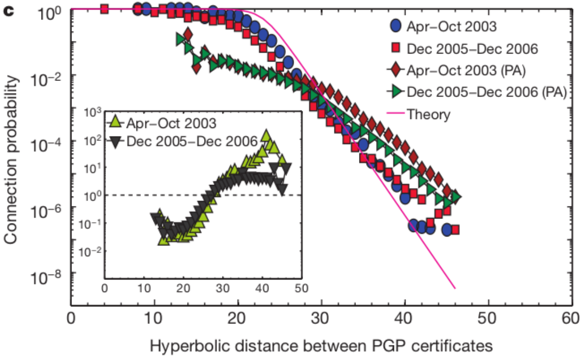

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HYPERMAP

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth.
IEEE/ACM Transactions on Networking. **23**, 198–211 (2015).

Concepts of HyperMap:

- Perform a likelihood optimisation with respect to the E-PSO model.
- However, instead of a "standard" MCMC method, replay the assumed network growth, and find the optimal coordinate "locally" for the new node at each step.

F. Papadopoulos, C. Psomas, D. Krioukov: Network Mapping by Replaying Hyperbolic Growth. *IEEE/ACM Transactions on Networking.* **23**, 198–211 (2015).

HyperMap

Important note:

- We are going to assign coordinates to the nodes that correspond to their position at the end of the network generation process.
- However, since popularity fading is pulling the nodes outward during every time step, the actual node coordinates when the connections arise are different from these!
- Luckily, the probability that s and t, having a distance x_{st} at the end of the network generation are connected can be given as

$$\tilde{p}(x_{st}) = \frac{1}{N - s_{\min} + 1} \sum_{s=s_{\min}}^{N} \frac{1}{1 + e^{\frac{\zeta}{2T}(x_{st} - R_N + \Delta_s)}} \simeq \frac{1}{1 + e^{\frac{\zeta}{2T}(x_{st} - R_N)}},$$

where $s_{\min} = \max(2, \lceil N e^{-\frac{\zeta x_{st}}{4(1-\beta)}} \rceil)$, and $\Delta_s = \frac{2}{\zeta} \ln \left[\left(\frac{N}{s} \right)^{2\beta - 1} \frac{m I_s}{m_s I_N} \right]$.

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HyperMap

- The likelihood of observing an adjacency matrix A_{ij} for given final hyperbolic distances x_{ij} can be written as

$$\mathcal{L}_A \equiv \mathcal{L}(A_{ij} \mid \{r_i(t=N), \theta_i\}, m, L, \zeta, \beta, T) = \prod_{1 \le j < i \le N} \tilde{p}(x_{ij})^{A_{ij}} \left[1 - \tilde{p}(x_{ij})\right]^{1-A_{ij}}.$$

- Bayes' theorem:

$$\mathcal{L}_{r,\theta} \equiv \mathcal{L}_{r,\theta}(\{r_i(N), \theta_i\} \mid A_{ij}, m, L, \zeta, \beta, T) = \frac{\mathcal{L}(\{r_i(N), \theta_i\} \mid m, L, \zeta, \beta, T) \cdot \mathcal{L}_A}{\mathcal{L}(A_{ij} \mid m, L, \zeta, \beta, T)},$$

where the conditional probability for obtaining the final node coordinates $\{r_i(N), \theta_i\}$ given the model parameters is

$$\mathcal{L}(\{r_i(N), \theta_i\} \mid m, L, \zeta, \beta, T) = \mathcal{L}(\{r_i(N), \theta_i\} \mid \zeta, \beta) = \frac{1}{(2\pi)^N} \prod_{i=1}^{N} \frac{\zeta}{2\beta} e^{\frac{\zeta}{2\beta}(r_i(N) - r_N(N))},$$

where $r_N(N) = \frac{2}{\zeta} \ln N$.

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HyperMap

- The logarithmic loss is

$$LL_{r,\theta} = -\ln \mathcal{L}_{r,\theta} = C - \frac{\zeta}{2\beta}\sum_{i=1}^{N} r_i(N) - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} A_{ij}\ln \tilde{p}(x_{ij}) - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(1-A_{ij})\ln\left[1-\tilde{p}(x_{ij})\right]$$

- The optimal value for the radial coordinates can be calculated analytically, resulting in

$$r_i^*(i) = \frac{2}{\zeta}\ln i^*, \qquad r_i^*(N) = \beta r_i^*(i) + (1-\beta) r_N^*(N),$$

where the optimal ordering of the nodes given by i^* is following the node degrees, with the largest degree node in the network obtaining $i^* = 1$.

- Thus, we have to optimise "only" the angular coordinates based on

$$LL_\theta = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} A_{ij}\ln \tilde{p}(x_{ij}) - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(1-A_{ij})\ln\left[1-\tilde{p}(x_{ij})\right].$$

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HyperMap

- Instead of a general MCMC method, we take advantage of that the degrees define both a radial and a time ordering:
  - 1^{st} hub: $i = 1$,
  - 2^{nd} hub: $i = 2$,
  - etc.
- We can replay the network growth as follows:
  - Add the nodes one by one at their starting radial coordinates,
  - update the radial coordinates (popularity fading),
  - and optimise the angular coordinate of the "new" node j based on its connections to previous nodes, using a local likelihood

$$LL_{\text{loc.}} = -\sum_{i=1}^{j-1} A_{ij} \ln p(x_{ij}) - \sum_{i=1}^{j-1} (1 - A_{ij}) \ln \left[1 - p(x_{ij})\right].$$

(Here we can use the original E-PSO connection probability).

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HyperMap

HyperMap embedding algorithm

- Set m, L, β and T according to the "global" properties of the network such as $\langle k \rangle$, $k_{\min}$, $\langle C \rangle$ and γ.
- Sort node degrees in decreasing order $k_1 > k_2 > \cdots > k_N$. (Break ties randomly).
- Assign node indices according to the degree order.
- Node $i = 1$ is born with initial radial coordinate $r_1(t = 1) = 0$ and a random $\theta_1 \in [0, 2\pi]$.
- for $i = 2$ to N do:
    - Node i is born with $r_i(t = i) = \frac{2}{\zeta}\ln(i)$.
    - Increase the radial coordinate of all previous nodes $j < i$ as $r_j(i) = \beta r_j(j) + (1-\beta) r_i(i)$.
    - Assign node i the θ_i that maximises the local likelihood

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HyperMap

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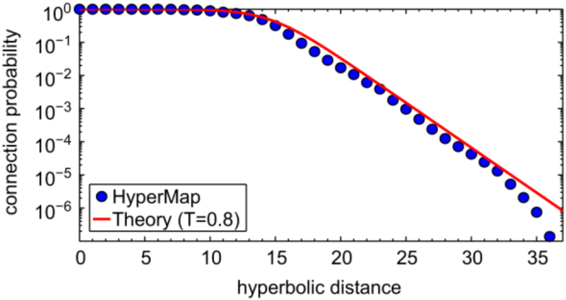

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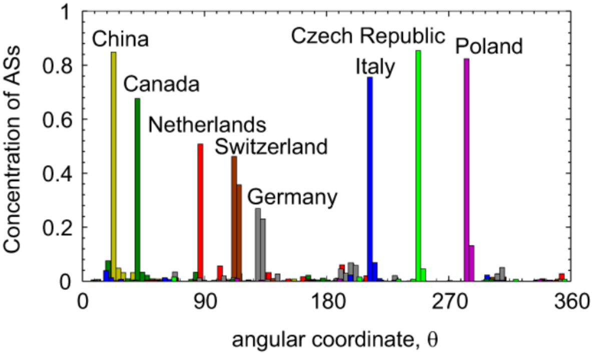

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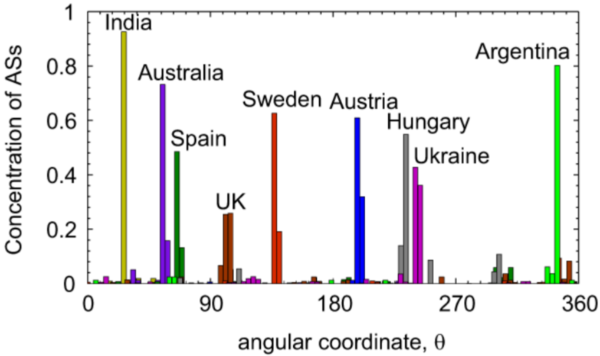

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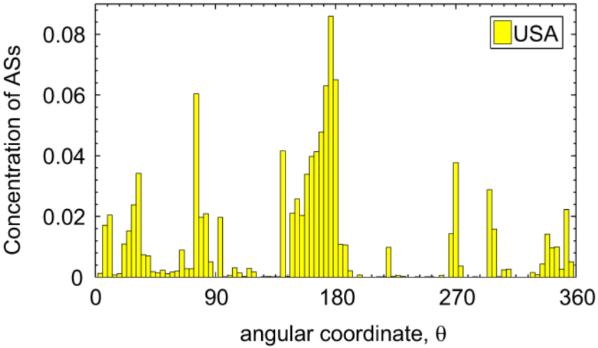

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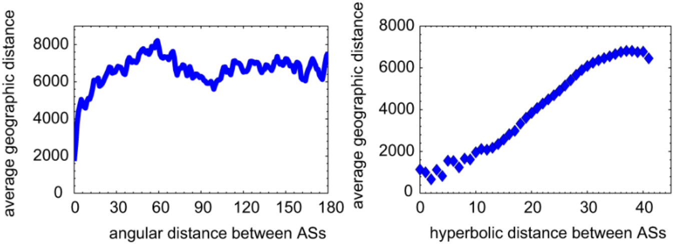

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EMBEDDING VIA DIMENSION REDUCTION

Embedding via dimension reduction

- Rough flowchart of this approach:

network

↓

"distance" matrix

↓

dimension reduction

↓

hyperbolic coordinates

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network
↓
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- How could this work?

Embedding via dimension reduction

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  When data is organised into some lower dimensional manifold embedded in higher dimensional space, revealing the manifold can be beneficial.

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  When applying manifold learning techniques on networks generated with hyperbolic models, they can provide a 1d manifold organised according to the original angular coordinates in the network.

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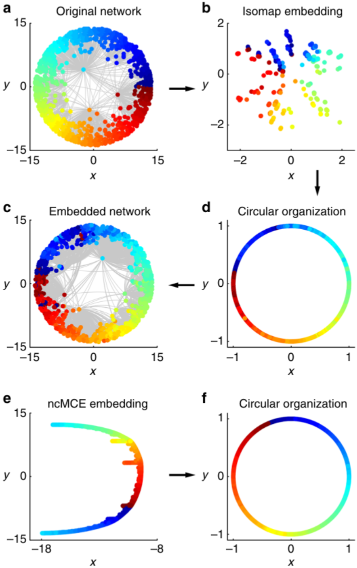
→ We can exploit this for inferring the angular coordinates!

Coalescent embeddings

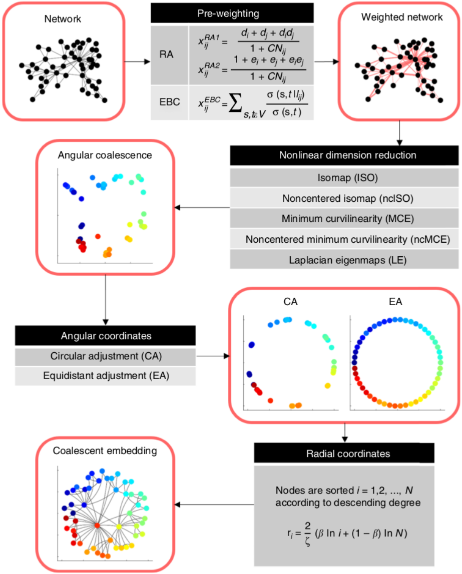
Angular coalescence



A. Muscoloni, J. M. Thomas, S. Ciucci, G. Bianconi, C. V. Cannistraci:
Machine learning meets complex networks via coalescent embedding in the hyperbolic space. *Nat. Commun.* **8**, 1615 (2017).

Coalescent embeddings

Flow chart



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ncMCE embedding

Non-centered minimum curvilinear embedding:

- The matrix $\mathbf{D}$ we prepare is trying to model the minimum curvilinear distances between the nodes.
- Otherwise we follow the general flowchart of coalescent embeddings with SVD dimension reduction.

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ncMCE embedding

- **Pre-weighting**: we prepare a matrix $\mathbf{W}$ with elements

$$W_{ij} = \frac{k_i + k_j + k_i k_j}{1 + CN_{ij}},$$

where CN_{ij} is the number of common neighbours between i and j.

→ This way nodes in different neighbourhoods obtain larger W_{ij}, i.e., they are less similar.

- We prepare the **minimum weight spanning tree** of $\mathbf{W}$, and define $\mathbf{D}$ based on the pairwise distance in the spanning tree.

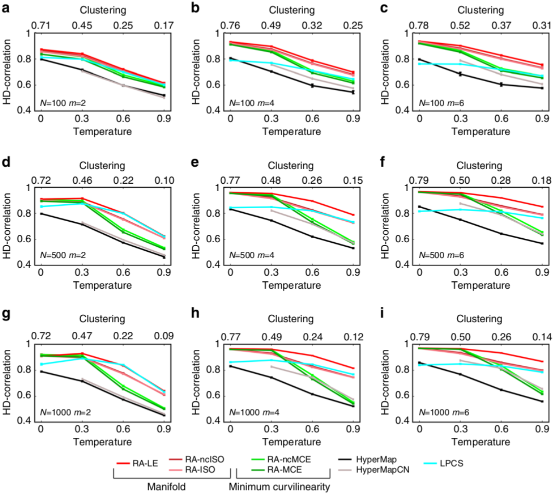
  D_{ij} is an estimate for the min. curvilinear distance between i and j

- The **dimension reduction** is carried out via singular value decomposition, $\mathbf{D} = \mathbf{U}\Sigma\mathbf{V}^T$, where Σ is a diagonal matrix, from which we keep only the two largest ones (the rest is put to 0).
- Angular coordinates are obtained from the 2nd column of $\mathbf{X} = \left(\sqrt{\Sigma}\cdot\mathbf{V}^T\right)^T$.
- These are then rescaled in an **equidistant** manner in $[0, 2\pi)$.
- Radial coords. are set based on the degree, similarly to Hypermap.

A. Muscoloni, J. M. Thomas, S. Ciucci, G. Bianconi, C. V. Cannistraci:
Machine learning meets complex networks via coalescent embedding in the hyperbolic space. *Nat. Commun.* **8**, 1615 (2017).

Coalescent embeddings
Results

Correlation between original and embedded hyperbolic distances for PSO networks:



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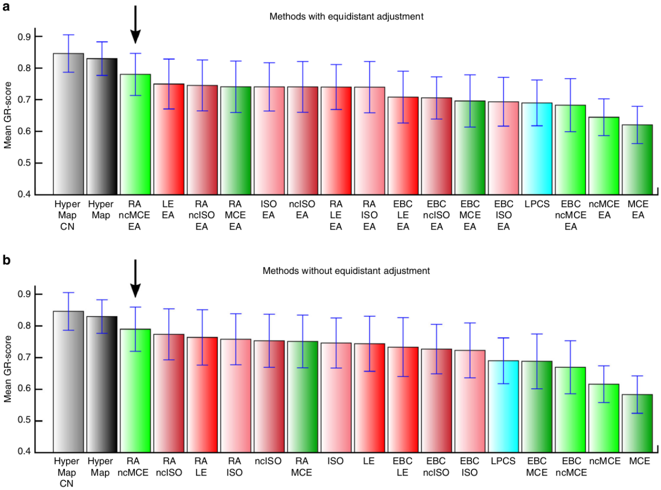
Average greedy routing scores for embedded real networks:



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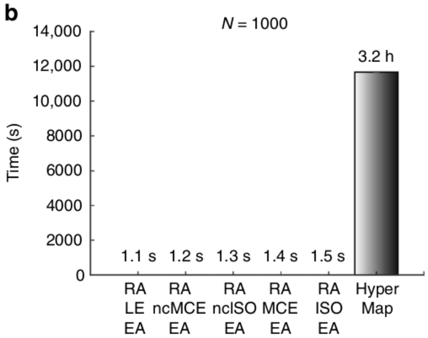
Running times on PSO networks



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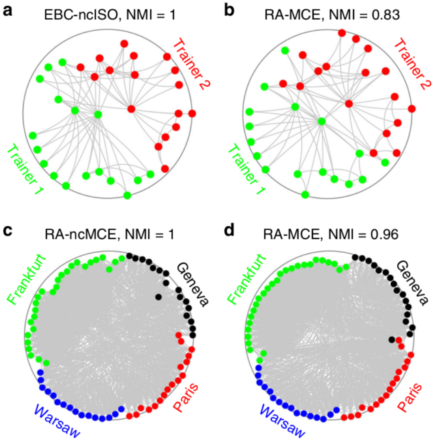
Machine learning meets complex networks via coalescent embedding in the hyperbolic space. *Nat. Commun.* **8**, 1615 (2017).

Embedded layouts for social networks:



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Machine learning meets complex networks via coalescent embedding in the hyperbolic space. *Nat. Commun.* **8**, 1615 (2017).

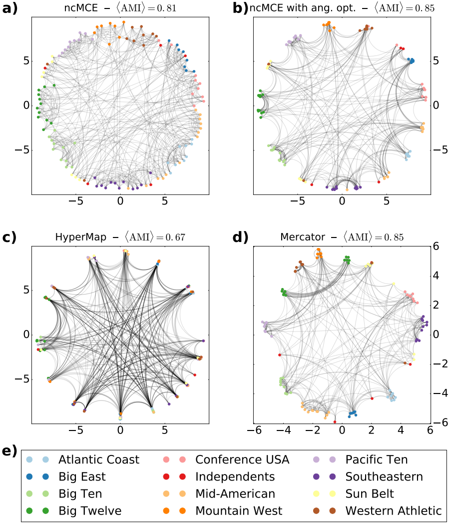
Optimised coalescent embedding

- Dimension reduction and likelihood optimisation can also be combined.

- Since radial coordinates are set according to the PSO model also in coalescent embeddings, it can make sense to apply a further angular optimisation (using likelihood optimisation) on the coordinates obtained from dimension reduction.

B. Kovács, G. Palla: Optimisation of the coalescent hyperbolic embedding of complex networks. *Sci. Rep.* **11**, 8350 (2021).

Optimised coalescent embedding



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Hyperbolic communities

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Communities, **modules**, **clusters**, or **cohesive groups**:
more highly interconnected parts in networks with no widely accepted unique definition.

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Modularity

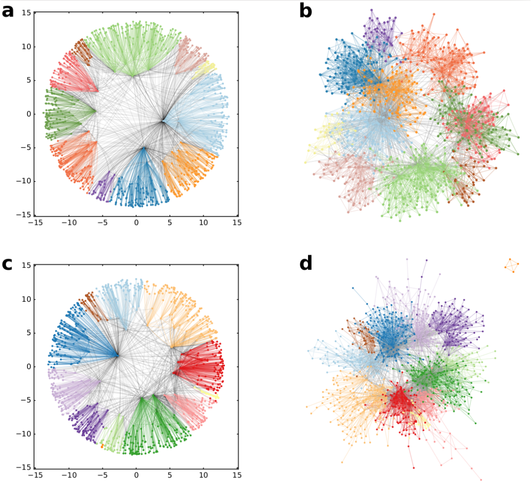
- Modularity is the most widely used quantity for measuring the "strength" of communities based on the network structure.
- It compares the observed fraction of links inside community c with expected fraction of inside links based on the configuration model:

$$Q = \sum_{c=1}^{n} \left[\frac{L_c}{L} - \left(\frac{k_c}{2L} \right)^2 \right]$$

Communities in PSO and RHG networks

Commuities found by maximising Q

Communities found by Louvain algorithm in PSO and RHG networks:



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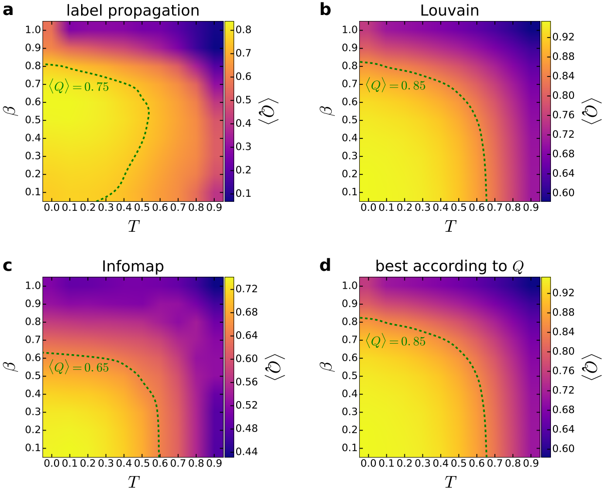
Communities in PSO and RHG networks



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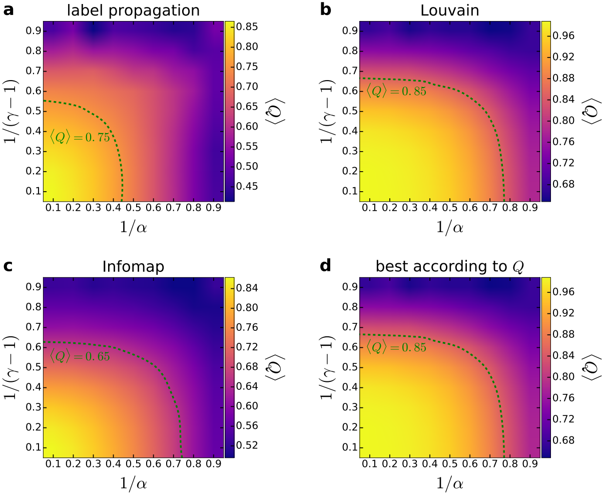

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