Higher-order networks

An introduction to simplicial complexes

Lesson I

Mathematics of Large Networks
Erdos Center, Budapest

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Outline of the lessons

1. Higher order networks structure

2. Higher-order network models and emergent geometry

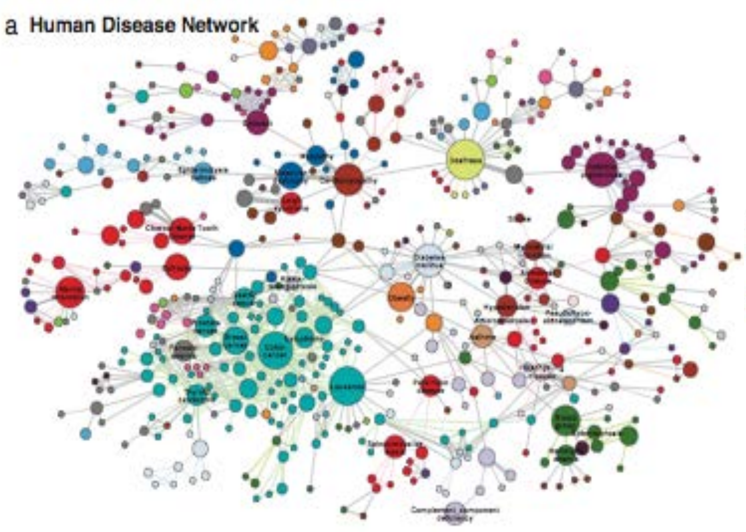
3. Interplay between higher-order topology and dynamics

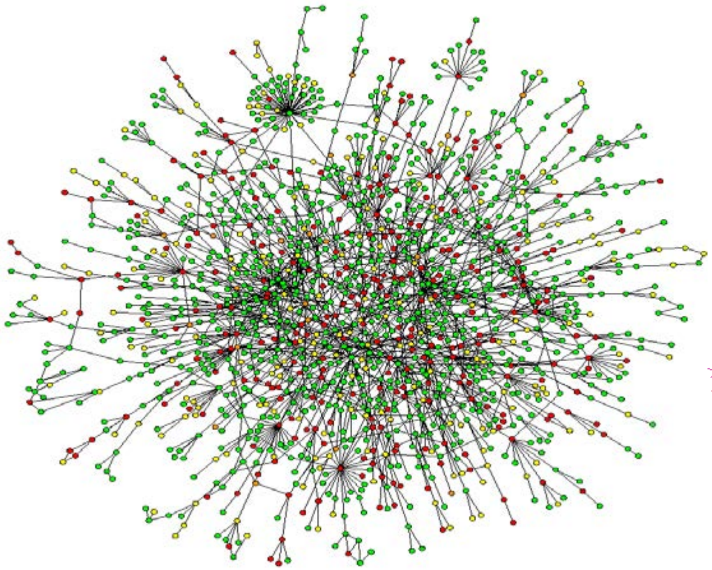
Lesson I:
Higher order networks structure

- **Background on networks and growing network models**
- **Higher-order networks**

  1. **Definitions**
  2. **Introduction to network geometry**

Networks





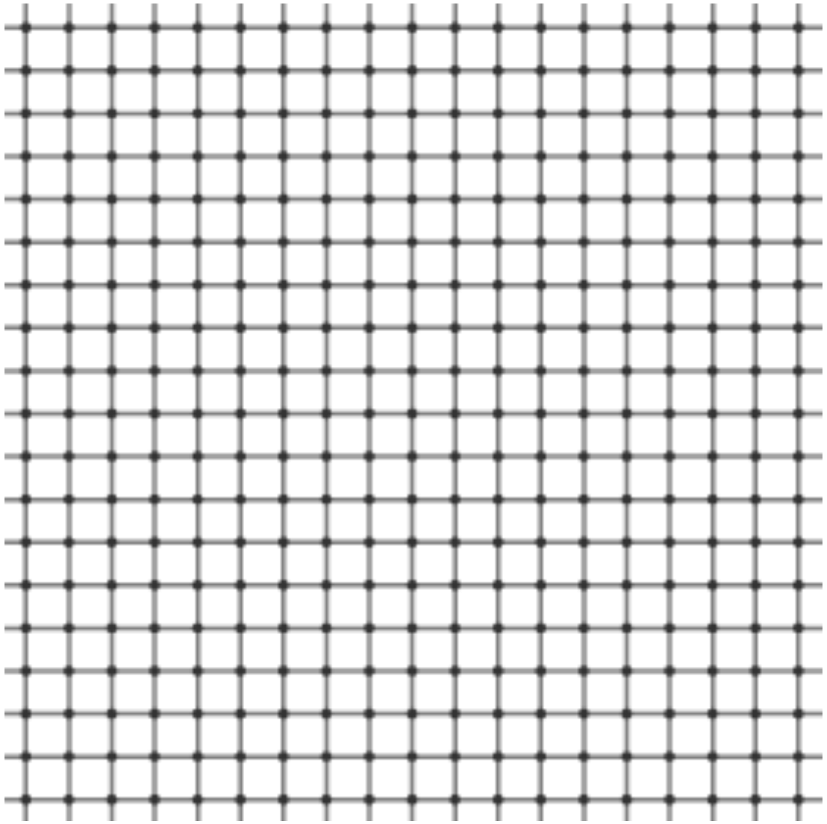
describe

the interactions between the elements

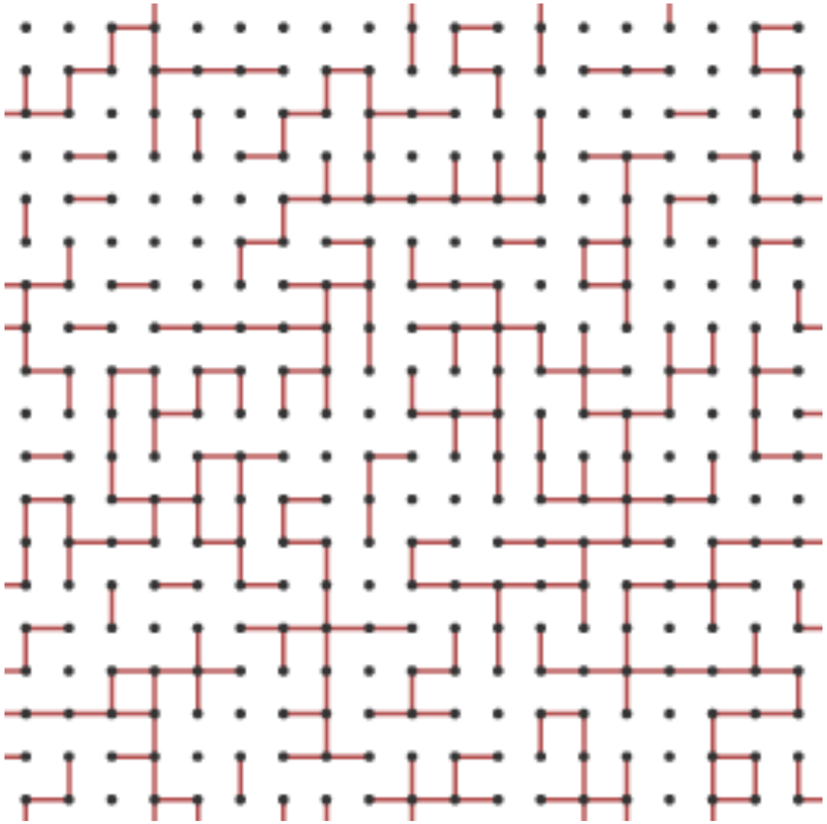
of large complex systems.

Randomness and order
Percolation

p=1

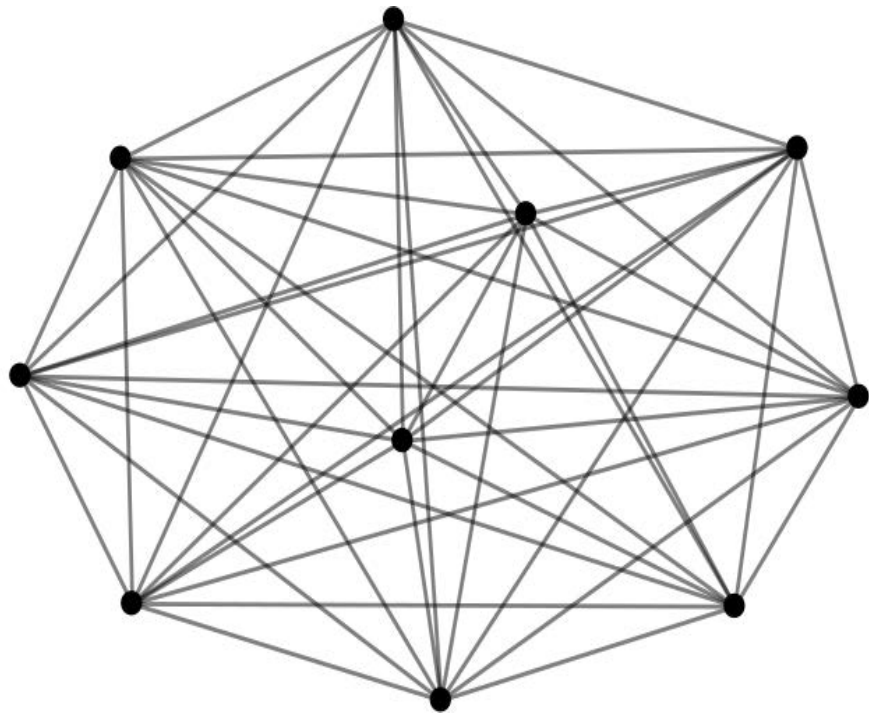


p=0.4



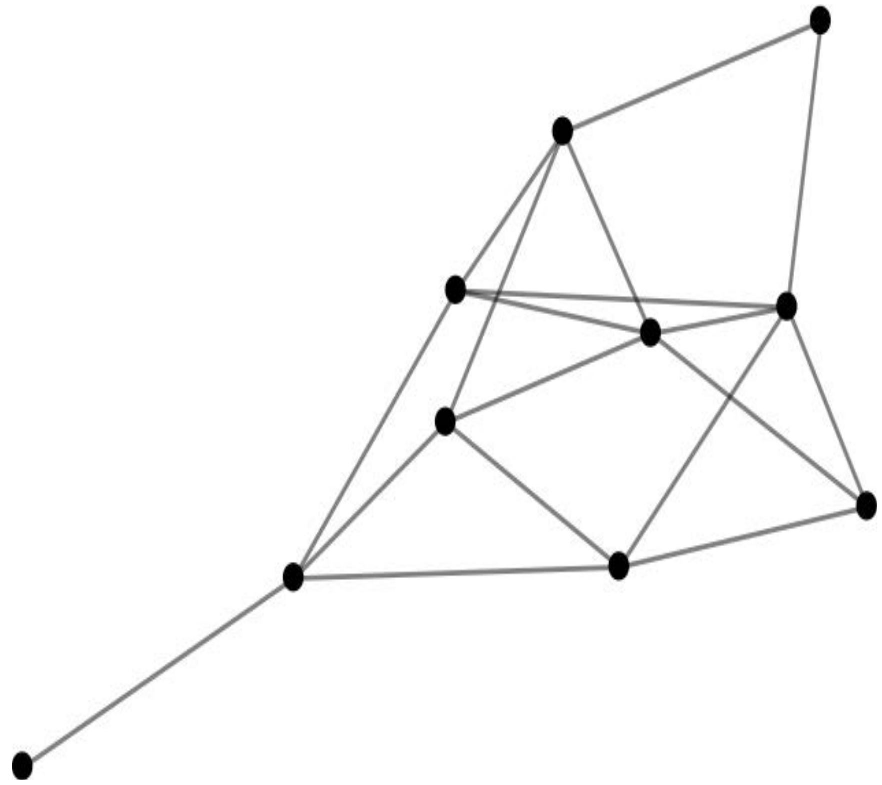
Randomness and order
Random graph

p=1



Complete graph

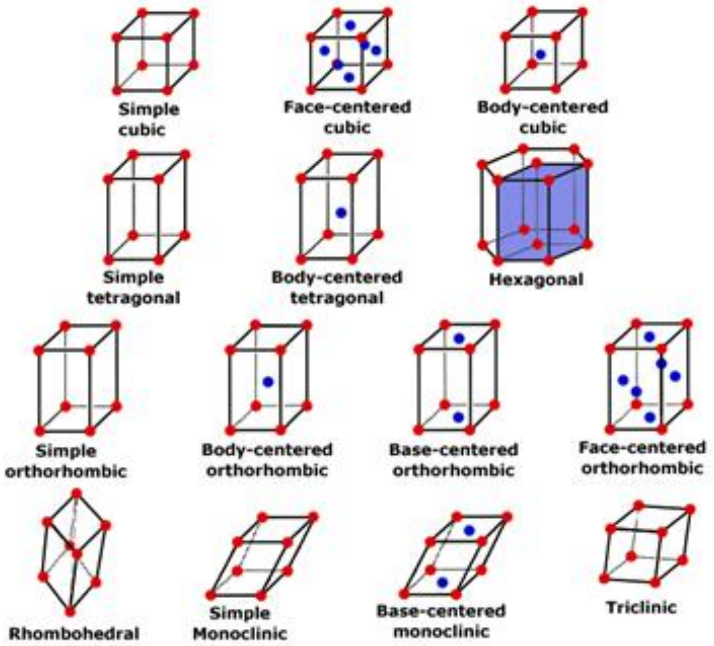
p=0.4



Erdos-Renyi graph

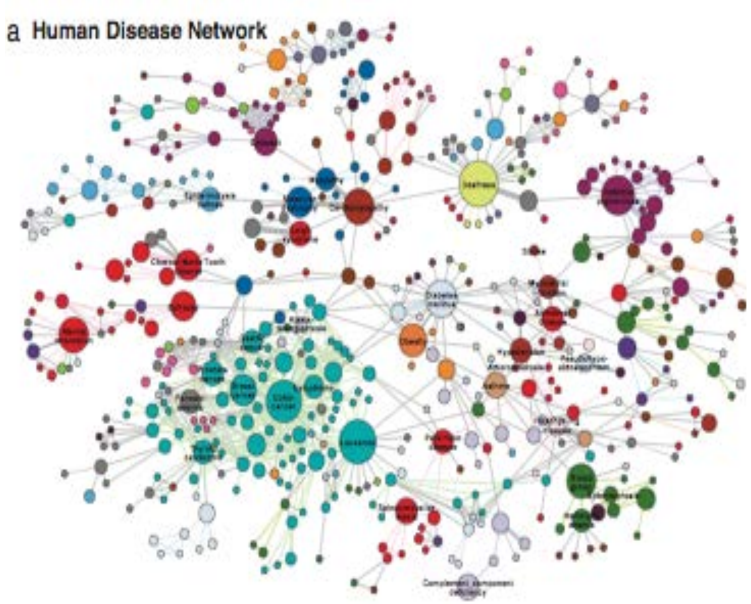
Randomness and order
Complex networks

LATTICES



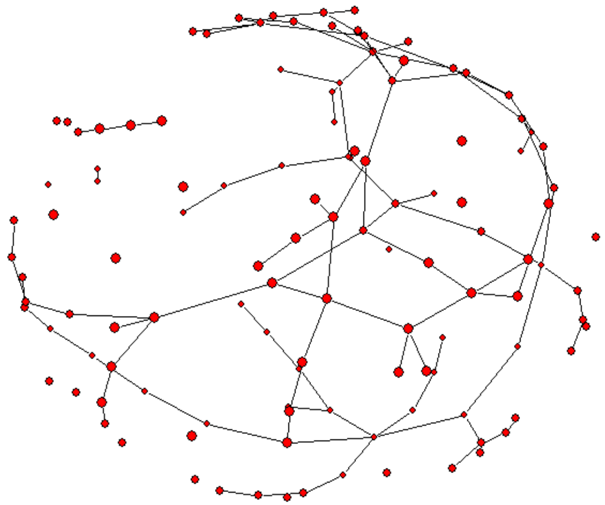
Regular networks
Symmetric

COMPLEX NETWORKS



Scale free networks
Small world
With communities
ENCODING INFORMATION IN THEIR STRUCTURE

RANDOM GRAPHS



Totally random
Binomial degree distribution

Universalities

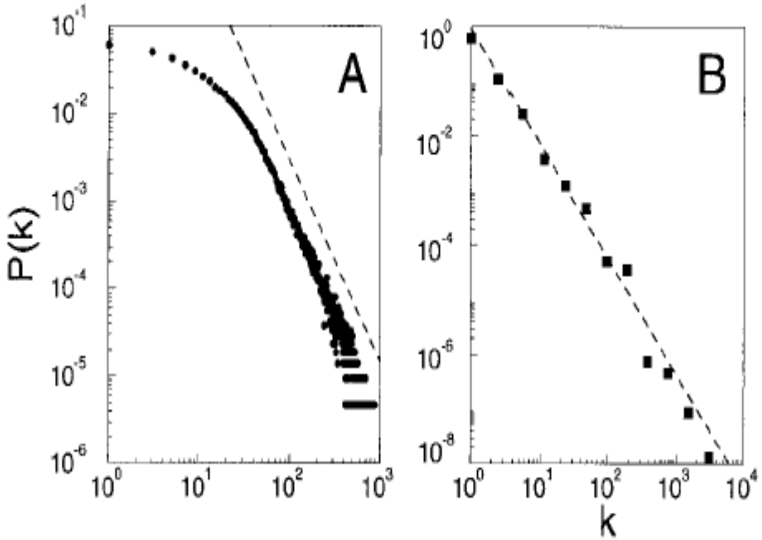
- Small-world: $d_H = \infty$

  [Watts & Strogatz 1998]

- Scale-free: $P(k) \sim k^{-\gamma}$ **for** $k \gg 1$

  [Barabasi & Albert 1999]

  with $\gamma \in (2,3]$

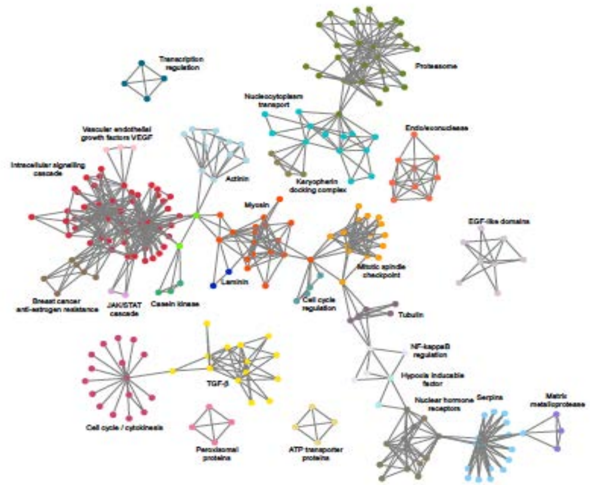


$\langle k \rangle \to$ **const** $\langle k^2 \rangle \to \infty$

for $N \to \infty$

- Modularity: Local communities of nodes

  [Fortunato 2010]



Models

- Non-equilibrium growing network models:

  Explanatory of emergent properties of complex networks

  -BA model, BB model

- Deterministic models:

  Hierarchical models

  -Apollonian network, Pseudo-fractal network

- Maximum entropy ensembles:

  Maximum random graphs satisfying a set of constraints

  -Configuration model, Exponential Random Graphs

Growth by uniform attachment of links

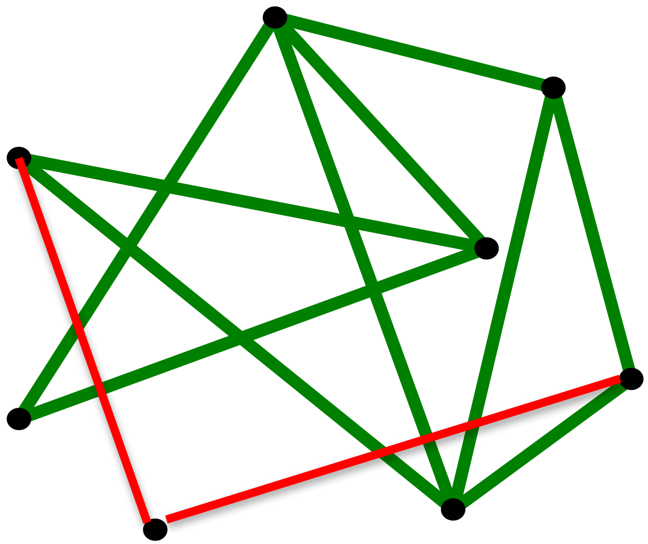
GROWTH :

At every timestep we add a new node with m edges (connected to the nodes already present in the system).

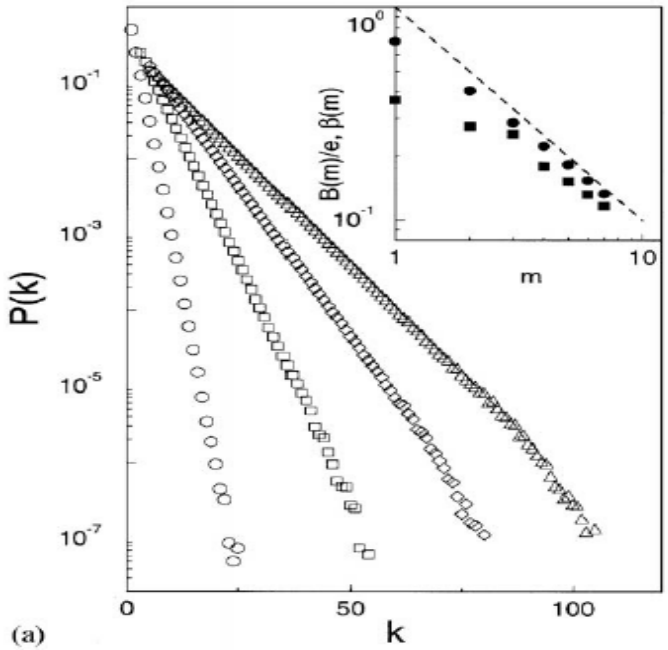
UNIFORM ATTACHMENT :

The probability Π_i that a new node will be connected to node *i is uniform*

$$\Pi_i = \frac{1}{N}$$



Exponential



[Barabási & Albert, Physica A (1999)]

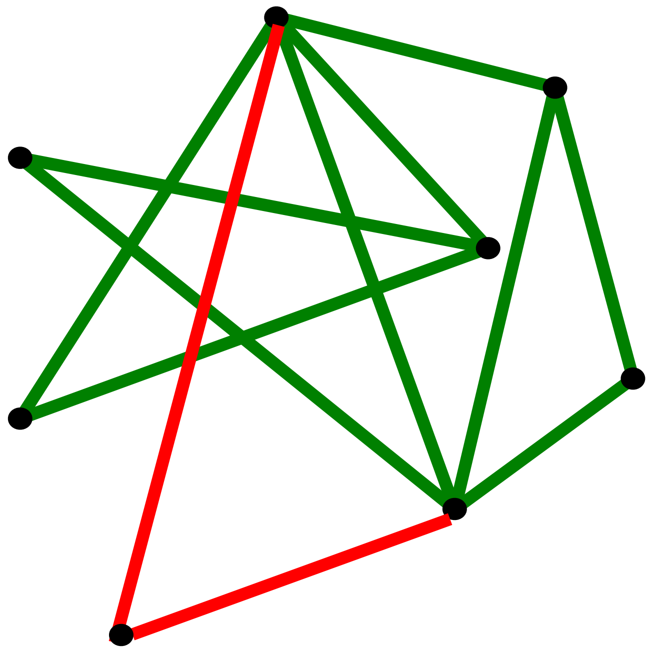
Barabasi-Albert model

GROWTH :

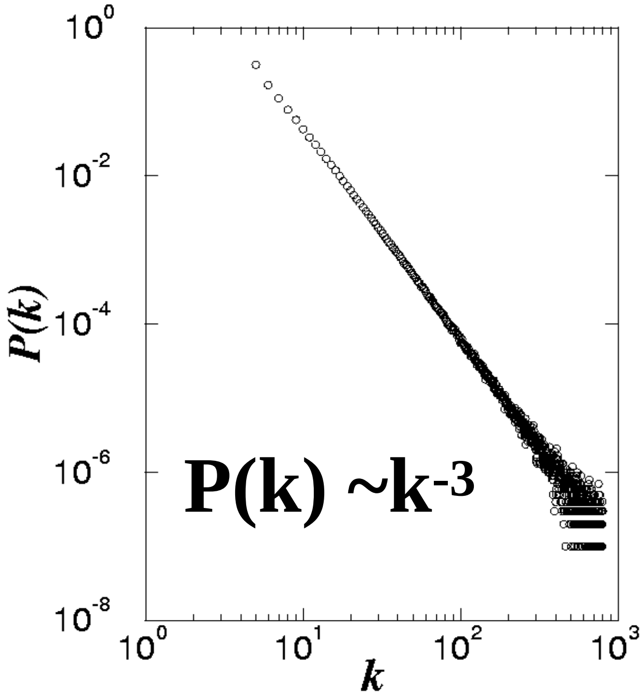
At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

PREFERENTIAL ATTACHMENT :

The probability Π_i that a new node will be connected to node *i* depends on the degree k_i of that node



$$\Pi_i = \frac{k_i}{\sum_j k_j}$$



[Barabási et al. Science (1999)]

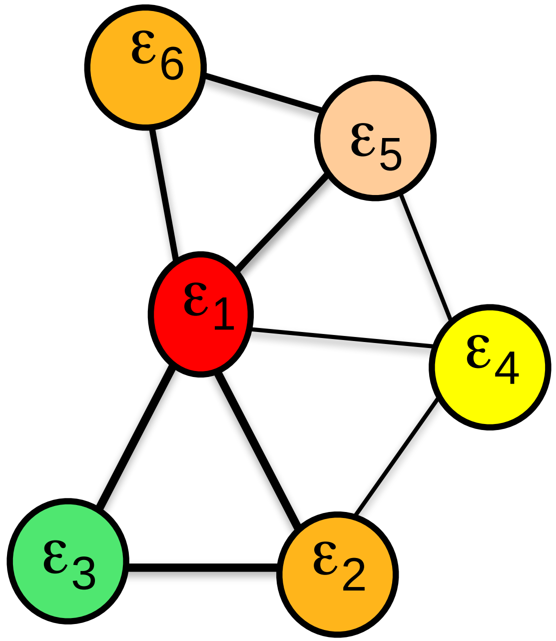
Energies of the nodes

Not all the nodes are the same!

Let assign to each node i

an energy ε *from a*

$g(\varepsilon)$ *distribution*



The Bianconi-Barabasi model

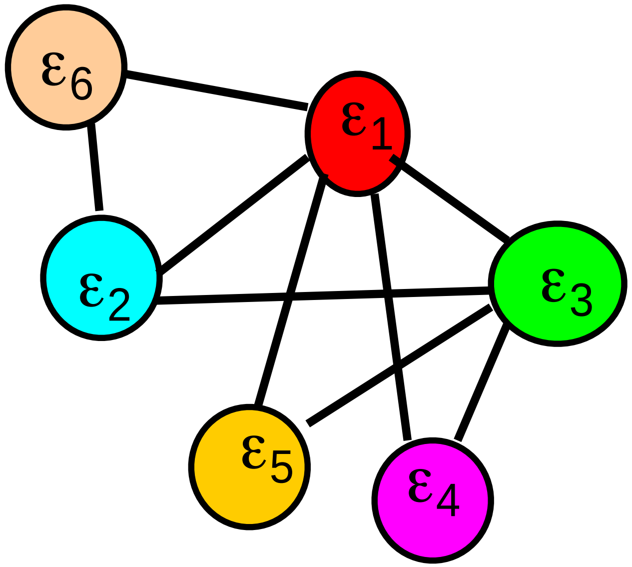
Growth:

–At each time a new node and *m* links are added to the network.

–To each node *i* we assign a energy ε_i from a $g(\varepsilon)$ distribution

Preferential attachment towards high degree low energy nodes:

–Each node connects to the rest of the network by *m* links attached preferentially to well connected, low energy nodes.



$$\Pi_i = \frac{e^{-\beta\epsilon_i}k_i}{\sum_j e^{-\beta\epsilon_j}k_j}$$

[G. Bianconi, A.-L. Barabási 2001]

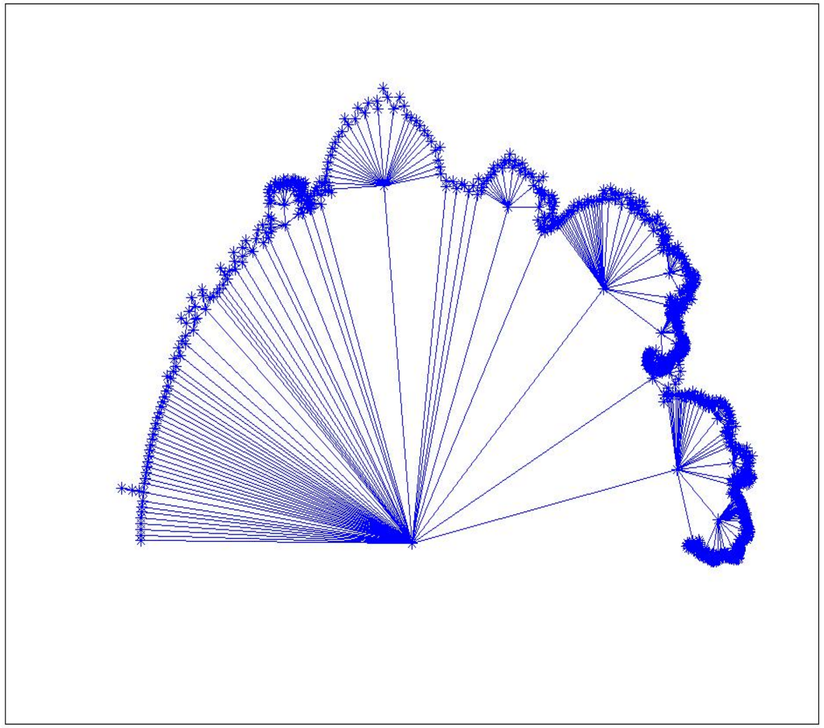
Bose-Einstein condensation in complex networks

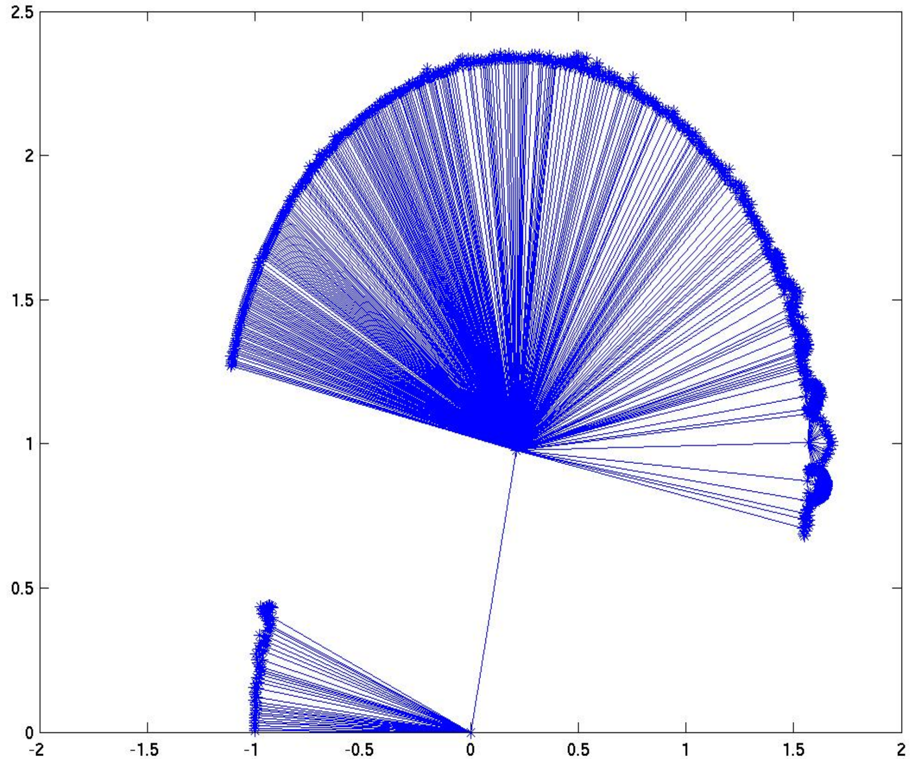
Scale-Free Phase

$$\beta < \beta_c$$

Bose-Einstein Condensate Phase

$$\beta > \beta_c$$



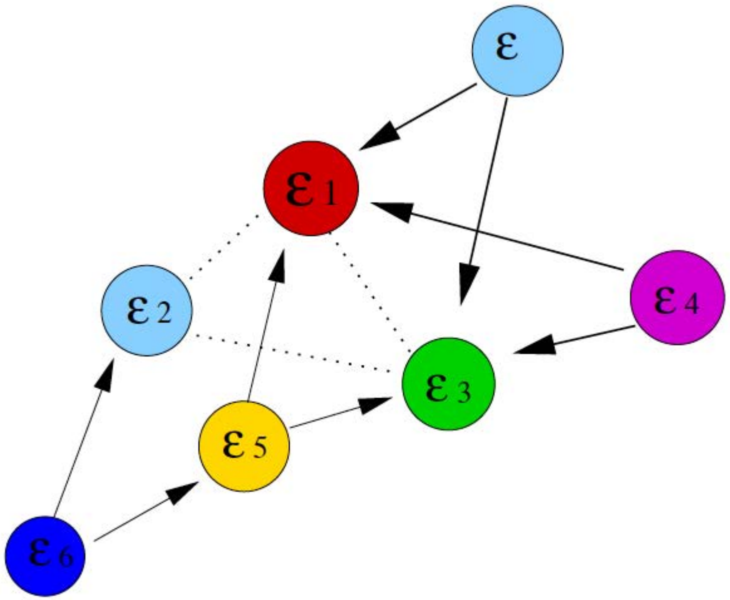


[G. Bianconi, A.-L. Barabási 2001]

Quantum statistics in growing networks

Scale-free network

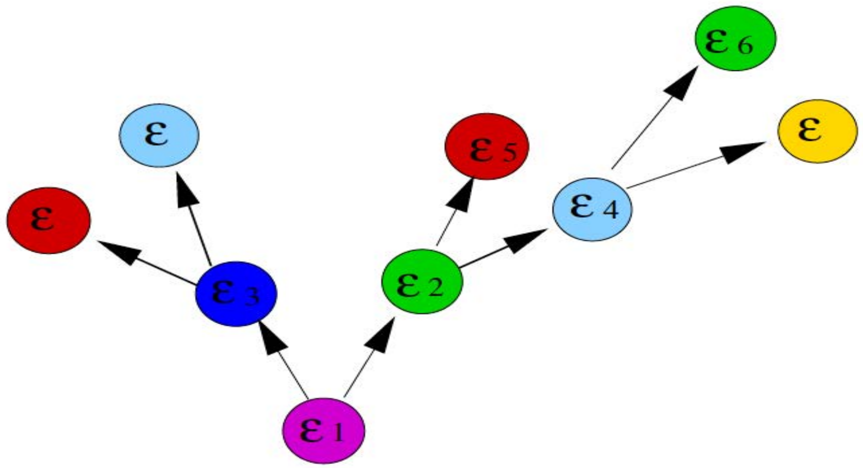
Bianconi-Barabasi model (2001)



Bose Einstein statistics

Complex Cayley tree

Bianconi (2002)



Fermi statistics

The Complex Growing Cayley tree model

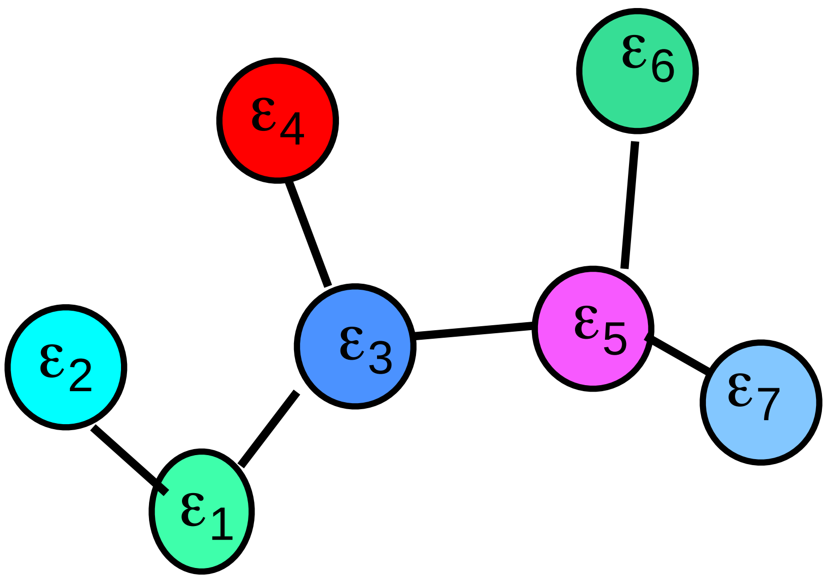
Growth:

– At each time attach a old node with $n_i=0$ to *m* links are added to the network and then we set $n_i=1$.

– To each node *i* we assign a energy ε_i from a $g(\varepsilon)$ distribution

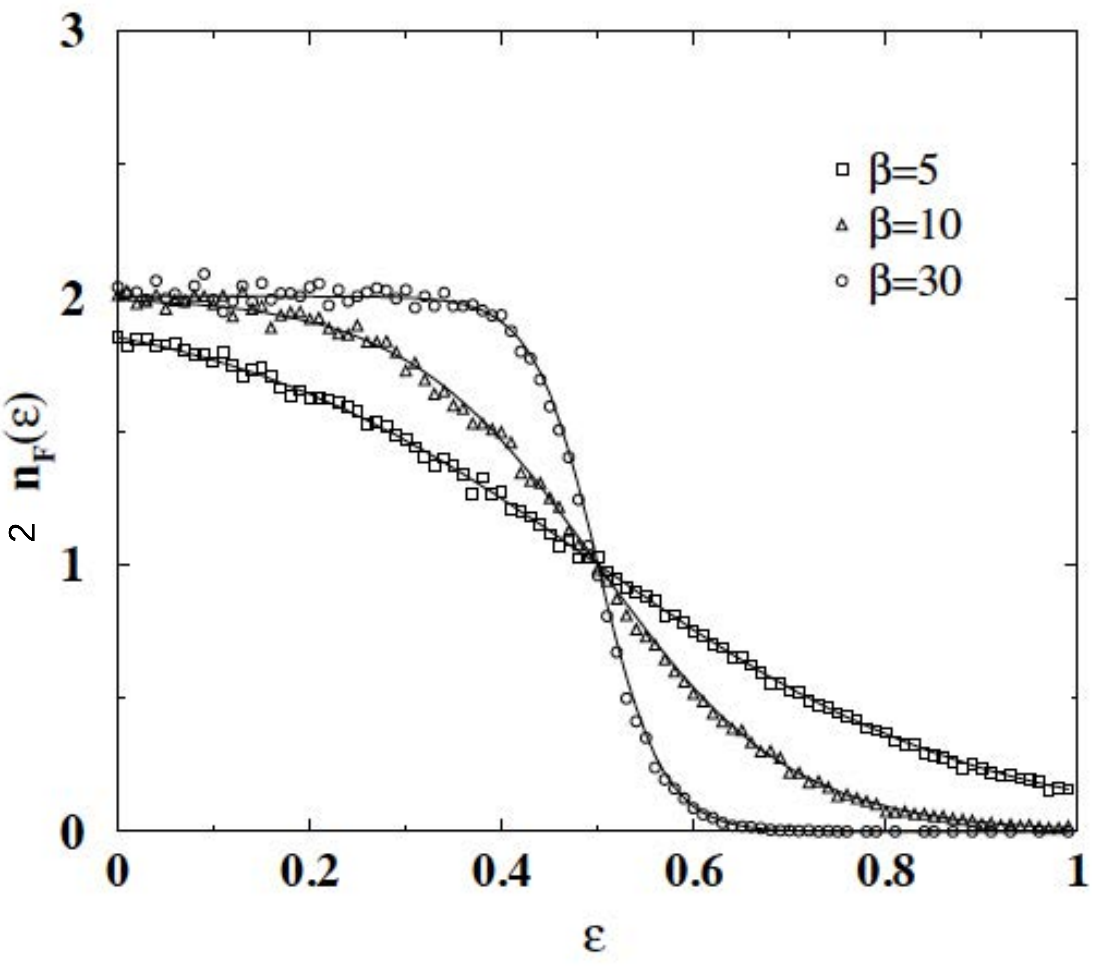
Attachment towards low energy nodes:

– The node *i* to which we attach the new “unitary cell” is chosen with probability



$$\Pi_i = \frac{e^{-\beta\epsilon_i}(1-n_i)}{\sum_j e^{-\beta\epsilon_j}(1-n_j)}$$

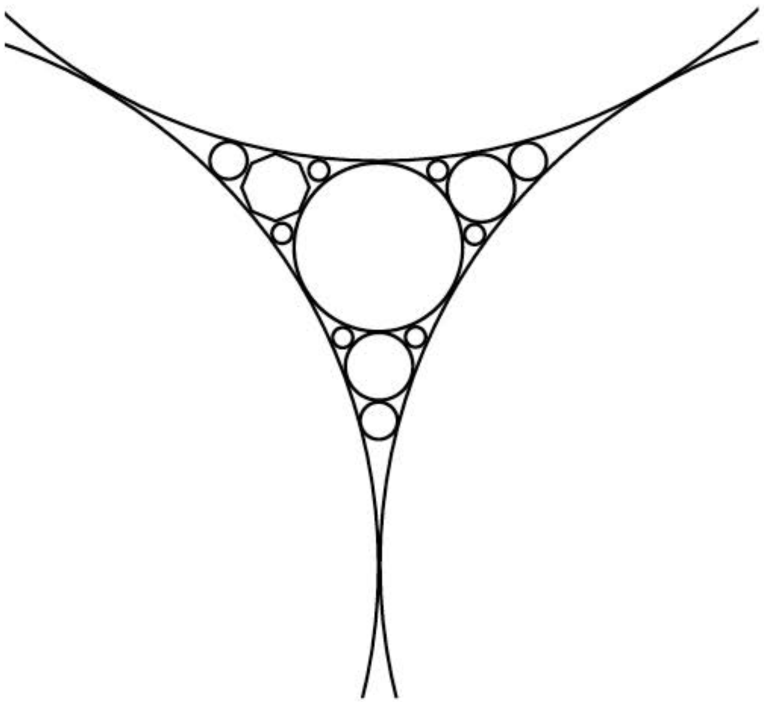
Energy distribution of the nodes at the bulk of the growing Cayley tree network

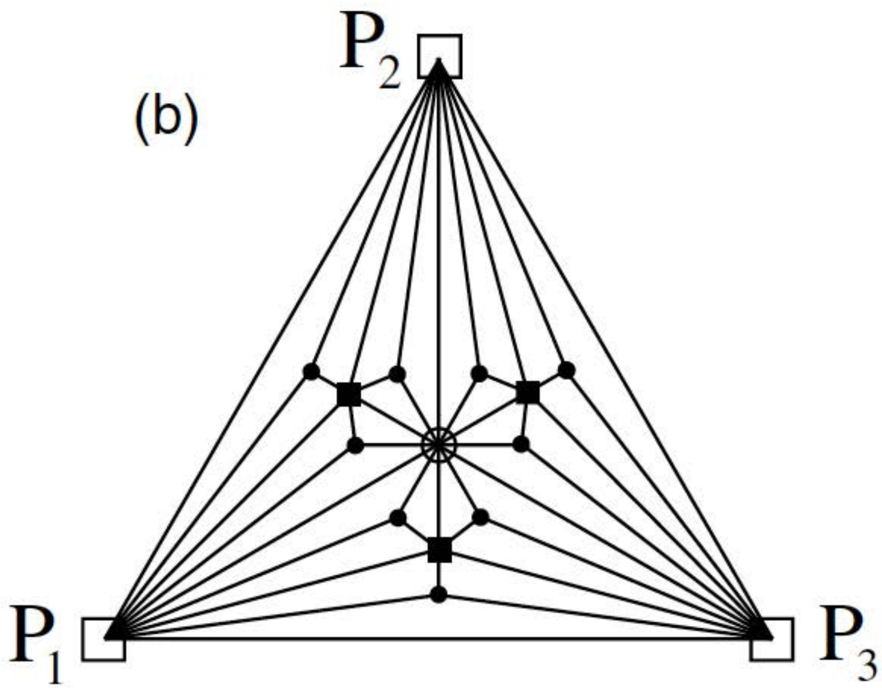


Apollonian networks

Apollonian networks are formed by linking the centers of an Apollonian sphere packing

They are scale-free and are described by the Apollonian group



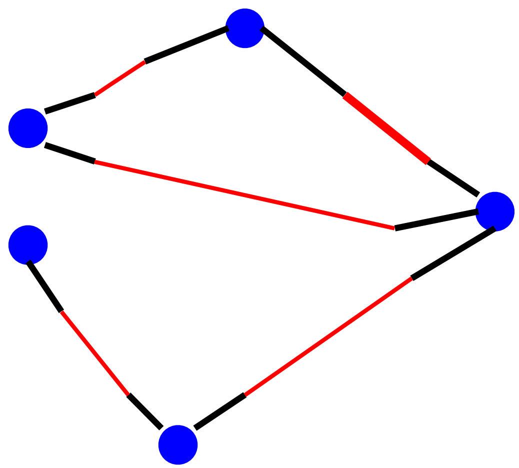


[Andrade et al. PRL 2005]
[Soderberg PRA 1992]

Microcanonical and canonical network ensembles

Microcanonical ensemble

$$P(G) = \frac{1}{Z}\prod_{i=1}^{N}\delta\left(k_i, \sum_{j=1}^{N} a_{ij}\right)$$

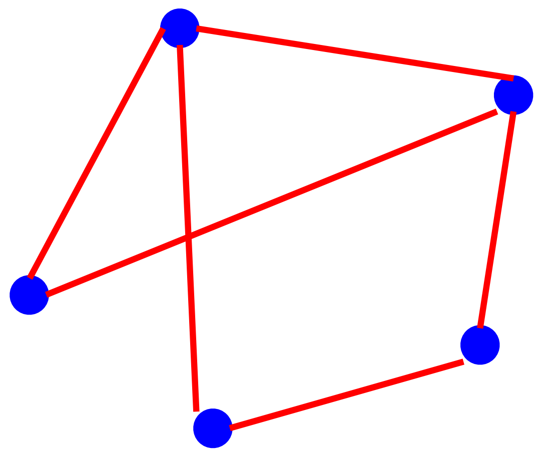


Ensemble of network with exact degree sequence

Configuration model

Canonical ensemble

$$P(G) = \frac{1}{Z} e^{-\sum_{i=1}^{N}\lambda_i \sum_{j=1}^{N} a_{ij}}$$



Ensemble of networks given expected degree sequence

Exponential Random Graph

K. Anand, G. Bianconi PRE 2009

No-equivalence of the network ensembles

There is no equivalence of the ensembles as long as the number of constraints is extensive

$$\boxed{\Sigma = S - \Omega}$$

K. Anand, G. Bianconi PRE 2009, PRE 2010

Network Ensembles and their non-equivalence

2000

Configuration model

A Random Graph Model for Massive Graphs

William Aiello, AT&T Labs, Florham Park, New Jersey, aiello@research.att.com
Fan Chung, University of California, San Diego, fan@ucsd.edu
Linyuan Lu, University of California, San Diego, llu@math.ucsd.edu

2001

ERG
Social network literature

THE STATISTICAL EVALUATION OF SOCIAL NETWORK DYNAMICS

*Tom A. B. Snijders**

A class of statistical models is proposed for longitudinal network data. The dependent variable is the changing (or evolving) relation network, represented by two or more observations of a directed

2004

ERG
Physics network literature

PHYSICAL REVIEW E **70**, 066117 (2004)

Statistical mechanics of networks

Juyong Park and M. E. J. Newman
Department of Physics and Center for the Study of Complex Systems, University of Michigan, Ann Arbor, Michigan 48109-1120, USA
(Received 2 June 2004; revised manuscript received 20 August 2004; published 7 December 2004)

2008

Ω
is extensive

PHYSICAL REVIEW E **78**, 016114 (2008)

Entropies of complex networks with hierarchically constrained topologies

Ginestra Bianconi,[1] Anthony C. C. Coolen,[2,3] and Conrad J. Perez Vicente[4]
[1]*Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy*

2009

Microcanonical and Canonical ensembles Non-equivalence of the ensembles

PHYSICAL REVIEW E **80**, 045102(R) (2009)

Entropy measures for networks: Toward an information theory of complex topologies

Kartik Anand[1] and Ginestra Bianconi[2]

2010

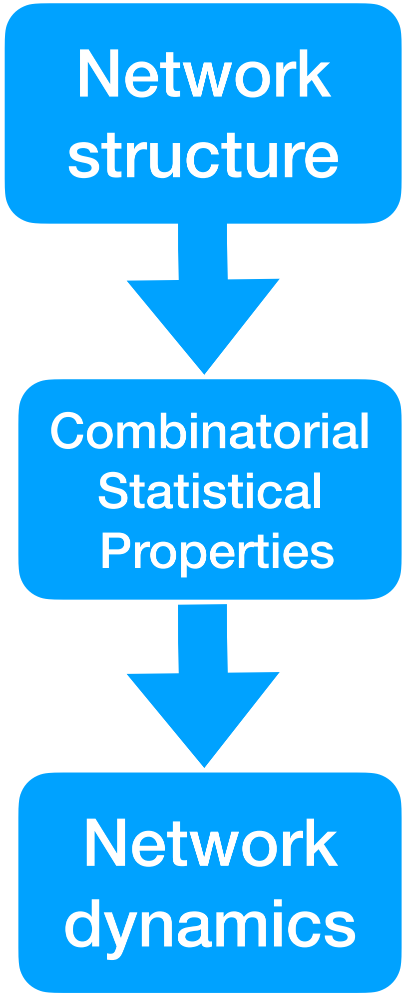
Non-equivalence of the ensembles of networks in general cases

PHYSICAL REVIEW E **82**, 011116 (2010)

Gibbs entropy of network ensembles by cavity methods

Kartik Anand[1] and Ginestra Bianconi[2]

Interplay between network structure and dynamics



Critical phenomena on scale-free networks

Scale free networks:

- **Percolation:**

Percolation threshold

$$\boxed{p_c \frac{\langle k(k-1)\rangle}{\langle k\rangle} = 1}$$

Cohen-Havlin 2001

Scale free networks are always percolating

- **Ising model:**

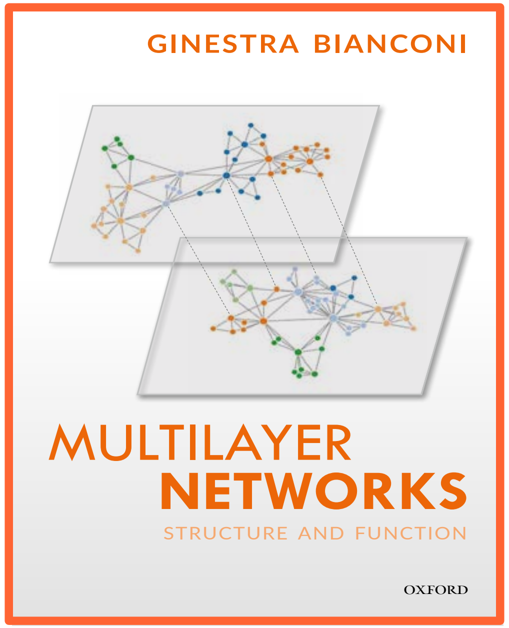
Critical temperature

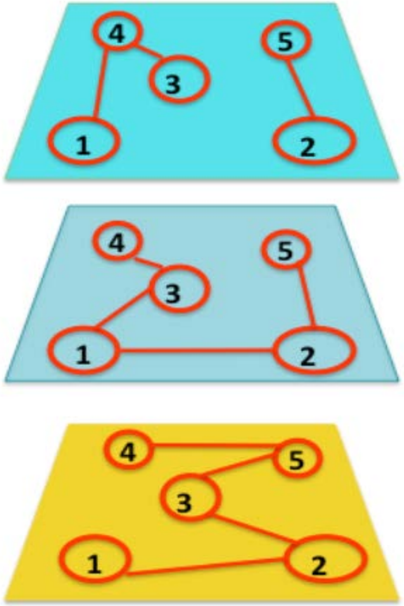
$$\boxed{\beta J \frac{\langle k(k-1)\rangle}{\langle k\rangle} = 1}$$

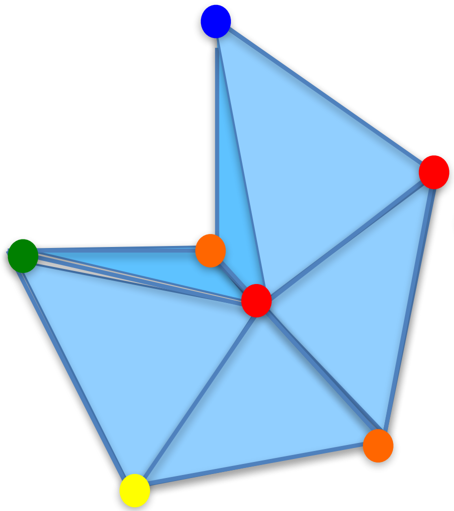
The Ising model on scale-free networks

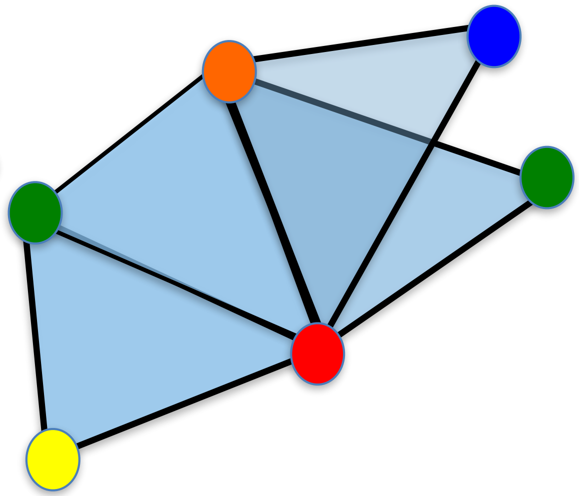
is always in the ferromagnetic phase

Generalized network structures









Going beyond the framework of simple networks

is of fundamental importance

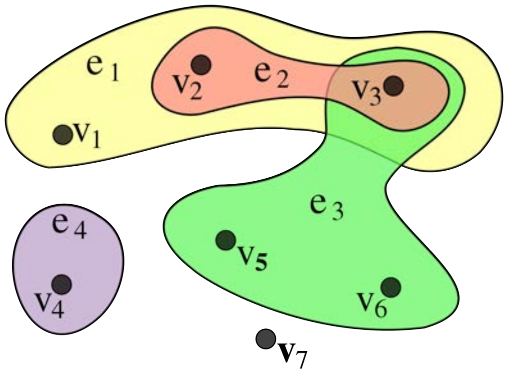
for understanding the relation between structure and

dynamics in complex systems

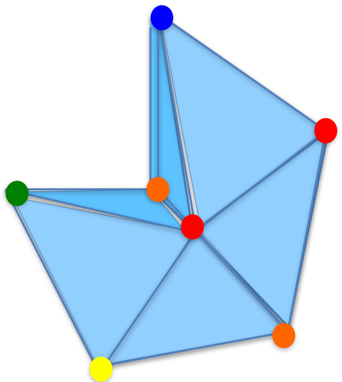
Higher-order networks

Higher-order networks

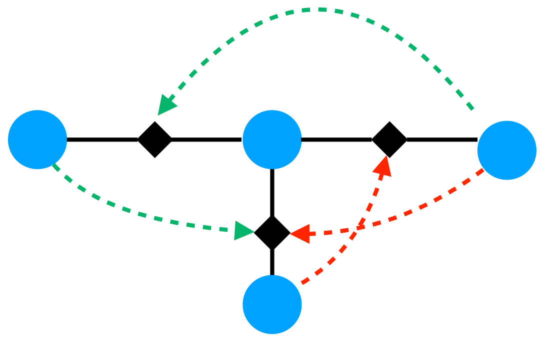
Higher-order networks are characterising the interactions between two or more nodes



Hypergraph



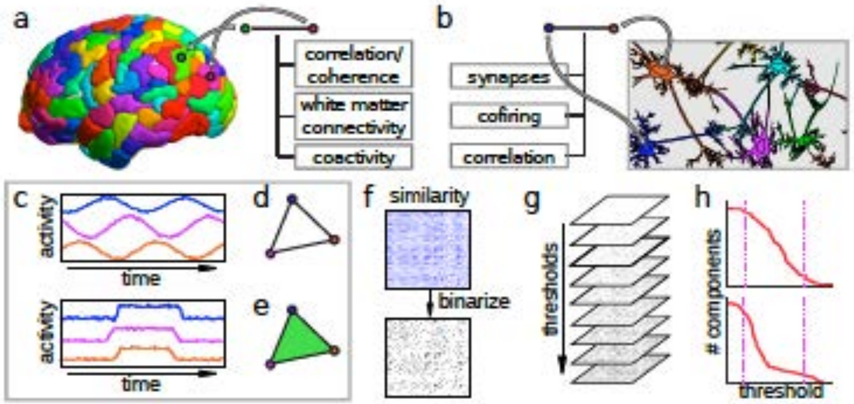
Simplicial complex



Network with triadic interactions

Higher-order network data

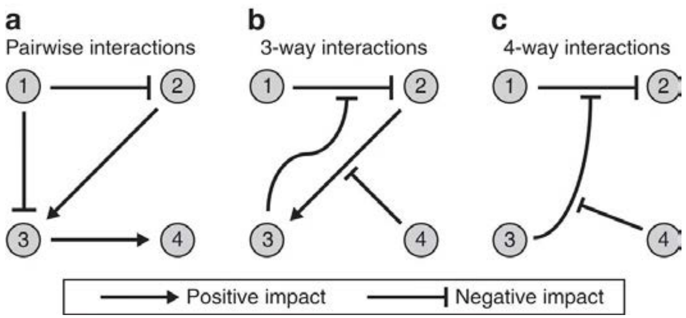
Brain data



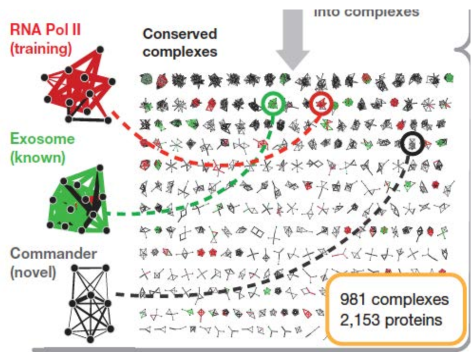
Face-to-face interactions

Collaboration networks

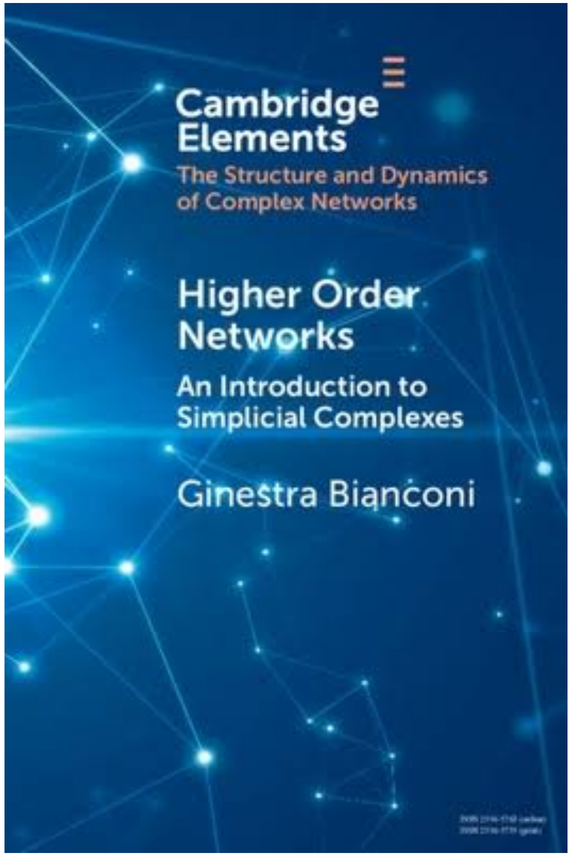
Ecosystems



Protein interactions



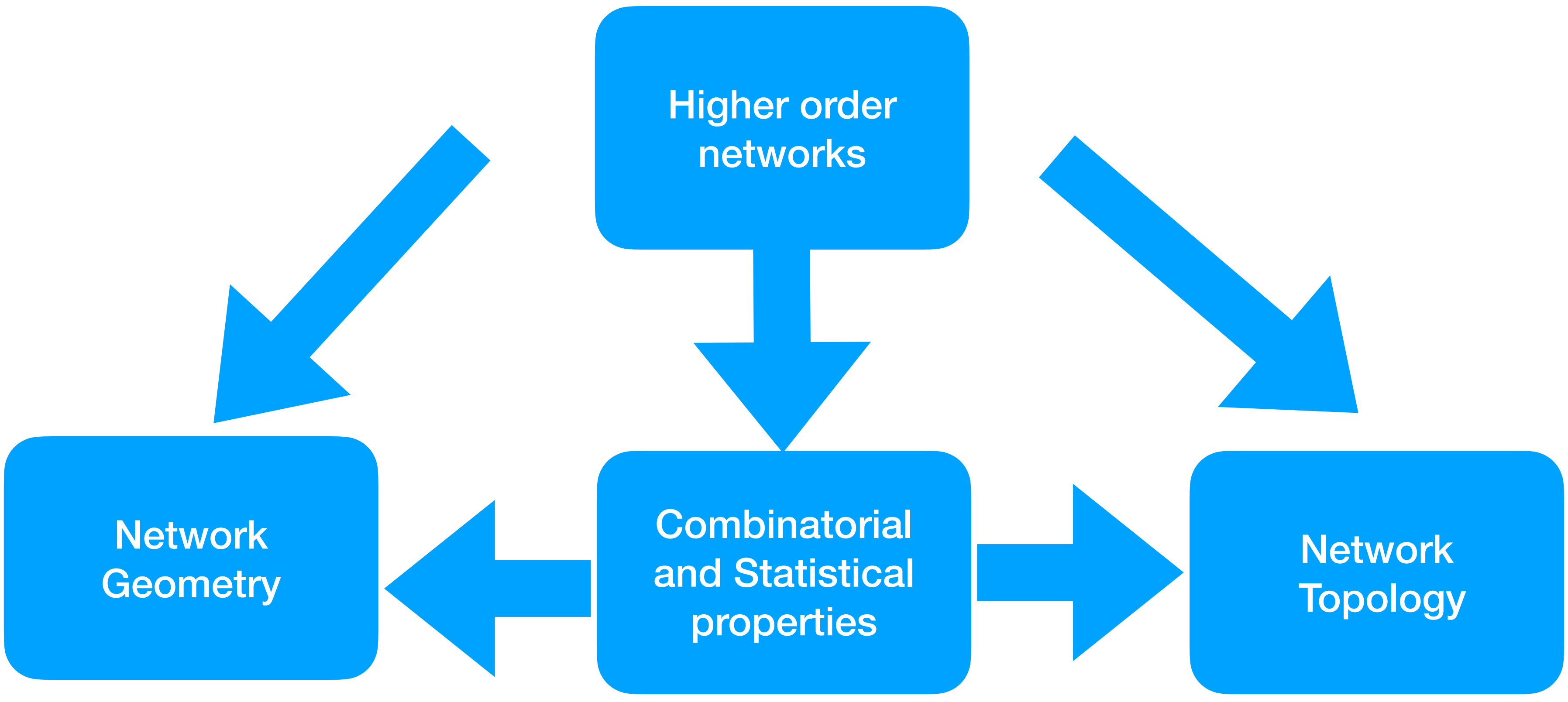
Higher-order networks



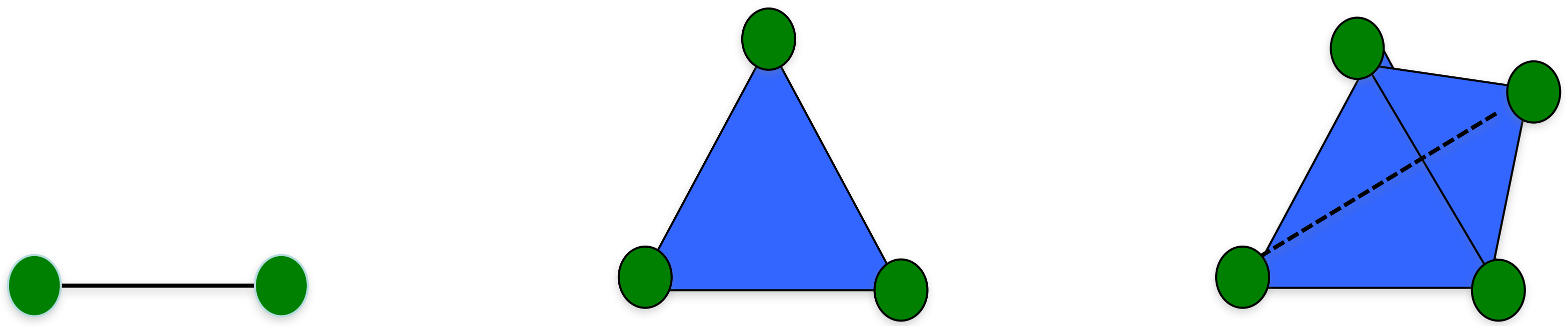
New book
by Cambridge University Press!!

Providing a general view of the interplay between topology and dynamics

Higher order networks Structure



Hyperedges



2-hyperedge 3-hyperedge 4-hyperedge

An m-hyperedge is set nodes

$$\alpha = [i_1, i_2, i_3, \ldots i_m]$$

-it indicates the interactions between the m-nodes

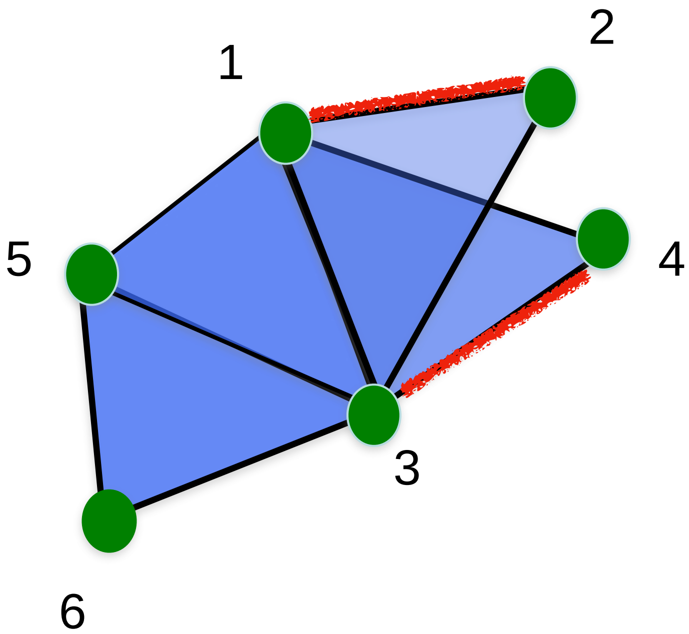
Hypergraphs

HYPERGRAPH

A hypergraph $\mathcal{G} = (V, E_H)$ is defined by a set V of N nodes and a set E_H of hyperedges, where a $(m+1)$-hyperedge indicates a set of $m+1$ nodes

$$e = [v_0, v_1, v_2, \ldots, v_m],$$

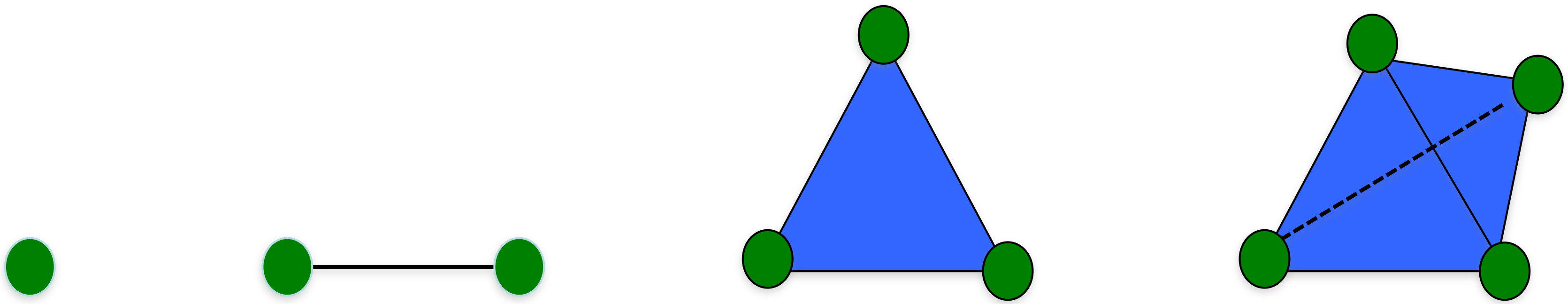
with generic value of $1 \leq m < N$.
An hyperdge describes the many-body interaction between the nodes.



Every hyperedge α formed by a subset of the nodes can belong or not to the hypergraph $\mathscr{H}$

$$\mathscr{H} = \{[1,2], [3,4], [1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$$

Simplices



0-simplex 1-simplex 2-simplex 3-simplex

SIMPLICES

A d-dimensional simplex α (also indicated as a d-simplex α) is formed by a set of $(d+1)$ interacting nodes

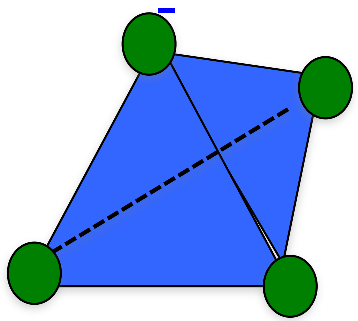
$$\alpha = [v_0, v_1, v_2 \ldots, v_d].$$

It describes a many body interaction between the nodes.
It allows for a topological and a geometrical interpretation of the simplex.

Faces of a simplex

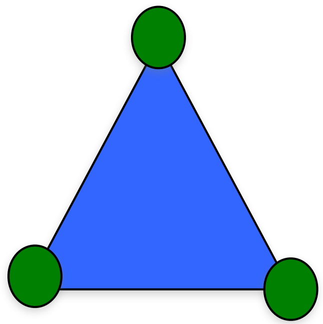
Faces

A face of a d-dimensional simplex α is a simplex α' formed by a proper subset of nodes of the simplex, i.e. $\alpha' \subset \alpha$.



3-simplex

Faces



4 0-simplices

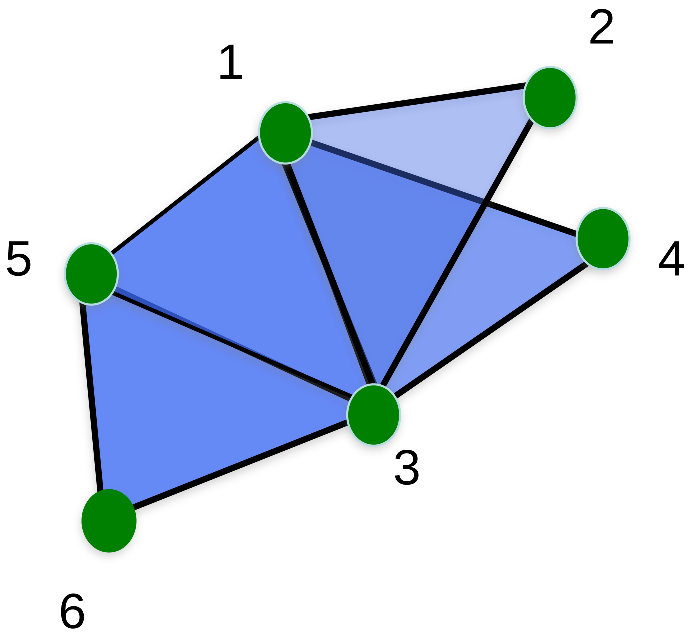
6 1-simplices

4 2-simplices

Simplicial complex

SIMPLICIAL COMPLEX

A simplicial complex $\mathcal{K}$ is formed by a set of simplices that is closed under the inclusion of the faces of each simplex.
The dimension d of a simplicial complex is the largest dimension of its simplices.

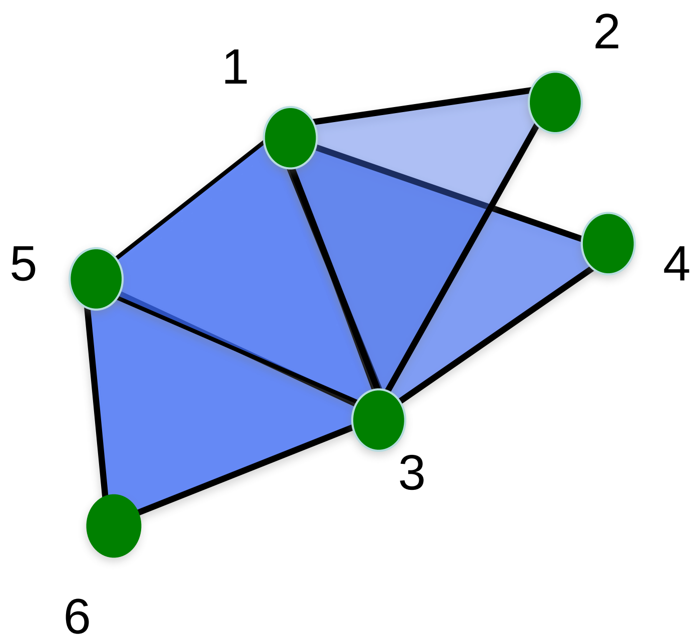


If a simplex α belongs to the simplicial complex $\mathscr{K}$ then every face of α must also belong to $\mathscr{K}$

$$\mathscr{K} = \{[1], [2], [3], [4], [5], [6], [1,2], [1,3], [1,4], [1,5], [2,3], [3,4], [3,5], [3,6], [5,6], [1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$$

Dimension of a simplicial complex

The dimension of a simplicial complex $\mathscr{K}$ is the largest dimension of its simplices



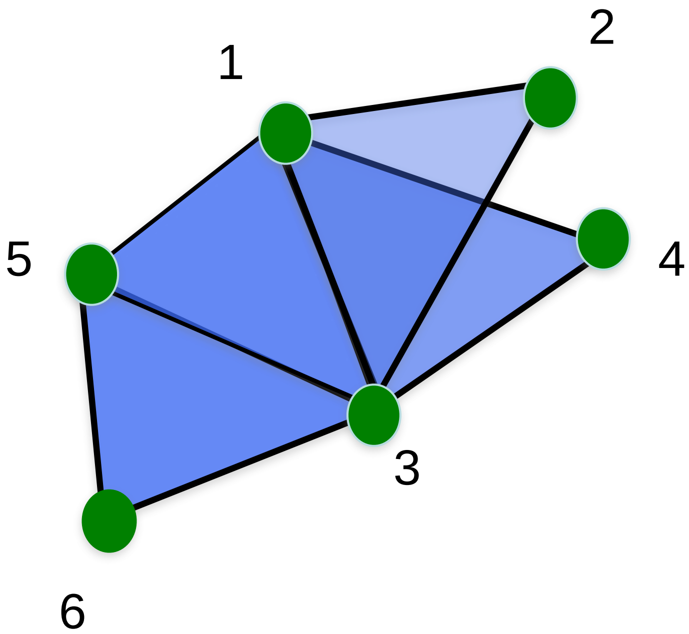
This simplicial complex has dimension 2

$$\mathscr{K} = \{[1], [2], [3], [4], [5], [6],\\ [1,2], [1,3], [1,4], [1,5], [2,3],\\ [3,4], [3,5], [3,6], [5,6],\\ [1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$$

Facets of a simplicial complex

FACET

A facet is a simplex of a simplicial complex that is not a face of any other simplex. Therefore a simplicial complex is fully determined by the sequence of its facets.



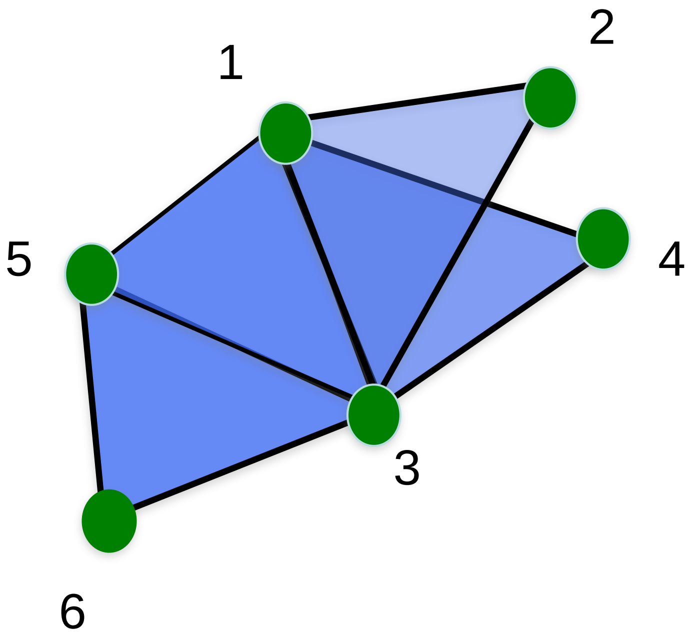
The facets of this simplicial complex are

$$\mathscr{K} = \{[1,2,3],[1,3,4],[1,3,5],[3,5,6]\}$$

Pure simplicial complex

PURE SIMPLICIAL COMPLEXES

A pure d-dimensional simplicial complex is formed by a set of d-dimensional simplices and their faces.
Therefore pure d-dimensional simplicial complexes admit as facets only d-dimensional simplices.



A pure d-dimensional simplicial complex is fully determined by an adjacency matrix tensor with (d+1) indices.
For instance this simplicial complex is determined by the tensor

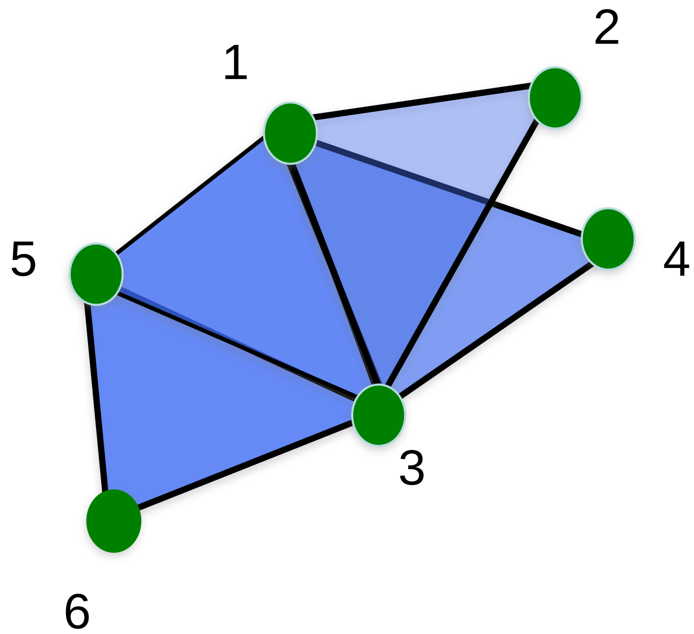
$$a_{rsp} = \begin{cases} 1 \ \textbf{if} \ \ (r,s,p) \in \mathscr{K} \\ 0 \ \textbf{otherwise} \end{cases}$$

Example

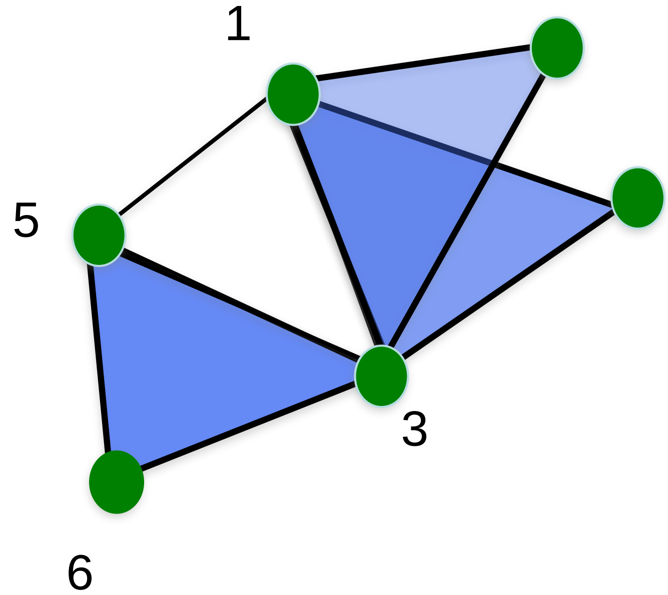
A simplicial complex $\mathscr{K}$ is pure

if it is formed by d-dimensional simplices

and their faces

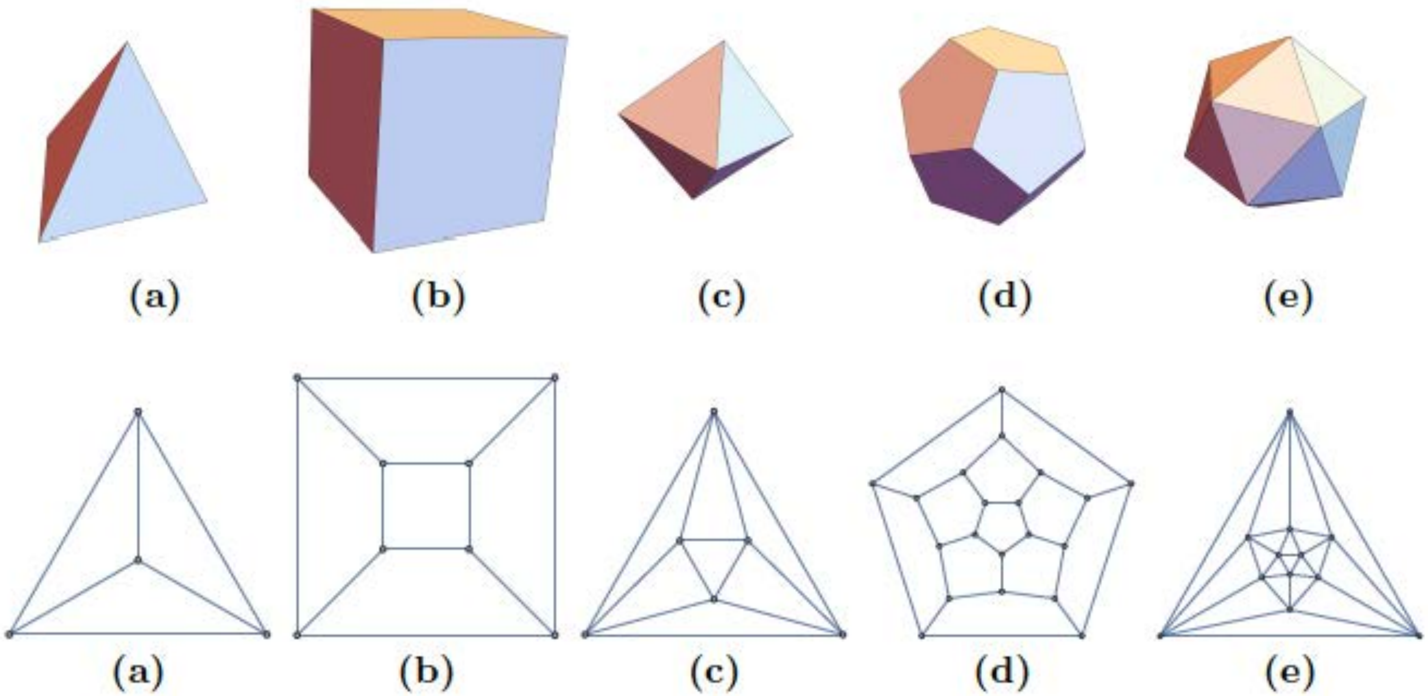


PURE SIMPLICIAL COMPLEX



SIMPLICIAL COMPLEX THAT IS NOT PURE

Cell complexes

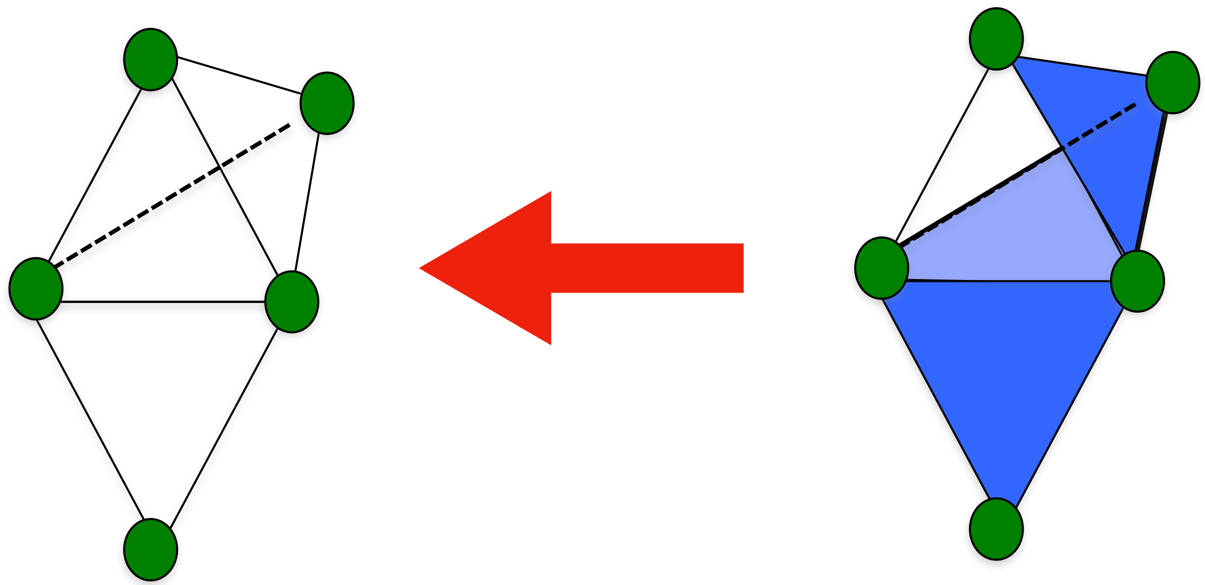


A cell complex $\hat{\mathcal{K}}$ has the following two properties:

(a) it is formed by a set of cells that is closure-finite, meaning that every cell is covered by a finite union of open cells;

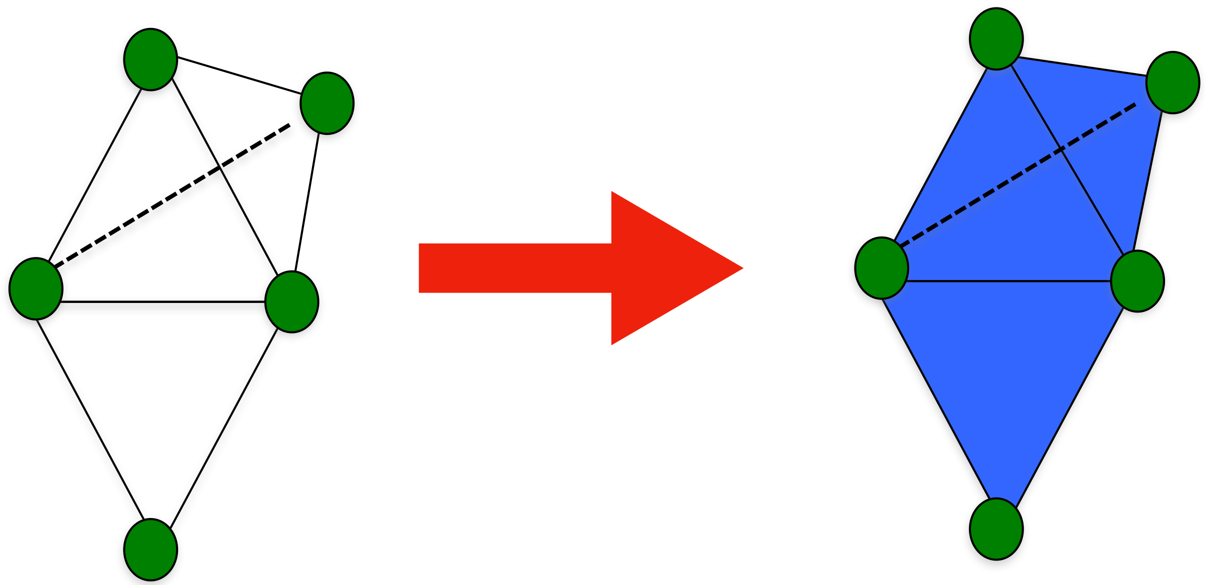
(b) given two cells of the cell complex $\alpha \in \hat{\mathcal{K}}$ and $\alpha' \in \hat{\mathcal{K}}$ then either their intersection belongs to the cell complex, i.e. $\alpha \cap \alpha' \in \hat{\mathcal{K}}$ or their intersection is a null set, i.e. $\alpha \cap \alpha' = \emptyset$.

Simplicial complex skeleton



From a simplicial complex is possible to generate a network salled the simplicial complex skeleton by considering only the nodes and the links of the simplicial complex

Clique complex



From a network is possible to generate a simplicial complex by Assuming that each clique is a simplex

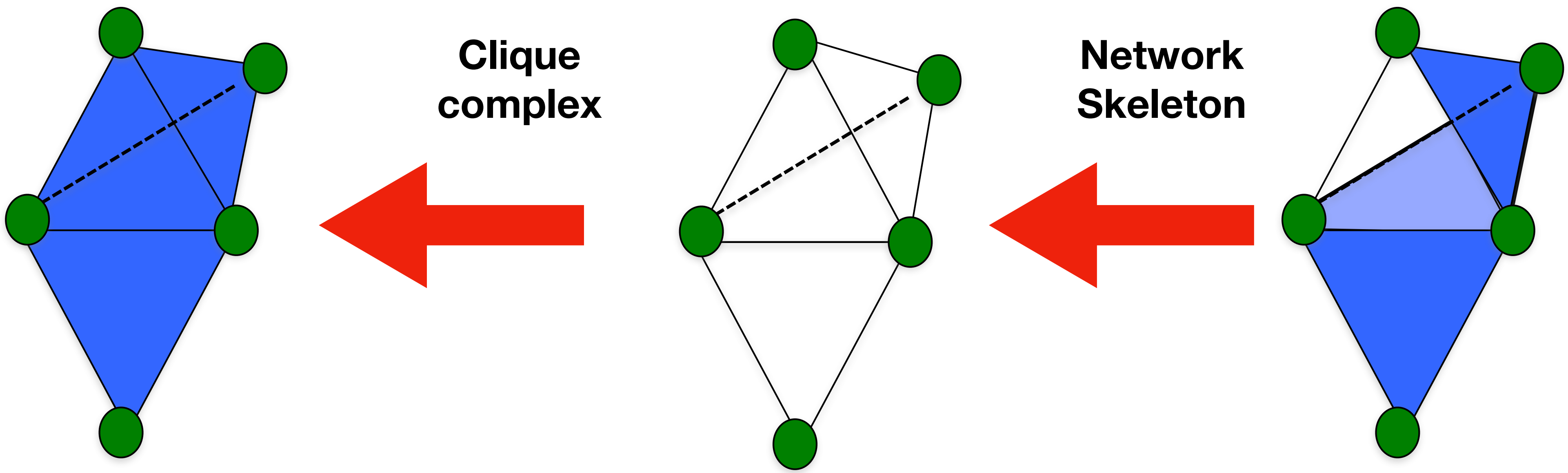
Note:

Poisson networks have a clique number that is 3 and actually on a finite expected number of triangles in the infinite network limit

However

Scale-free networks have a diverging clique number, therefore the clique complex of a scale-free network has diverging dimension. (Bianconi,Marsili 2006)

Concatenation of the operations

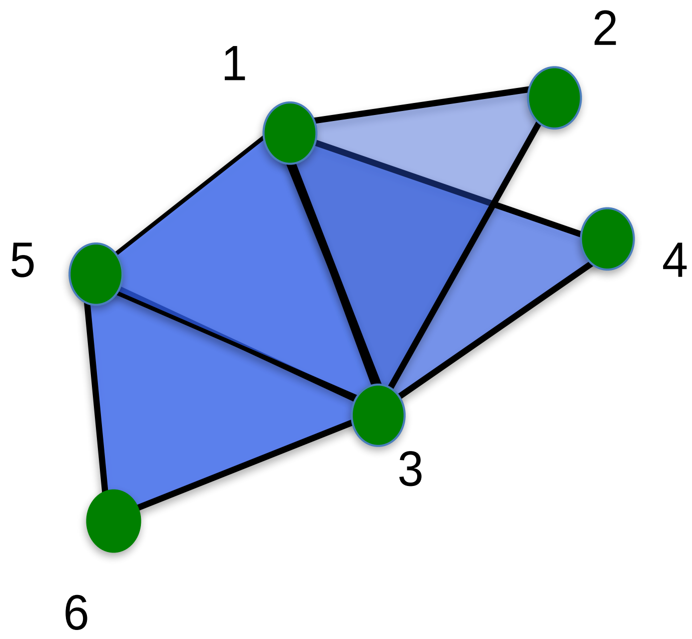


Attention!

By concatenating the operations you are not guaranteed to return to the initial simplicial complex

Generalized degrees

The generalized degree $k_{d,m}(\alpha)$ of a m-face α in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the m-face α.



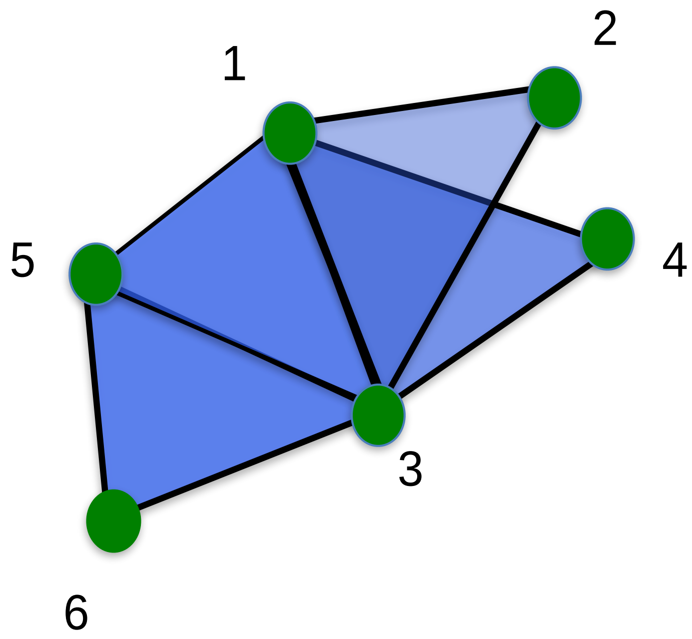
$k_{2,0}(\alpha)$ Number of triangles incident to the node α

$k_{2,1}(\alpha)$ Number of triangles incident to the link α

[Bianconi & Rahmede (2016)]

Generalized degree

The generalized degree $k_{d,m}(\alpha)$ of a m-face α in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the m-face α.

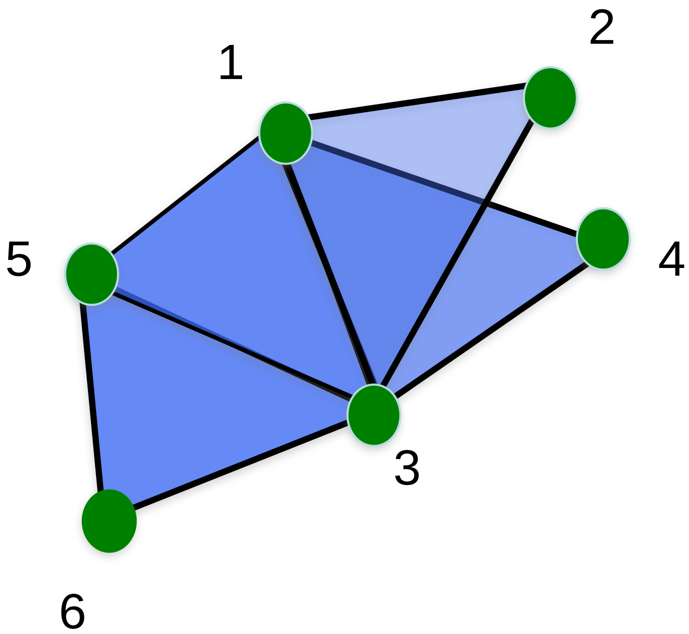


i	$k_{2,0}(i)$
1	3
2	1
3	4
4	1
5	2
6	1

(i,j)	$k_{2,1}(i,j)$
(1,2)	1
(1,3)	3
(1,4)	1
(1,5)	1
(2,3)	1
(3,4)	1
(3,5)	2
(3,6)	1
(5,6)	1

Pure simplicial complex

A simplicial complex $\mathscr{K}$ is pure
if it is formed by d-dimensional simplices
and their faces



A pure d-dimensional simplicial complex is fully determined by an adjacency matrix tensor with (d+1) indices. For instance this simplicial complex is determined by the tensor

$$a_{rsp} = \begin{cases} 1 \text{ if } (r,s,p) \in \mathscr{K} \\ 0 \text{ otherwise} \end{cases}$$

Combinatorial properties of the generalised degrees

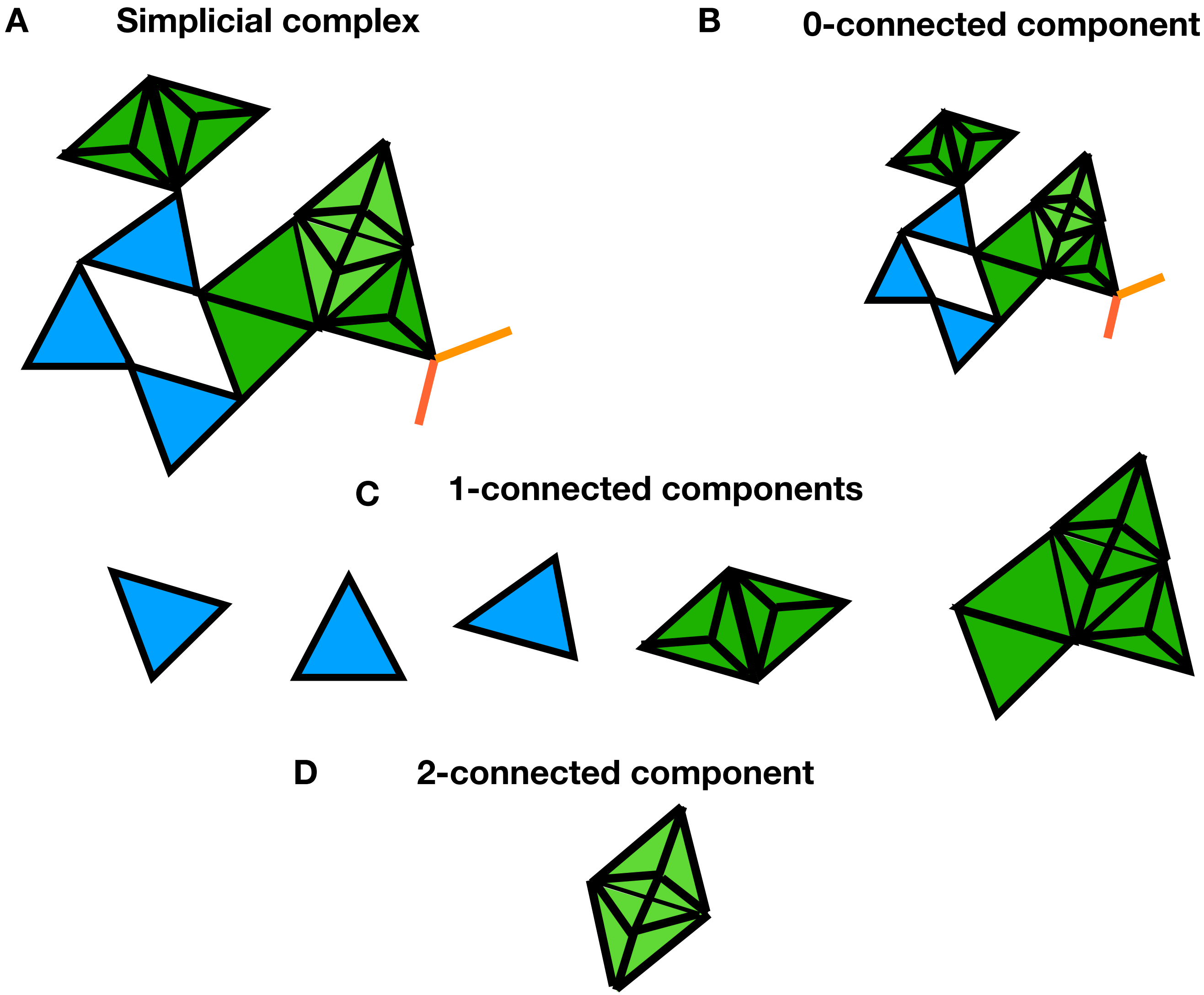
The generalized degrees $k_{d,m}(\alpha)$ of a pure d-dimensional simplicial complex can be defined in terms of the adjacency tensor $\mathbf{a}$ as

$$\boxed{k_{d,m}(\alpha) = \sum_{\alpha' \in \mathcal{Q}_d(N) | \alpha' \supseteq \alpha} a_{\alpha'}}$$

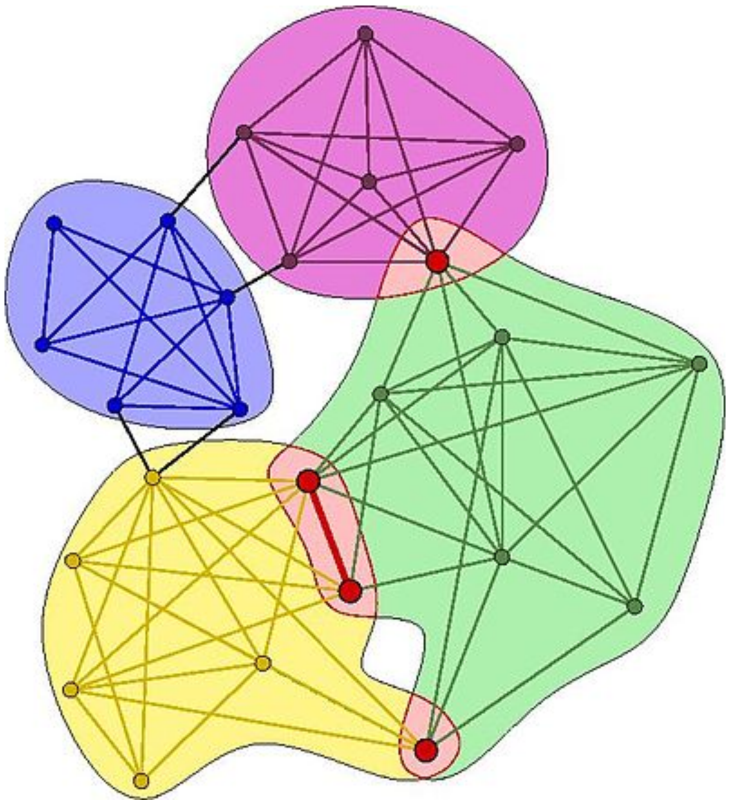
The generalized degrees obey a nice combinatorial relation as they are not independent of each other.
In fact for *m'>m* we have

$$\boxed{k_{d,m}(\alpha) = \frac{1}{\binom{d-m}{m'-m}} \sum_{\alpha' \in \mathcal{Q}_d(N) | \alpha' \supseteq \alpha} k_{d,m'}(\alpha')\,.}$$

m-connected components



Clique communities



Palla et al. Nature 2005

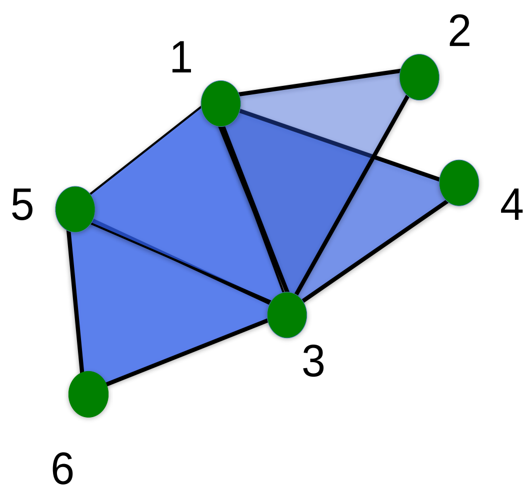
The m-clique communities are the m-connected components of the clique complex of the network

Geometrical properties of simplicial complexes

Incidence number

To each (d-1)-face α we associate the incidence number

$$\boxed{n_\alpha = k_{d,d-1}(\alpha) - 1}$$



[Bianconi & Rahmede (2016)]

(i,j)	$n_{(i,j)}$
(1,2)	0
(1,3)	2
(1,4)	0
(1,5)	0
(2,3)	0
(3,4)	0
(3,5)	1
(3,6)	0
(5,6)	0

Discrete manifolds

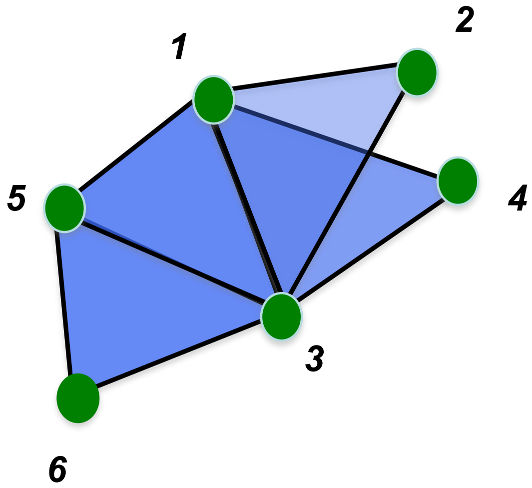
COMBINATORIAL CONDITIONS FOR DISCRETE MANIFOLDS

A discrete manifold $\mathcal{M}$ of dimension d is a pure simplicial complex that satisfies the following two conditions:

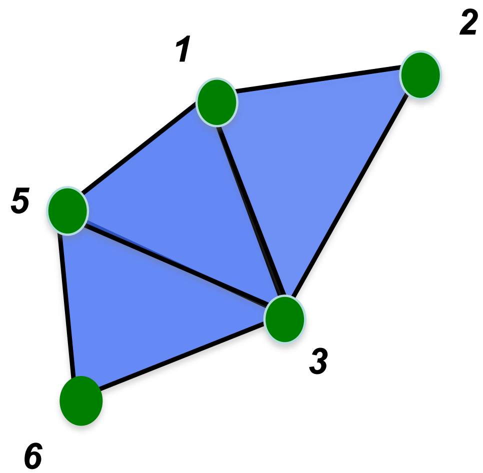
- it is $(d-1)$-connected;
- every two d-simplices α, α' belonging to the simplicial complex $\mathcal{K}$ either overlap on a $(d-1)$-face of $\mathcal{K}$, i.e. $\alpha \cap \alpha' \in S_{d-1}(\mathcal{K})$ or do not overlap, i.e. $\alpha \cap \alpha' = \emptyset$.
- all its $(d-1)$-faces α have an incidence number $n_\alpha \in \{0, 1\}$.

Discrete manifolds

If n_α takes only values $n_\alpha \in \{0,1\}$
each (d-1)-face is incident at most to two d-dimensional simplices.



NOT A MANIFOLD



MANIFOLD

Regge curvature

REGGE CURVATURE

The Regge curvature (Regge (1961)) is associated to each $(d-2)$-dimensional face $\alpha \in S_{d-2}(\mathcal{M})$ of a discrete d dimensional manifold $\mathcal{M}$. The Regge curvature R_α for a face $\alpha \in S_{d-2}(\mathcal{M})$ is defined as

$$R_\alpha = \begin{cases} 2\pi - \theta_\alpha & \text{if } \alpha \in \mathcal{B}, \\ \pi - \theta_\alpha & \text{otherwise,} \end{cases} \tag{42}$$

where θ_α is the sum of all dihedral angles of the d-dimensional simplices incident to the face α.

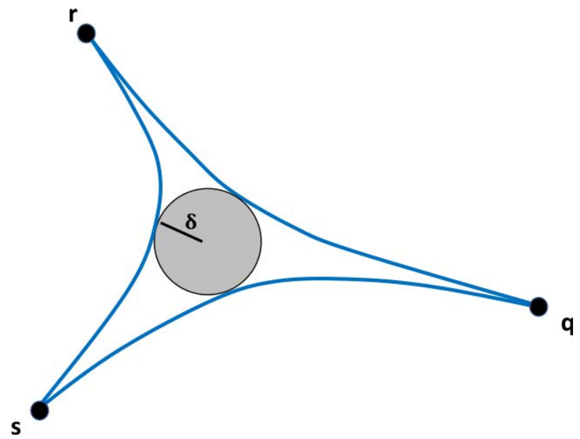
Regge curvature and generalized degrees

If the discrete manifold is formed
by a set of geometrically identical d-simplices
the Regge curvature
is simply related to the generalized degree of the (d-2)-faces, i.e.

$$\boxed{R_\alpha = \begin{cases} 2\pi - \theta_0 k_{d,d-2}(\alpha) & \text{if } \alpha \in \mathscr{B}, \\ \pi - \theta_0 k_{d,d-2}(\alpha) & \text{otherwise,} \end{cases}}$$

where θ_0 indicates the dihedral angle of each d-simplex.

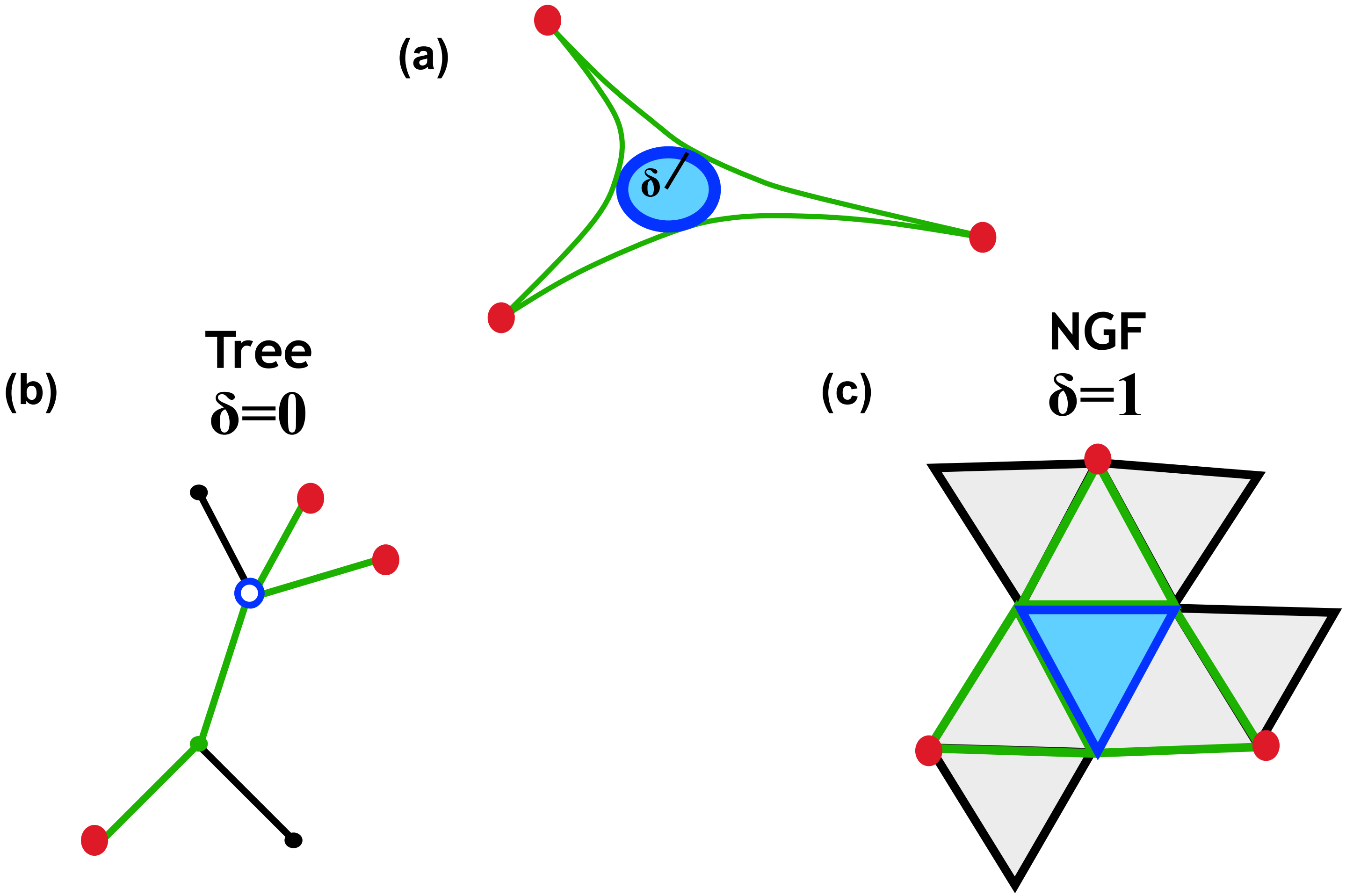
Gromov hyperbolicity



GROMOV δ-HYPERBOLICITY

A network is said to be δ-hyperbolic, if it obeys the δ-slim property, i.e. if there is a $\delta > 0$ such that for any triple of nodes r, s, q connected by the shortest paths $\mathcal{P}_{rs}, \mathcal{P}_{sq}, \mathcal{P}_{rq}$ the union of the δ-neighbourhood of any pair of shortest paths, say $N_\delta(\mathcal{P}_{rs}) \cup N_\delta(\mathcal{P}_{sq})$ includes nodes belonging to the third path, i.e.$\mathcal{P}_{rq}$.

Examples of δ-hyperbolic networks



Graph Laplacian

The graph Laplacian matrix is defined as

$$L_{ij} = \delta_{ij} k_i - a_{ij}$$

The graph Laplacian is a semi-positive matrix that in a connected network has eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \lambda_N$$

The Laplacian is key for describing diffusion processes and the Kuramoto model on networks and constitutes a natural link between topology and dynamics

The Fiedler eigenvalue λ_2 is also called **spectral gap**

Spectral dimension

In geometrical network models

$\lambda_2 \to 0$ **for** $N \to \infty$

and we say that the spectral gap "closes"

If the density of eigenvalues $\rho(\lambda)$ scales like

$$\boxed{\rho(\lambda) \sim \lambda^{d_S/2-1} \textbf{ for } \lambda \ll 1}$$

d_S is called the ***spectral dimension***

Square d-dimensional lattice

The eigenvalues μ of the Laplacian

of a d-dimensional lattice are given by

$$\mu = \sum_{i \in \{1,2,3,\ldots,d\}} 4\sin^2(k_i/2) \simeq |\mathbf{k}|^2$$

where $\mathbf{k}$ is the wave-number characterising the eigenvectors of the Laplacian (Fourier basis) with

$$k_i = \frac{2\pi n_i}{L}$$

It follows that $d_s = d$ for d-dimensional lattices.

Conclusions

- **Simplicial complexes capture the many-body interactions of complex systems and reveal the hidden geometry and topology of data**
- **The hyperbolicity of a network can be defined using Gromov delta-hyperbolicity**
- **A finite spectral dimension is a fundamental property of simplicial complexes with intrinsic geometrical character**

Higher-order networks

An introduction to simplicial complexes

Lesson II:

Mathematics of Large Networks
Erdos Center, Budapest

29 May 2022

Ginestra Bianconi

School of Mathematical Sciences, Queen Mary University of London
Alan Turing Institute

Queen Mary
University of London

The Alan Turing Institute

Lesson II:
Higher-order networks growing models

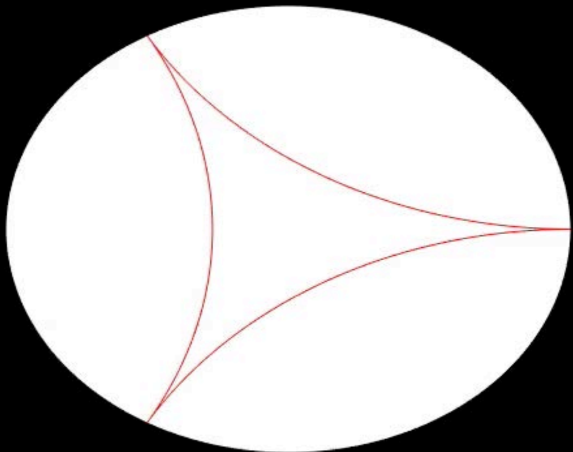
- **Emergent community structure**
- **Emergent geometry and preferential attachment**
- **Network Geometry with Flavor (NGF)**

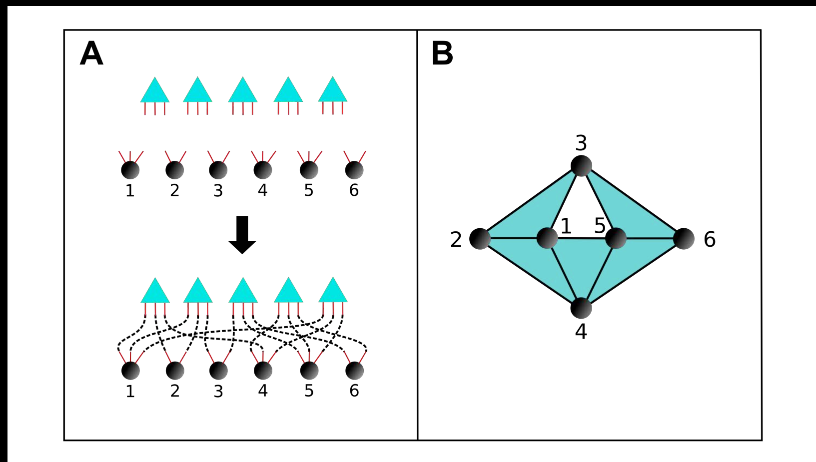
1. **Emergent hyperbolic geometry and quantum statistics**
2. **Statistical properties depending on dimension**
4. **Topological phase transitions in NGF with fitness**

Simplicial complex models of arbitrary dimension

Emergent Hyperbolic Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





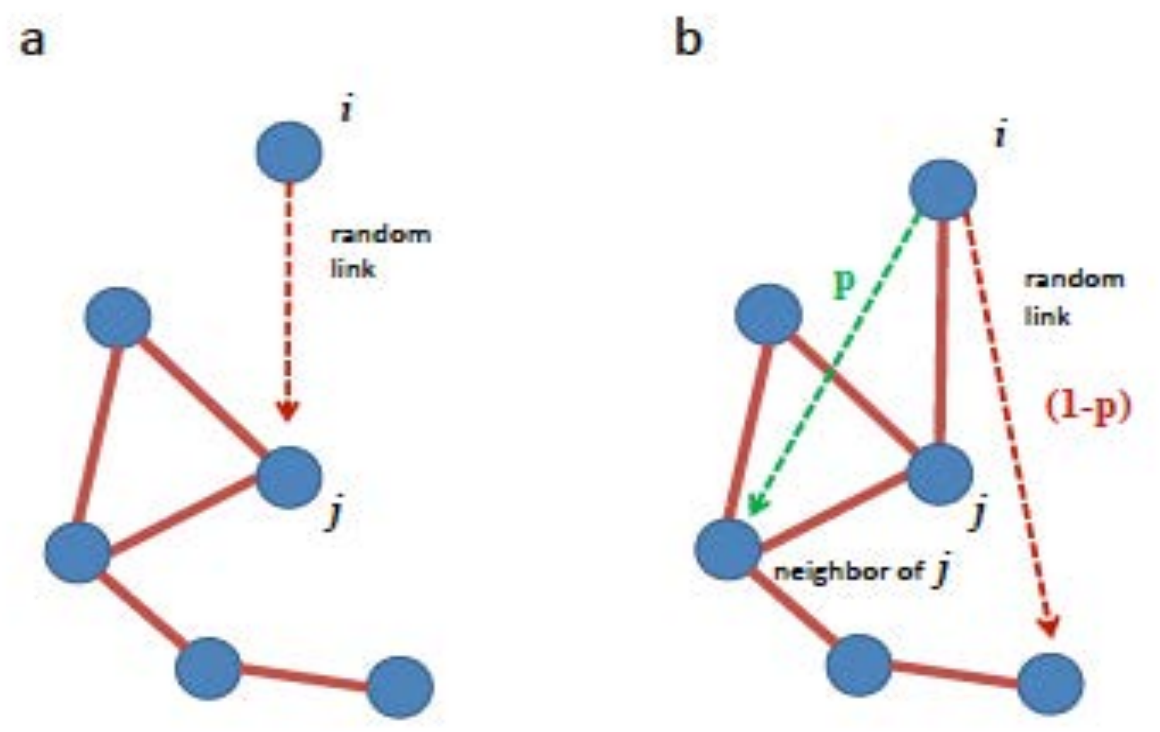
CODES AVAILABLE AT GITHUB **ginestrab**

Emergent properties of simplicial complexes

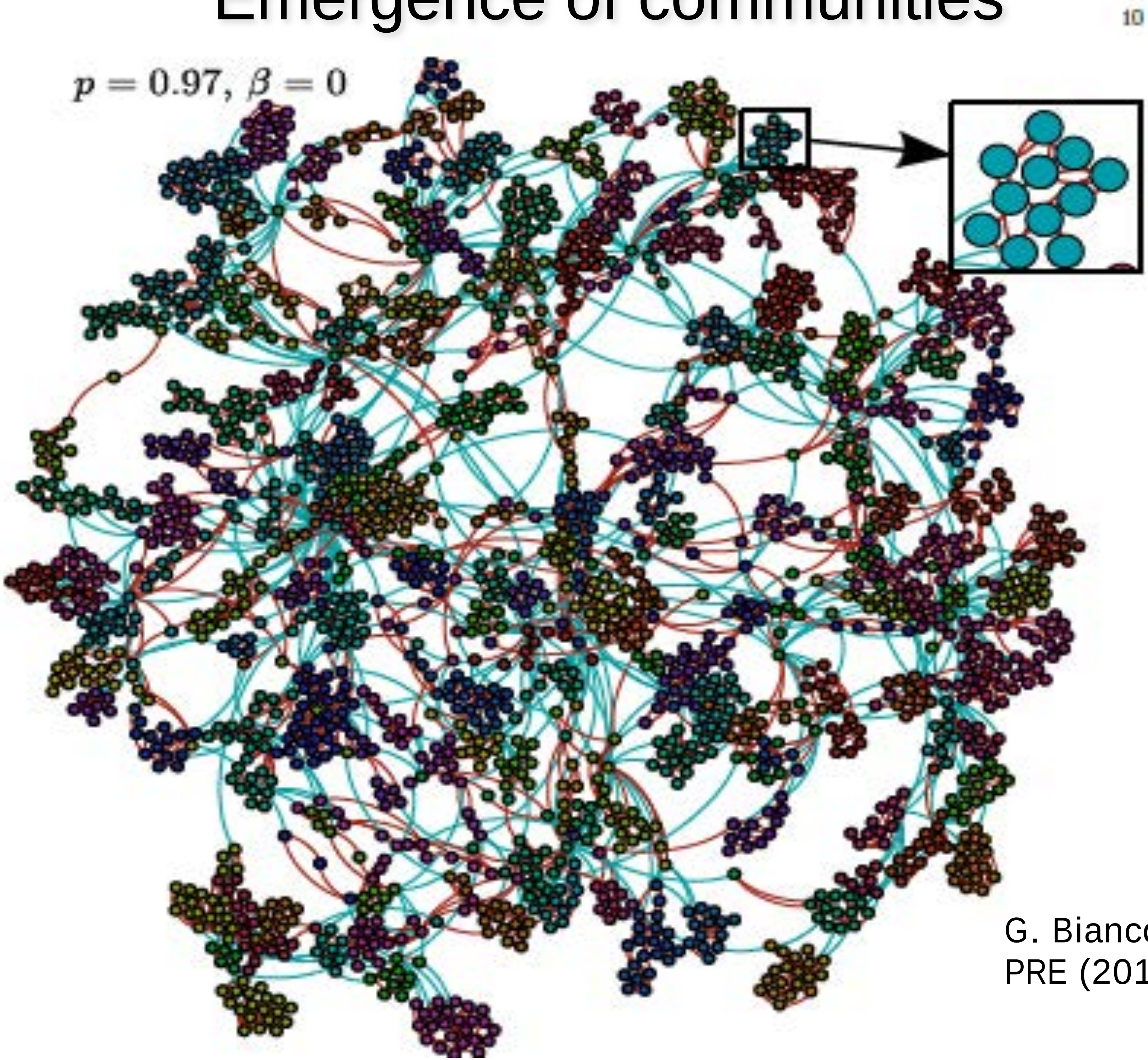
Emergence of communities

Triadic closure

- Starting from a finite connected network with $n_0>2$ nodes
- **(1) GROWTH** : At every timestep we add a new node with 2 edges (connected to the nodes already present in the system).
- **(2) TRIADIC CLOSURE:** The first link is attached to a random node, the second link with probability p closes a triangle and with probability 1-p is connected randomly



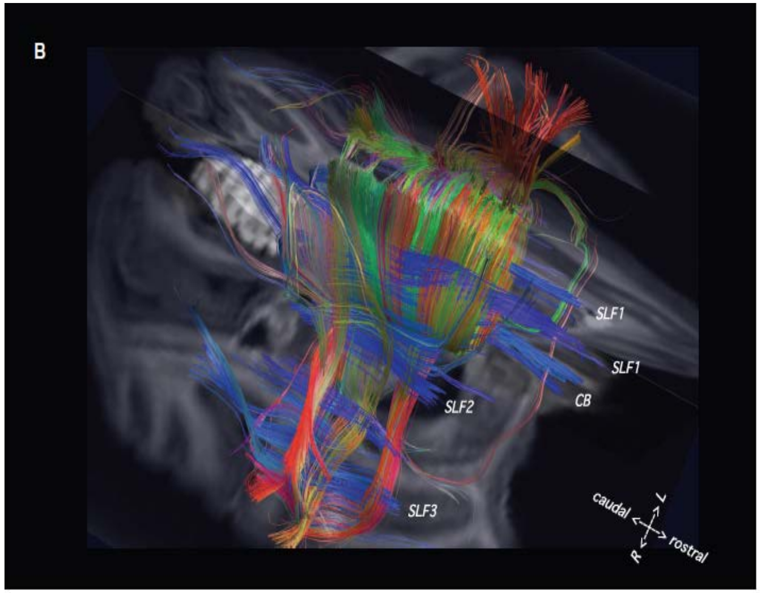
Emergence of communities

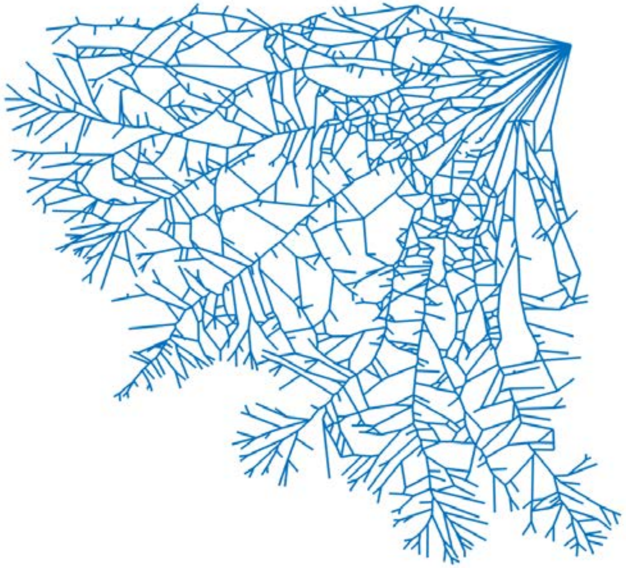


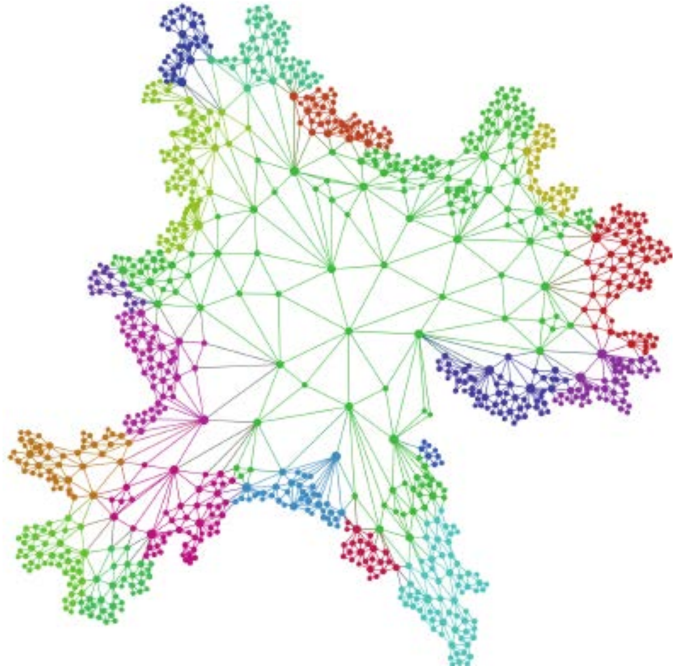
G. Bianconi et al.
PRE (2014)

Emergent geometry

Network Topology and Geometry







are expected to have impact in a variety of applications,

ranging from

brain research to biological transportation networks

Is the network geometry of complex systems
an a priori pre-requisite
for the network evolution
or is an emergent phenomenon of the
network dynamics?

Emergent geometry

In the framework of emergent geometry
networks with a geometry
are generated
by non-equilibrium dynamics
that is purely combinatorial,
i.e. is independent of the network geometry

Discrete and combinatorial space-time

My own view is that ultimately physical laws
should find their most natural expression
in terms of essentially combinatorial principles…
Thus, in accordance with such a view,
should emerge some form
of
discrete or combinatorial spacetime.

Roger Penrose
in
On the Nature of Quantum Geometry

Motivation

- Which is the basic mechanism for emergent geometry?
- What are the combinatorial/statistical properties of emergent geometry?
- What are the geometrical and topological properties that emerge?

Emergent geometry in 2-dimensional simplicial complexes

Emergent network geometry

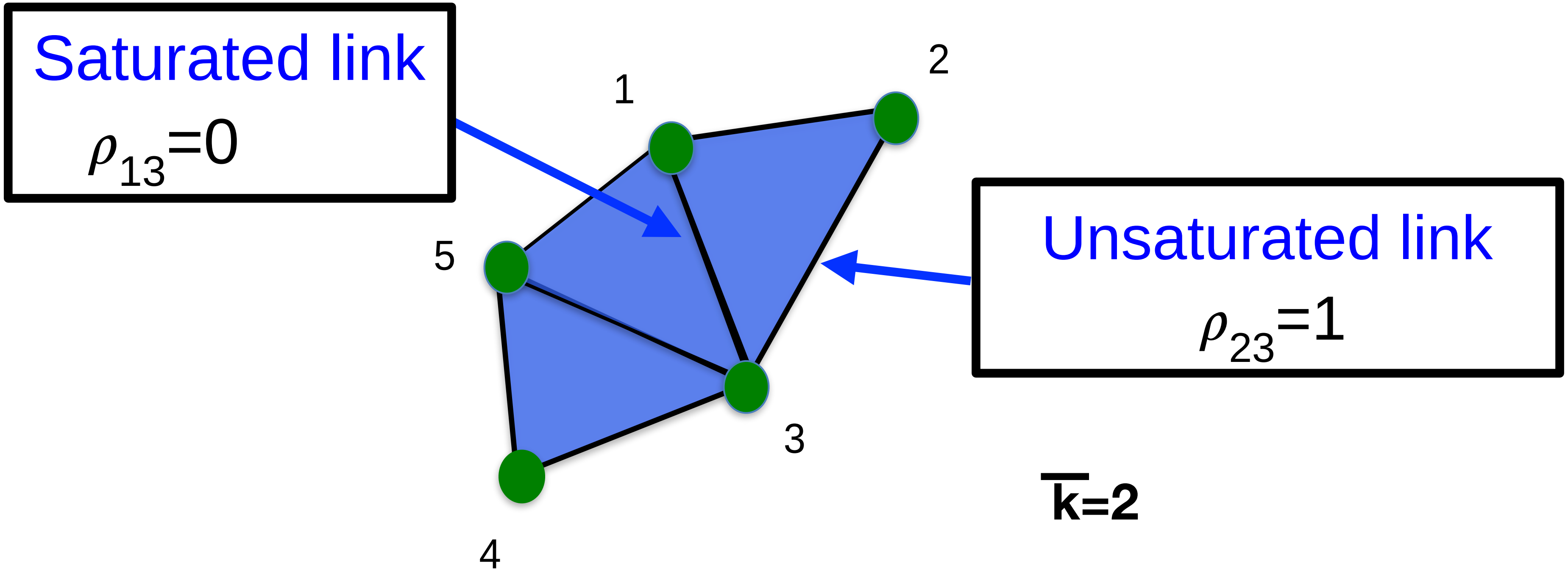
The model describes
the underlying structure of a simplicial complex
constructed by gluing together triangles by a
non-equilibrium dynamics.

Every link is incident to

at most $\bar{k}$ triangles with $\bar{k}>1$.

Wu, Menichetti, Rahmede, Bianconi, Scientific Reports (2015)

Saturated and unsaturated links



We classify links $[r, s]$ as *unsaturated* and *saturated* depending on the value of the auxiliary variable ρ_{rs} defined as

$$\rho_{rs} = \begin{cases} 0 & \text{if} \quad k_{2,1}([r,s]) < \bar{k}, \\ 1 & \text{if} \quad k_{2,1}([r,s]) = \bar{k}. \end{cases} \tag{5.1}$$

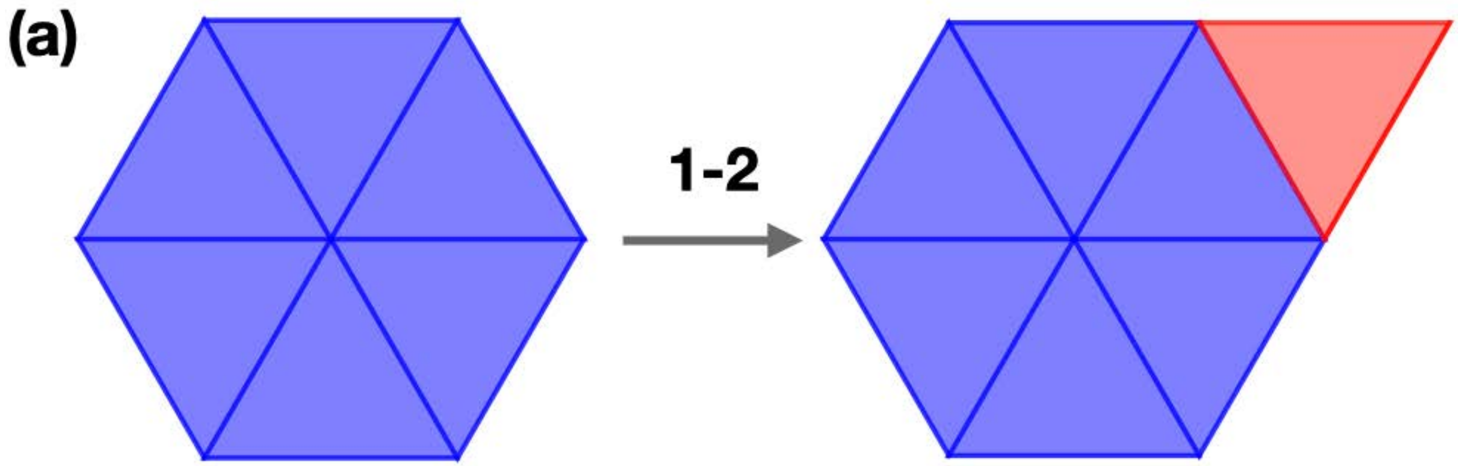
Therefore for each link $[r, s]$ there are two possibilities:

- if $\rho_{rs} = 0$ the link is *unsaturated* , i.e. less than $\bar{k}$ triangles are incident on it;
- if $\rho_{rs} = 1$ if the link is *saturated*, i.e. the number of incident triangles is given by $\bar{k}$.

Process (a)

We choose a link (i,j) with probability and glue a new triangle the link

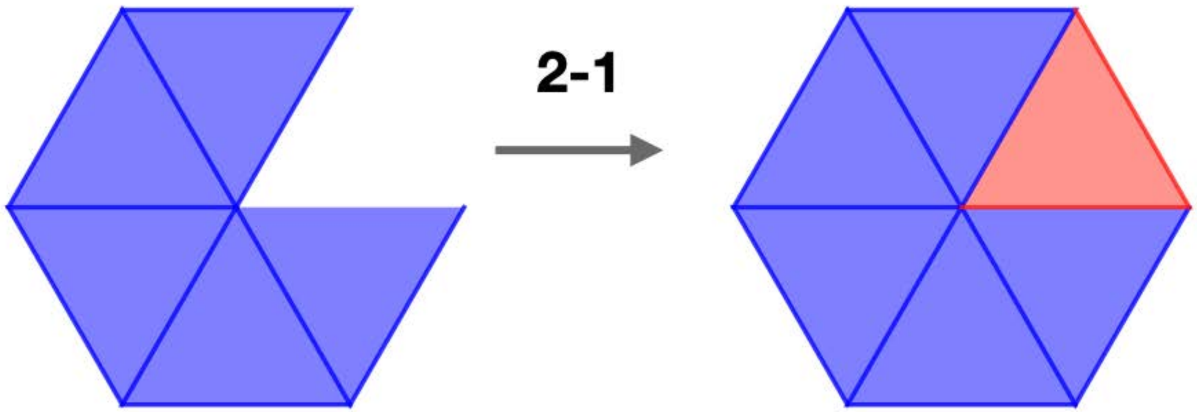
$$\Pi_{(i,j)} = \frac{\rho_{ij}}{\sum_{r,s} \rho_{rs}}$$



Process (b)

We choose a two adjacent unsaturated links and we add the link between the nodes at distance 2 and all triangles that this link closes as long that this is allowed.

(b)



The model

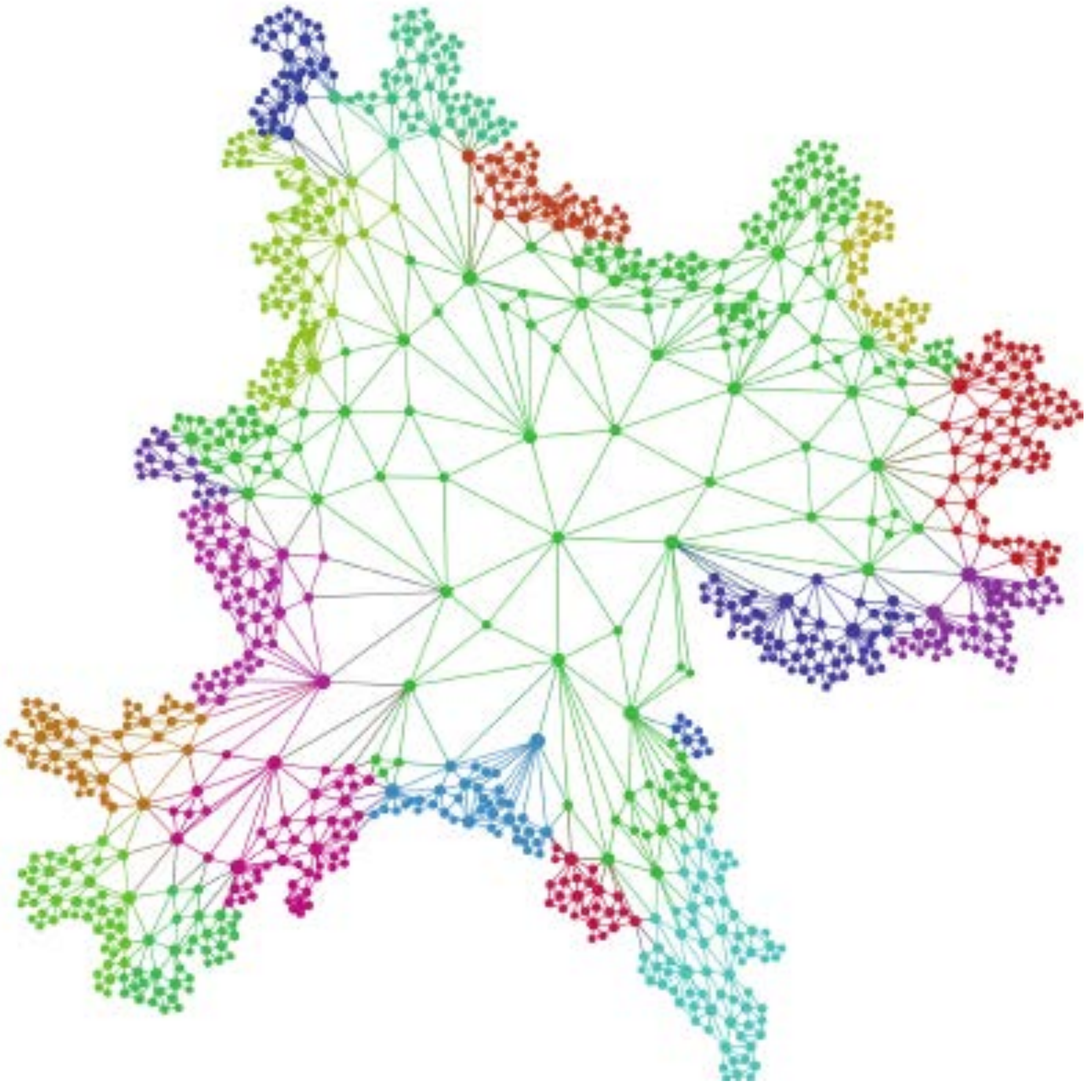
Starting from an initial triangle,
At each time

- **process (a) takes place**

and

- **process (b) takes place**
with probability p<1.

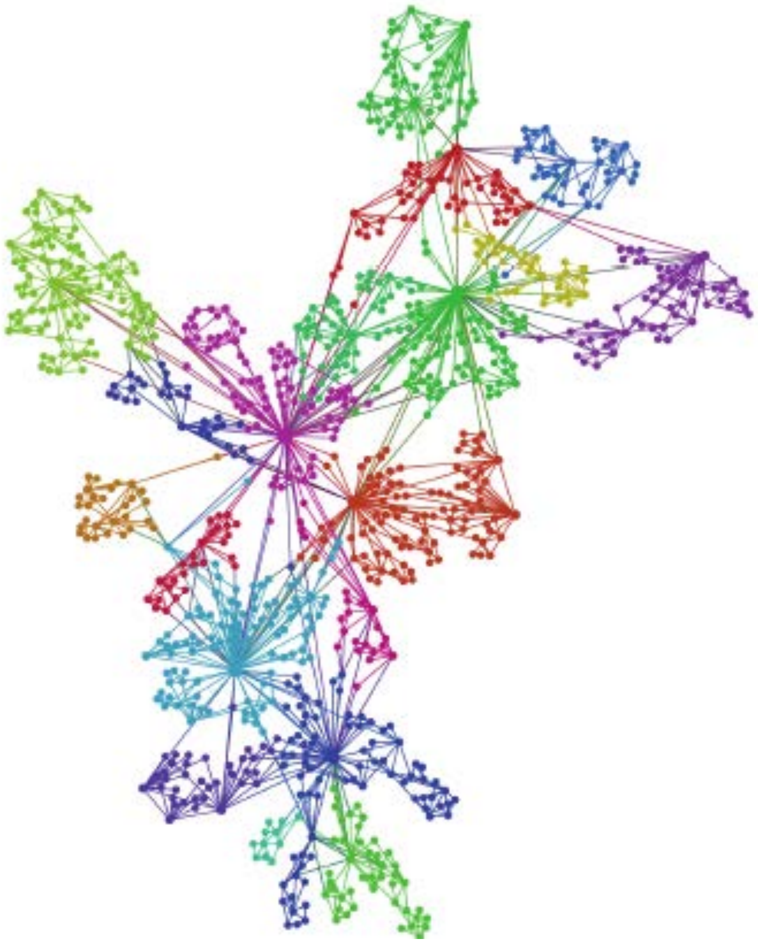
Discrete Manifolds



A discrete manifold of dimension d=2 is a simplicial complex formed by triangles such that every link is incident to at most two triangles.

Therefore the emergent network geometry for our model with m=2 is a discrete 2d manifold.

Scale-free networks



In the case $m = \infty$ a scale-free network with high clustering, significant community structure, finite spectral dimension is generated.

Planar for p=0.

Properties of emergent network geometries

- **Small world**
- **Finite clustering**
- **High modularity**
- **Finite spectral dimension**

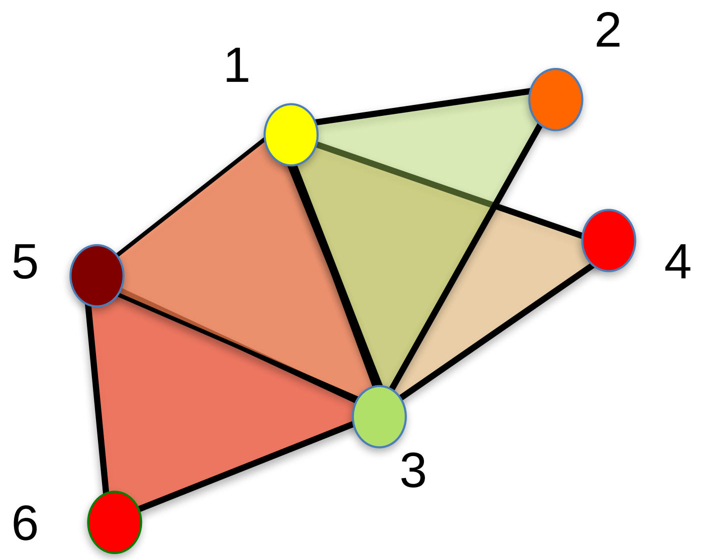
Which are properties of many real network datasets.

Properties of real datasets

Datasets	N	L	$\langle \ell \rangle$	C	M	d_S
1L8W (protein)	294	1608	5.09	0.52	0.643	1.95
1PHP (protein)	219	1095	4.31	0.54	0.638	2.02
1AOP chain A (protein)	265	1363	4.31	0.53	0.644	2.01
1AOP chain B (protein)	390	2100	4.94	0.54	0.685	2.03
Brain-(coactivation) [45]	638	18625	2.21	0.384	0.426	4.25
Internet [46]	22963	48436	3.8	0.35	0.652	5.083
Power-grid[38]	4941	6594	19	0.11	0.933	2.01
Add Health (school61)[47]	1743	4419	6	0.22	0.741	2.97

Network Geometry with Flavor

Network Geometry with Flavor



NETWORK GEOMETRY WITH FLAVOR (NEUTRAL MODEL) [29]

At time $t = 1$ the NGF is formed by a single d-dimensional simplex. At each time $t > 1$ the model evolves according to the following principles.

- GROWTH : At every timestep a new d-dimensional simplex formed by one new node and an existing $(d-1)$-face is added to the simplicial complex.
- ATTACHMENT: The probability that the new d-simplex is glued to a $(d-1)$-dimensional face α depends on the *flavor* $s \in \{-1, 0, 1\}$ and is given by

$$\Pi_{\alpha}^{[s]} = \frac{(1 + s n_{\alpha})}{\sum_{\alpha'}(1 + s n_{\alpha'})}. \tag{5.6}$$

Bianconi & Rahmede (2016)

Attachment probability

The attachment probability to (d-1)-dimensional faces is given by

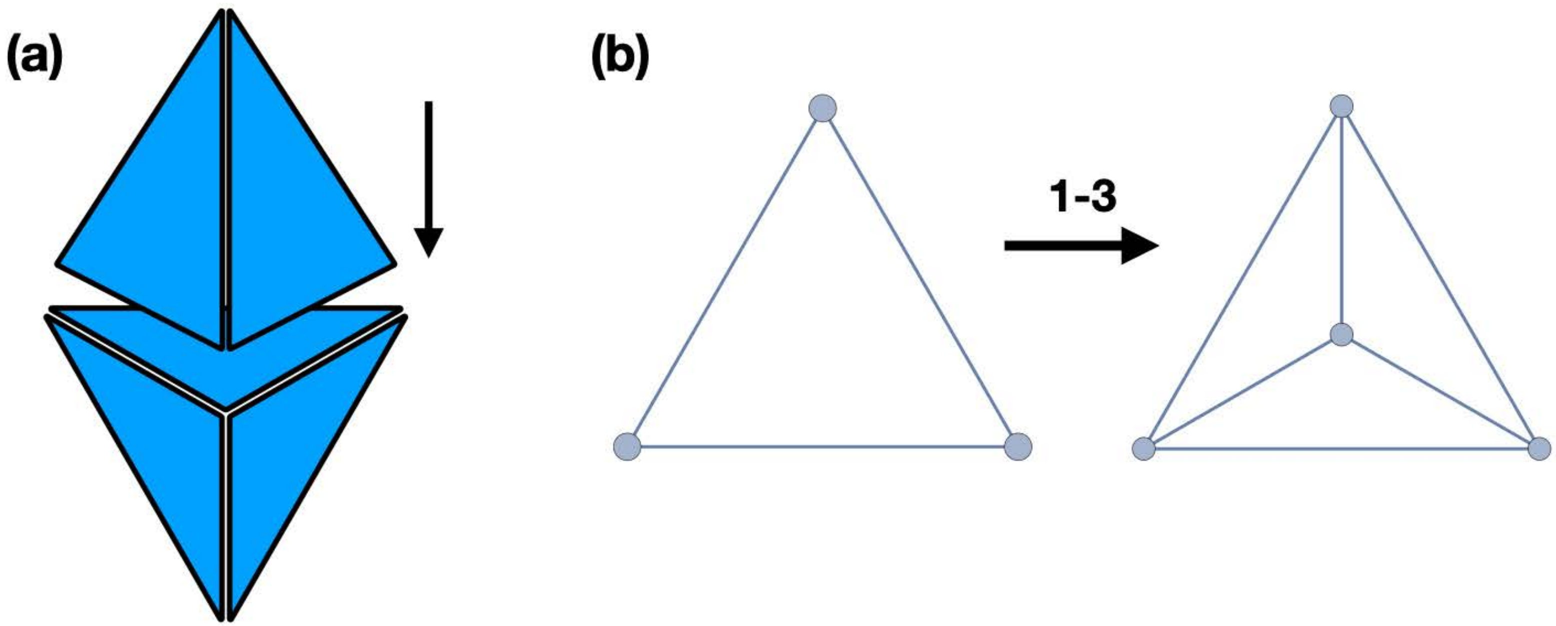
$$\Pi_{\alpha}^{[s]} = \frac{(1 + sn_{\alpha})}{\sum_{\alpha'}(1 + sn_{\alpha'})} \propto \begin{cases} 1 - n_{\alpha} & \textbf{if } s = -1 \\ 1 & \textbf{if } s = 0 \\ k_{d,d-1}(\alpha) & \textbf{if } s = 1 \end{cases}$$

For s=-1 we obtain discrete manifolds $n_{\alpha} = 0,1$

For s=0 we have uniform attachment $n_{\alpha} = 0,1,2,3,4...$

For s=1 we have a generalised preferential attachment $n_{\alpha} = 0,1,2,3,4...$

Pachner move 1-d for NGF with s=-1



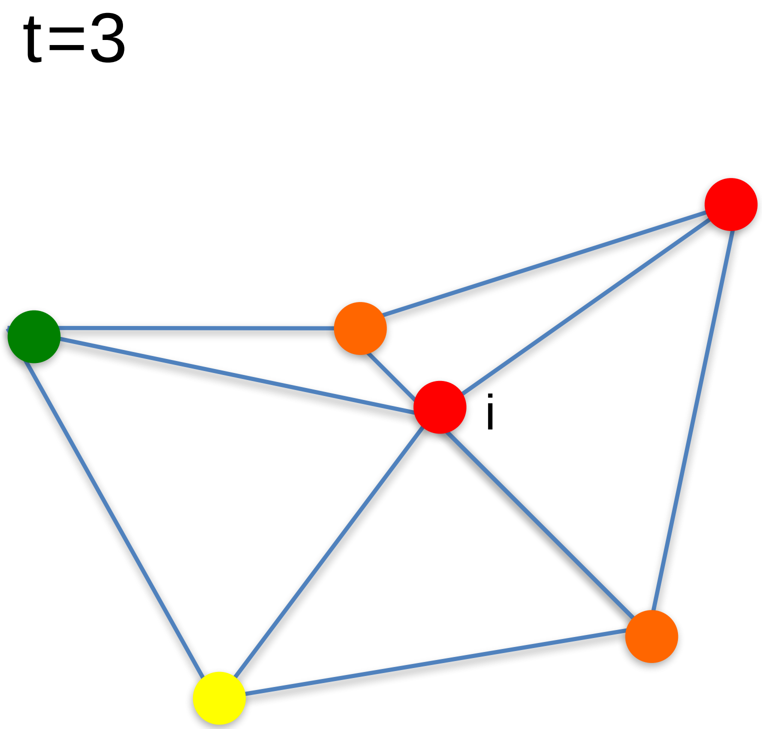
Emergence of preferential attachment

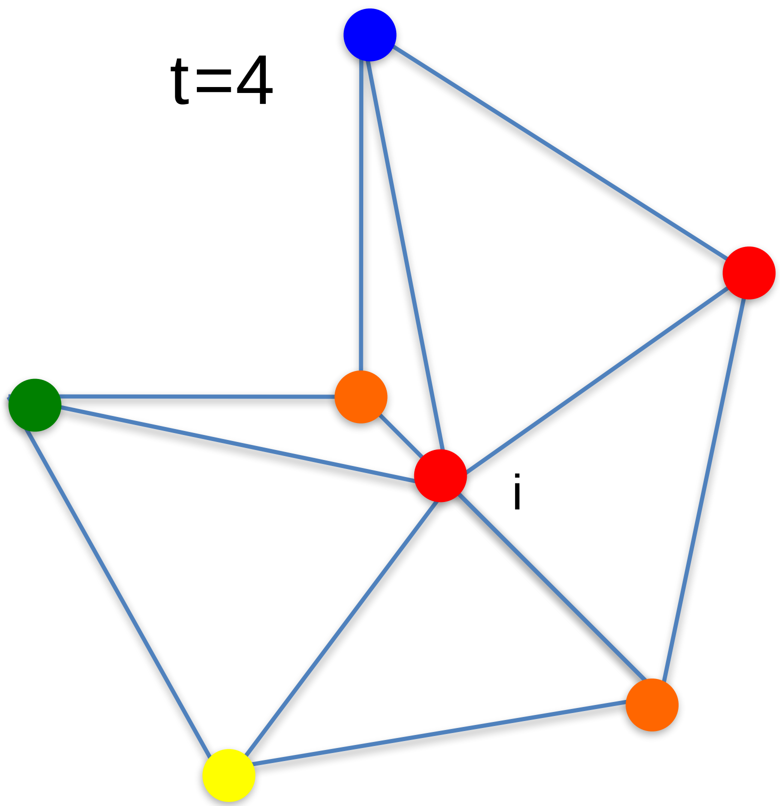
The probability of attaching a d-dimensional simplex to a δ-dimensional face is given by

$$\Pi_{d,\delta}(k) = \begin{cases} \dfrac{2-k}{(d-1)t} \text{ for } d+s-\delta-1=-1 \\ \dfrac{(d-\delta-1+s)k+1-s}{(d+s)t} \text{ for } d+s-\delta-1 \geq 0 \end{cases}$$

Therefore for $d-\delta > 1-s$ we observe a generalised preferential attachment as a consequence of the geometry and dimensionality of of the NGF

Effective preferential attachment in d=3 s=-1





Node i has generalized degree 3
Node i is incident to 5 faces with n=0

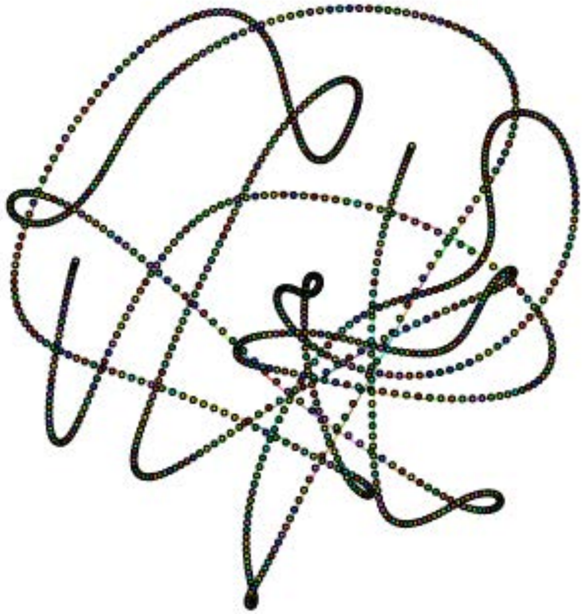
Node i has generalized degree 4
Node i is incident to 6 faces with n=0

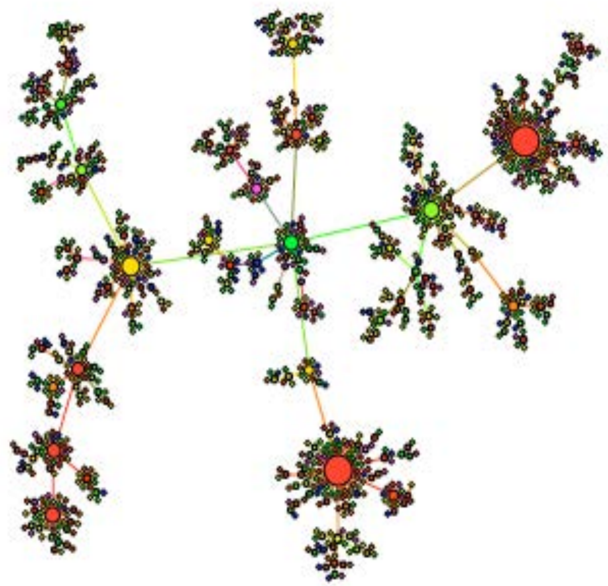
Dimension d=1

Manifold

Uniform attachment

Preferential attachment





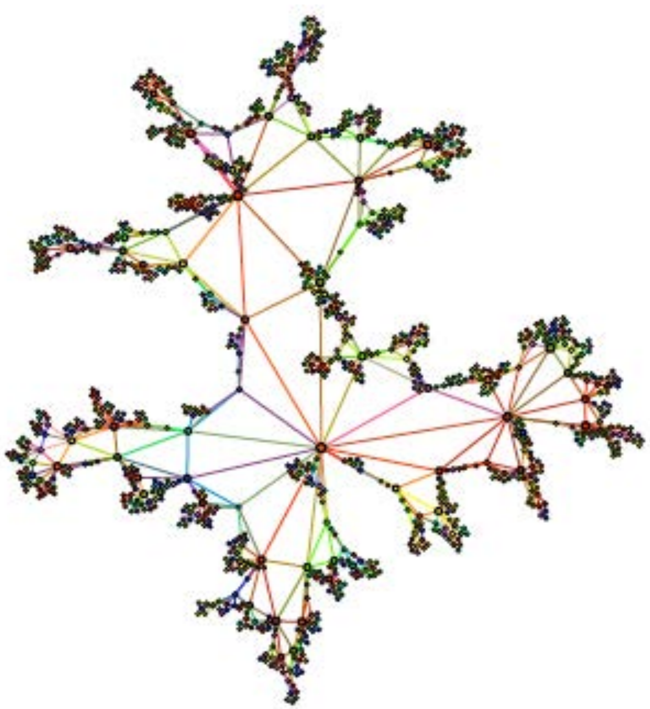
Chain

Exponential

BA model

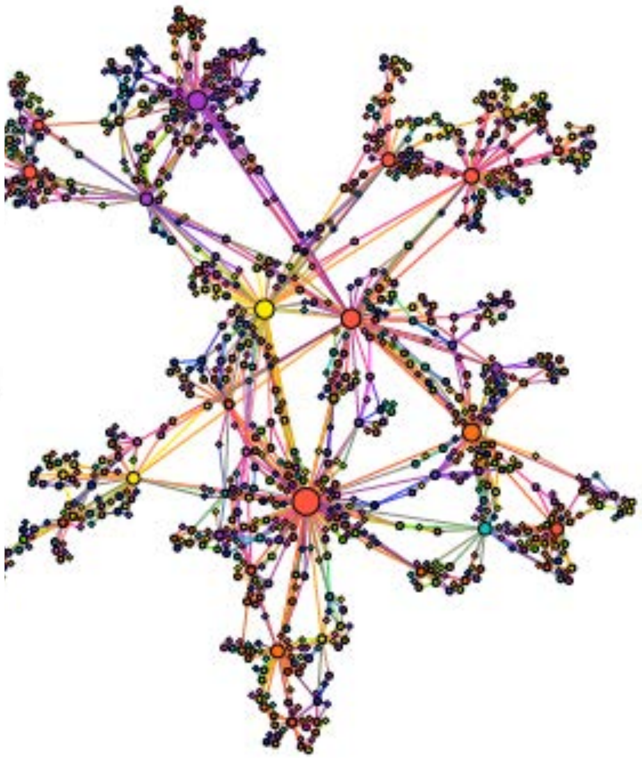
Dimension d=2

Manifold



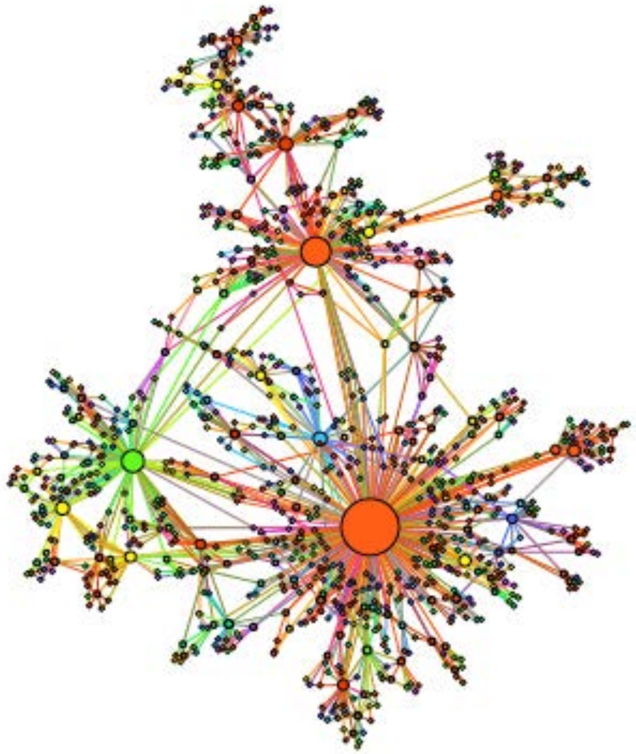
Exponential

Uniform attachment



Scale-free

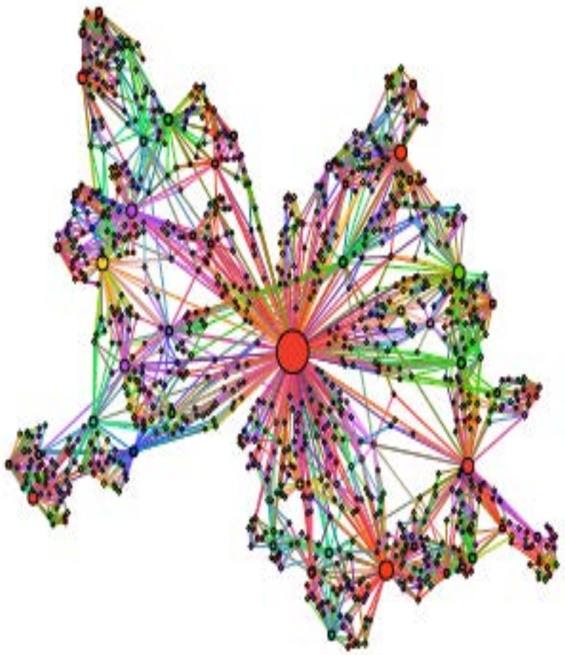
Preferential attachment



Scale-free

Dimension d=3

Manifold

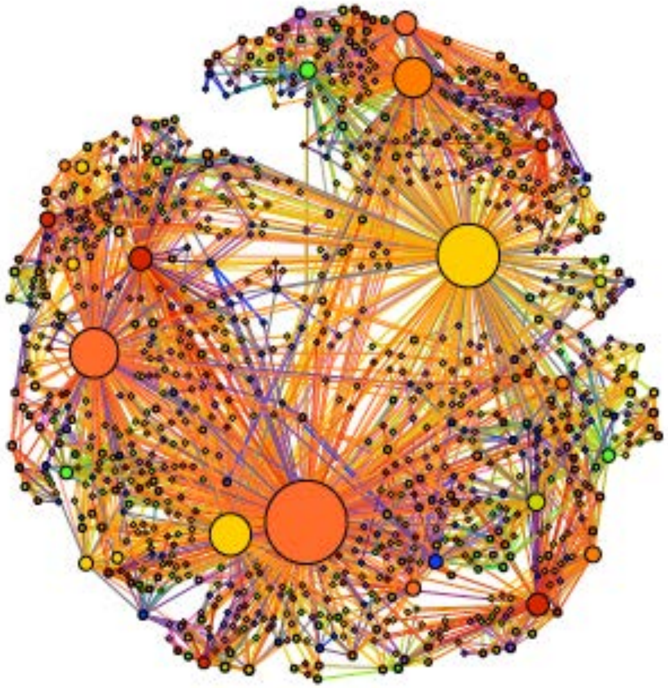


Scale-free

Uniform attachment

Scale-free

Preferential attachment



Scale-free

Degree distribution

For d+s=1

$$P_d^{[s]}(k) = \left(\frac{d}{d+1}\right)^{k-d} \frac{1}{d+1}$$

For d+s>1

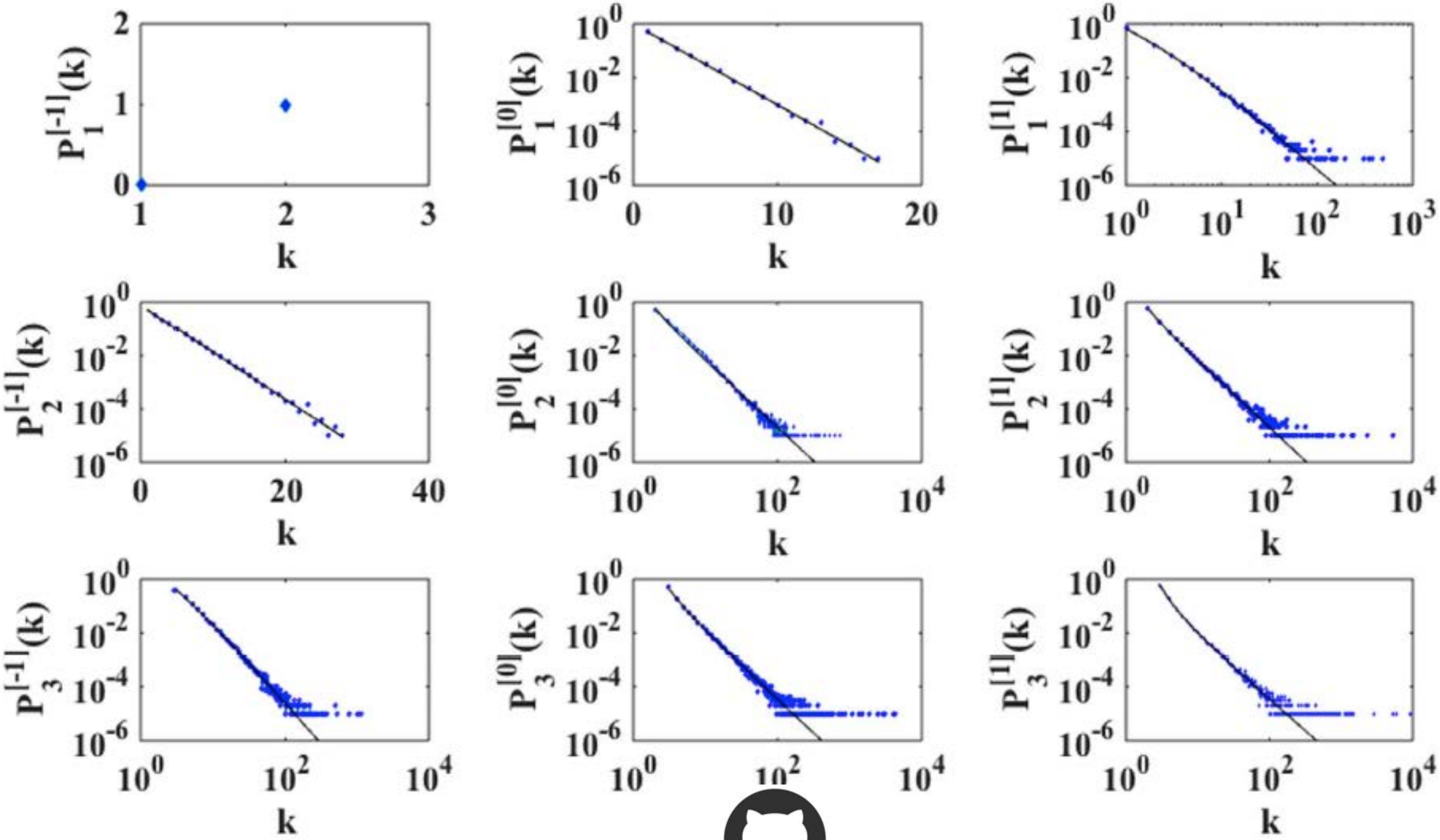
$$P_d^{[s]}(k) = \frac{d+s}{2d+s} \frac{\Gamma[1+(2d+s)/(d+s-1))]}{\Gamma[d/(d+s-1)]} \frac{\Gamma[k-d+d/(d+s-1)]}{\Gamma[k-d+1+(2d+s)/(d+s-1)]}$$

NGF are always scale-free for d>1-s

- For s=1 NGF are always scale free
- For s=0 and d>1 the NGF are scale-free
- For s=-1 and d>2 the NGF are scale-free

[Bianconi & Rahmede (2016)]

Degree distribution of NGF



CODE AVAILABLE AT GITHUB PAGE ginestrab

Generalized degree distribution

Simplicial complexes can have generalised degree distribution following different statistics depending on the dimension of the faces considered

Flavor	$s = -1$	$s = 0$	$s = 1$
$m = d - 1$	Bimodal	Exponential	Power-law
$m = d - 2$	Exponential	Power-law	Power-law
$m \leq d - 3$	Power-law	Power-law	Power-law

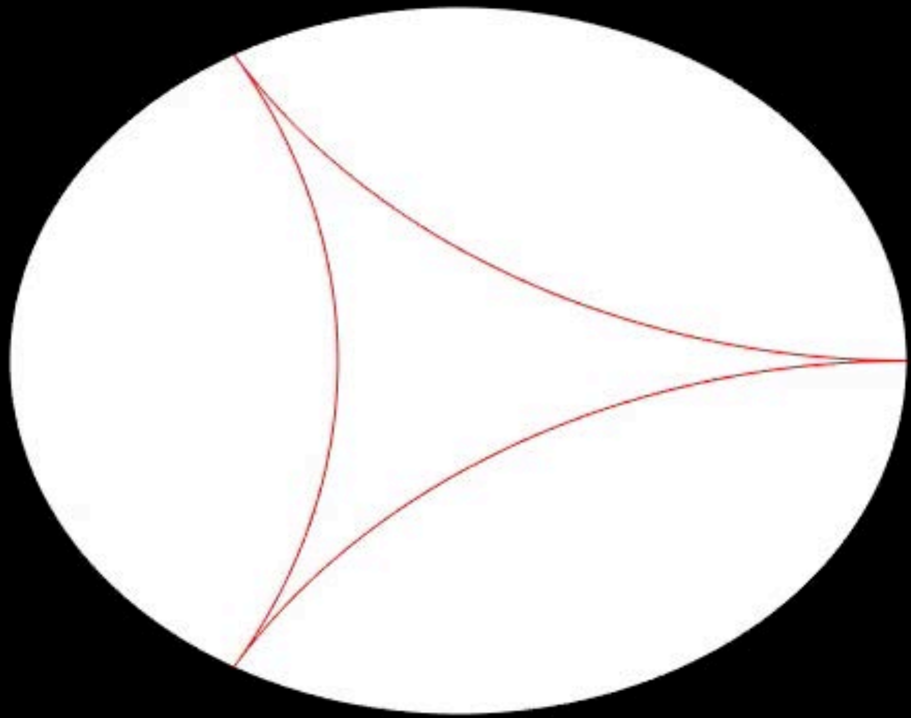
The generalized degree distribution depends on the flavor s and on the dimension m of the faces

[Bianconi & Rahmede (2016)]

Emergent Hyperbolic geometry

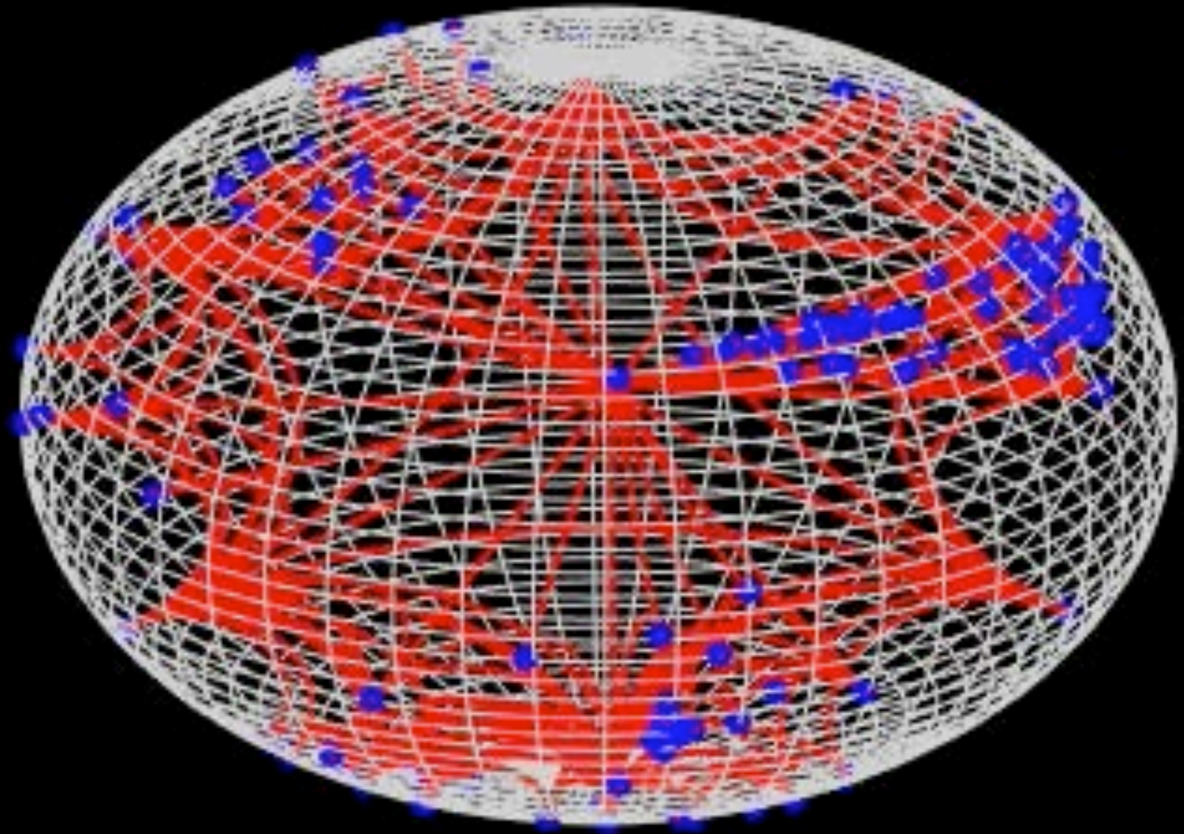
The emergent hidden geometry is the hyperbolic H^d space

Here all the links have equal length



d=2

Emergent hyperbolic geometry



d=3

NGF an hyperbolic network geometry

NGF for flavor s=-1 are discrete hyperbolic manifolds

NGF of any flavor and any dimension are δ-hyperbolic networks

[with δ=1]

What is a “natural” random geometry?

- Random graph

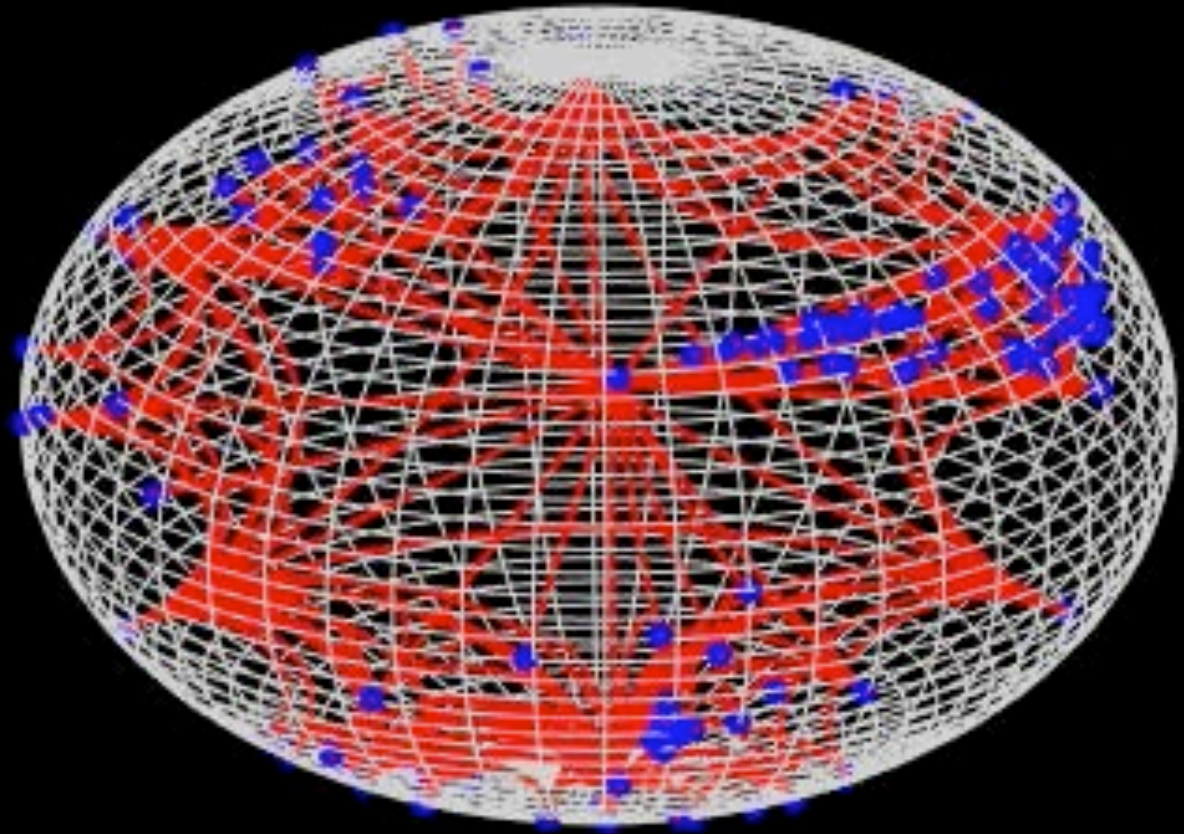
(fully connected network -trivial/no geometry- where some random links are selected)

- A percolation cluster in 2d

(square lattice -known given geometry- where only few links are preserved)

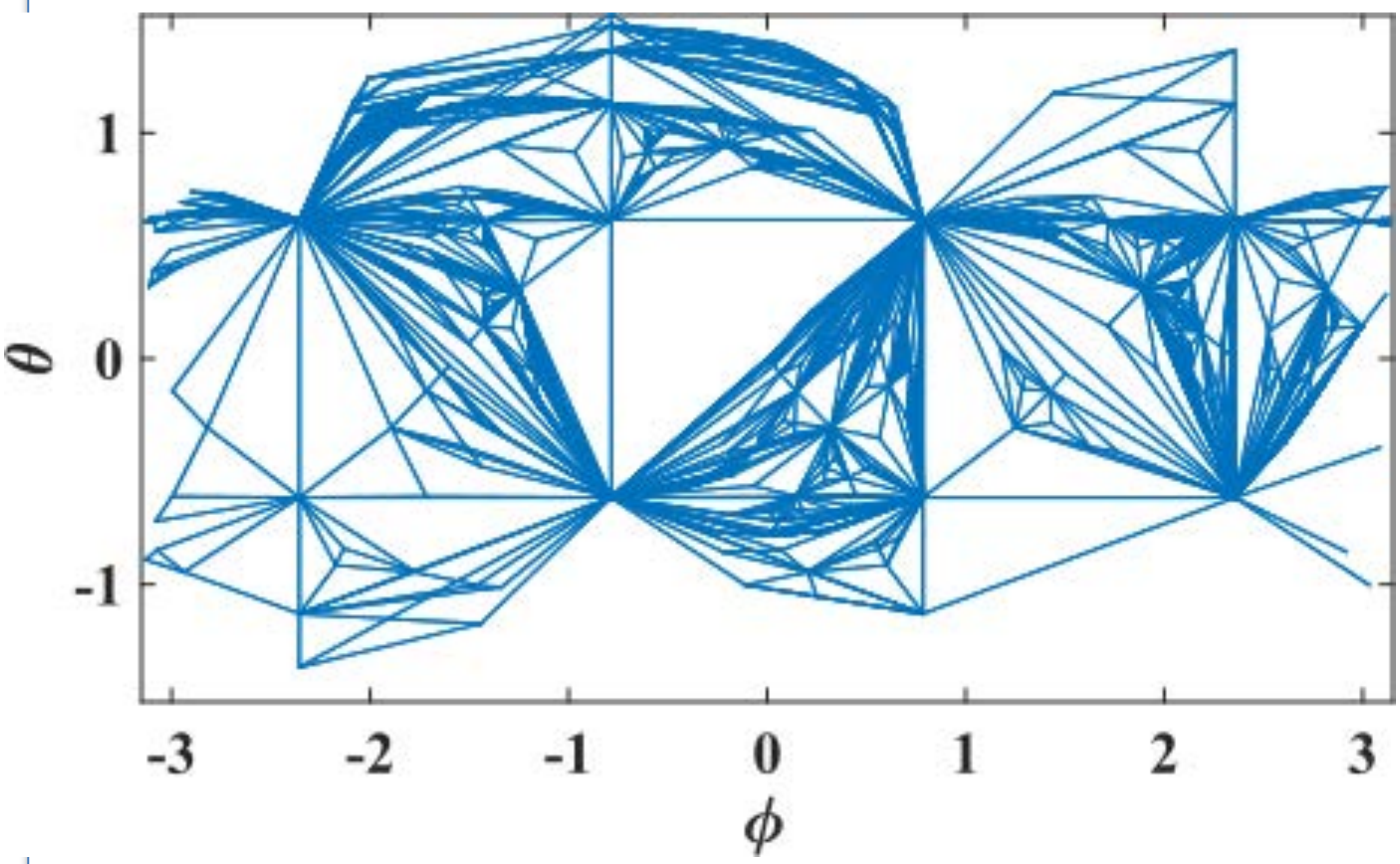
- A growing cluster on -emergent- hyperbolic lattice

Emergent hyperbolic geometry

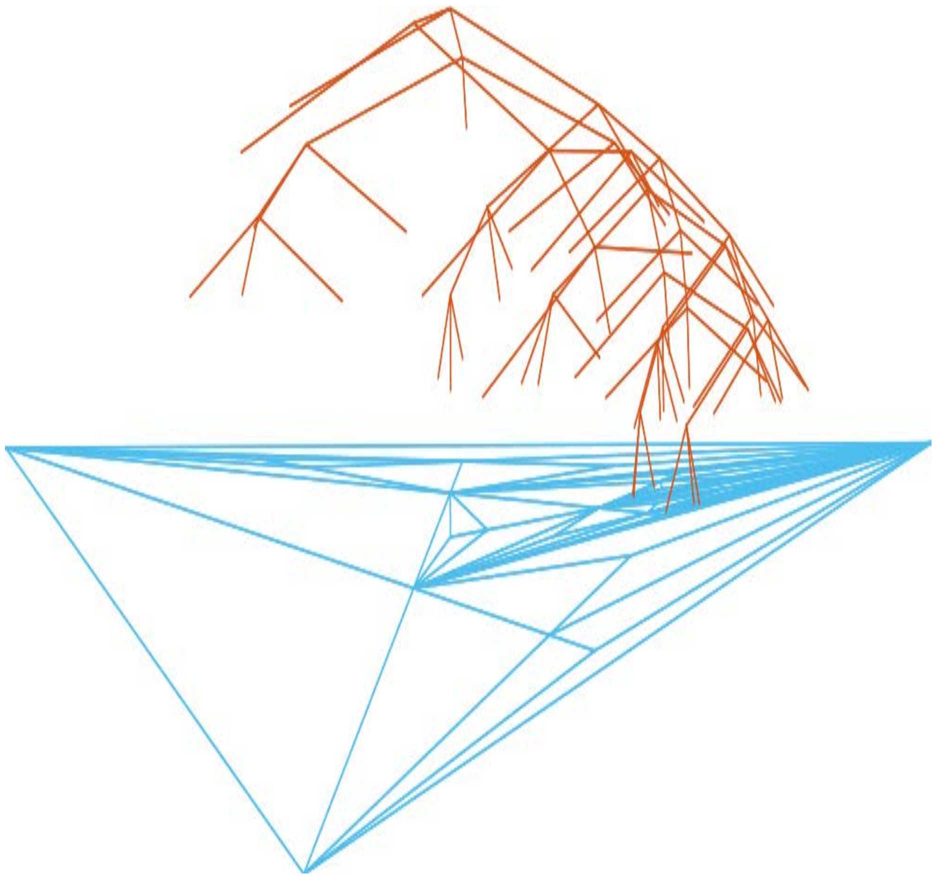


d=3

Planar projection of the d=3 NGF with s=-1

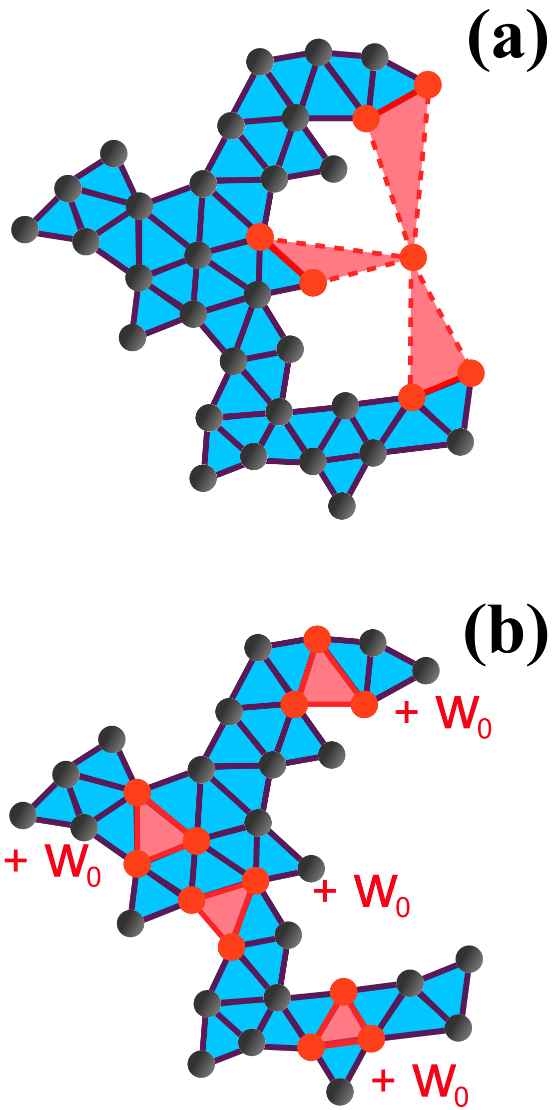


The relation to Trees



Line graph of the NGF

Growing weighted simplicial complex



We considered a weighted network model in which we assume:

- that each new node can attach m simplices to the rest of network
- that simplices can increase their weight in time

We found deep correlations between the weights of the simplices and the network topology.

Courtney Bianconi (2017)

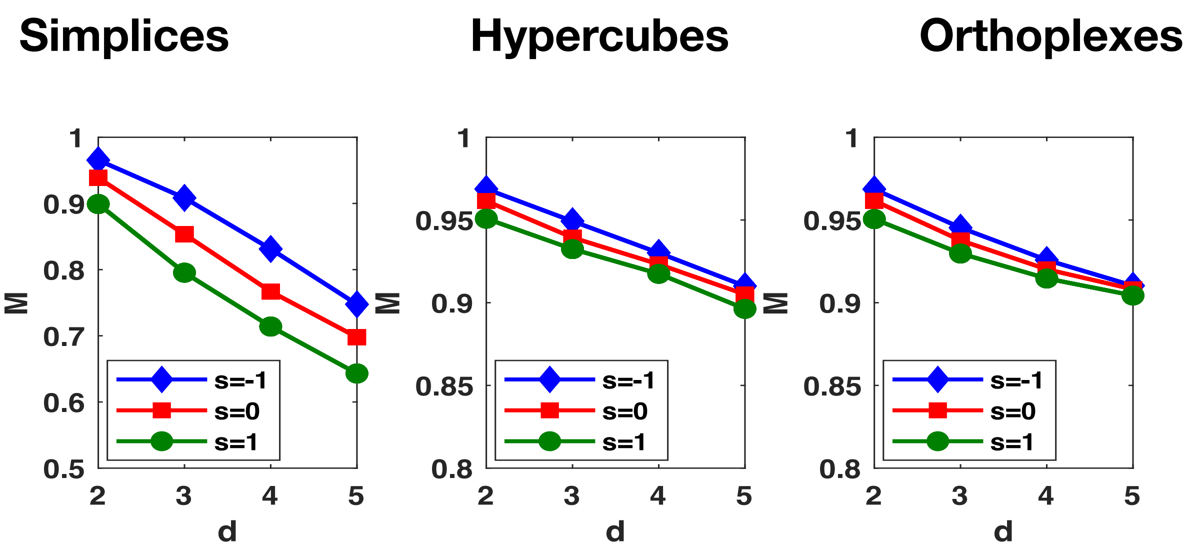
NGF cell complexes

The power-law exponent γ depends on the nature of the regular polytope that constitute the building block of the cell complex

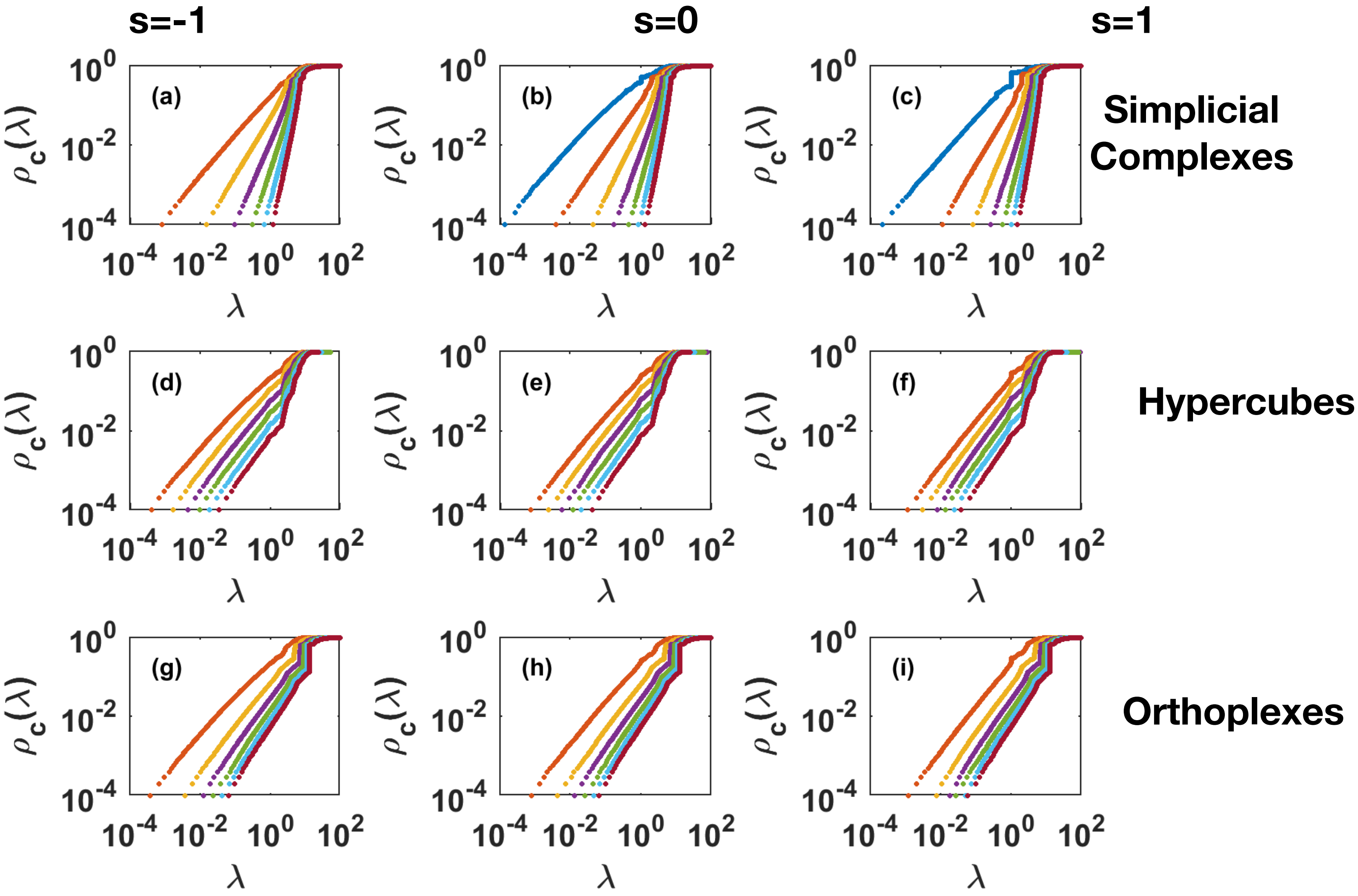
γ	$s=-1$	$s=0$	$s=1$
$d=1$			
link	N/A	N/A	3
$d=2$			
p-polygon	N/A	p	$1+\frac{p}{2}$
$d=3$			
tetrahedron	3	$2\frac{1}{2}$	$2\frac{1}{3}$
cube	5	$3\frac{1}{2}$	3
octahedron	4	$3\frac{1}{3}$	3
dodecahedron	11	$6\frac{1}{2}$	5
icosahedron	7	$5\frac{3}{4}$	5
$d=4$			
pentachoron	$2\frac{1}{2}$	$2\frac{1}{3}$	$2\frac{1}{4}$
tesseract	4	$3\frac{1}{3}$	3
hexadecachoron	$3\frac{1}{3}$	$3\frac{1}{7}$	3
24-cell	$6\frac{1}{2}$	$5\frac{3}{5}$	5
120-cell	60	$40\frac{2}{3}$	31
600-cell	$34\frac{2}{9}$	$32\frac{10}{19}$	31
$d>4$			
simplex	$2+\frac{1}{d-2}$	$2+\frac{1}{d-1}$	$2+\frac{1}{d}$
cube	$3+\frac{2}{d-2}$	$3+\frac{1}{d-1}$	3
orthoplex	$3+\frac{1}{2^{(d-2)}-1}$	$3+\frac{1}{2^{d-1}-1}$	3

Modularity of NGFs

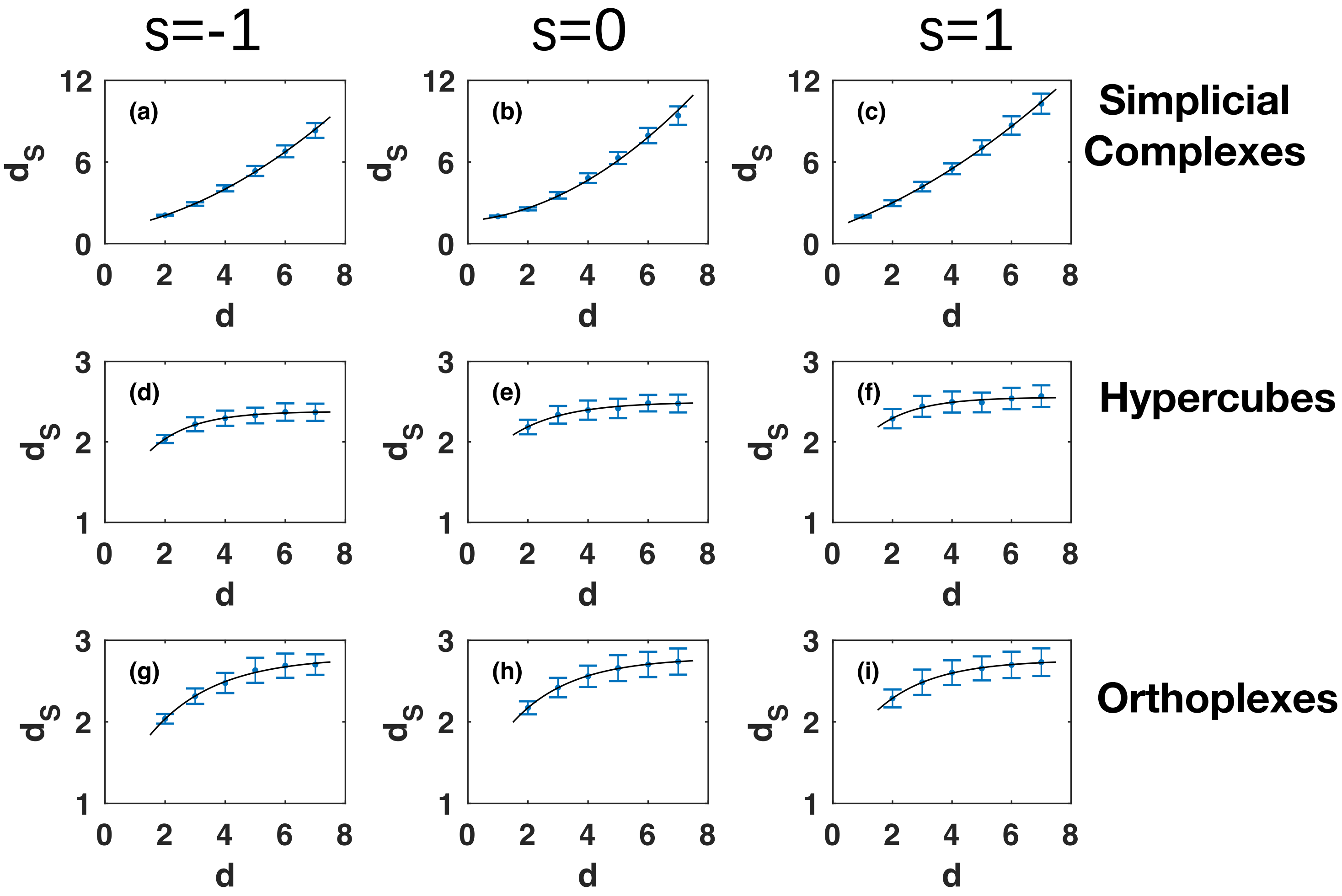
Network Geometry with Flavor displays emergent community structure



Laplacian spectrum of NGFs



Spectral dimension of NGF



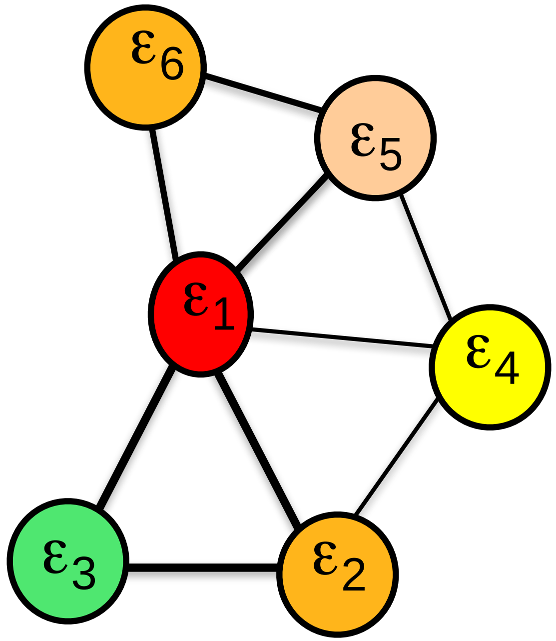
Energies of the nodes

Not all the nodes are the same!

Let assign to each node i

an energy ε *from a*

g(ε) *distribution*



Energy of the m-faces

ENERGY AND FITNESS OF THE FACES OF THE NGF SIMPLICIAL COMPLEXES [29]

The energy ε_α of the m-dimensional face α indicates its intrinsic (non-topological) properties. The energy $\varepsilon_{[r]}$ of a node r is a non negative number drawn from a given distribution $g(\varepsilon)$. The energy of a face α of dimension $m > 0$ is the sum of the energies of the nodes belonging to it, i.e.

$$\varepsilon_\alpha = \sum_{r \subset \alpha} \varepsilon_{[r]}. \tag{5.14}$$

The *fitness* associated to a m-dimensional face α describes the rate at which the face increases its generalized degree and is given by

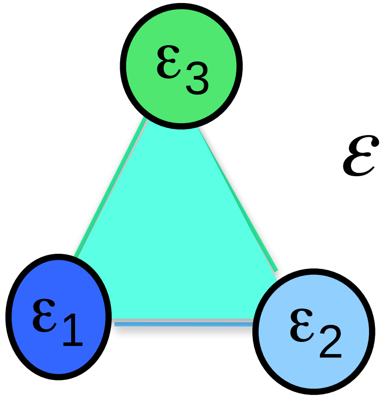
$$\eta_\alpha = e^{-\beta \varepsilon_\alpha} \tag{5.15}$$

where $\beta > 0$ is a parameter called *inverse temperature*. For $\beta = 0$ all the fitnesses are the same, and equal to one, while for $\beta \gg 1$ the small difference in energy leads to big differences in the fitnesses of the faces.

Energy of a link

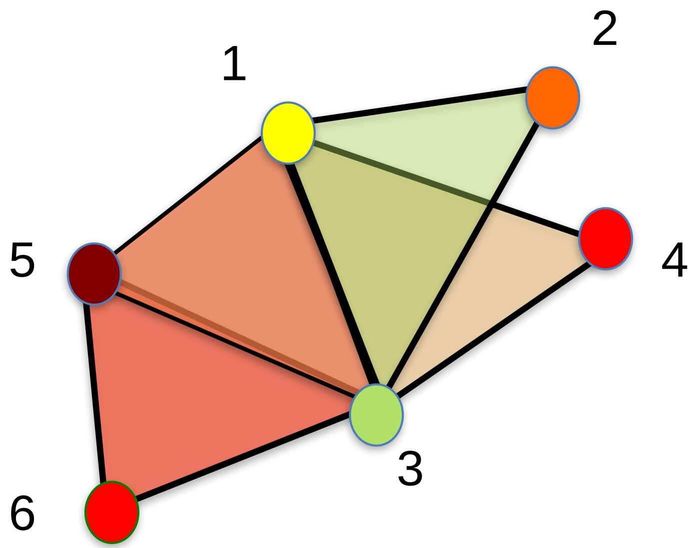
$\varepsilon_1 + \varepsilon_2$

Energy of a triangle



$\varepsilon_1 + \varepsilon_2 + \varepsilon_3$

Network Geometry with Flavor



$$\Pi_\alpha^{[s]} = \frac{e^{-\beta\varepsilon_\alpha}\,(1+sn_\alpha)}{\sum_{\alpha'} e^{-\beta\varepsilon_{\alpha'}}\,(1+sn_{\alpha'})}$$

NETWORK GEOMETRY WITH FLAVOR (WITH FITNESS) [29]

At time $t = 1$ the simplicial complex is formed by a single d-dimensional simplex. Each node r of this simplex has energy $\varepsilon_{[r]}$ drawn from a $g(\varepsilon)$ distribution. The energies of the higher-dimensional faces are calculated according to Eq. (5.14).

- GROWTH : At every timestep a new d-dimensional simplex formed by one new node and an existing $(d-1)$-face is added to the simplicial complex. Each new node r has energy $\varepsilon_{[r]}$ drawn from a $g(\varepsilon)$ distribution. The energies of the new higher-dimensional faces are calculated according to Eq. (5.14).
- ATTACHMENT: At every timestep the probability that the new d-simplex is connected to the existing $(d-1)$-dimensional face α depends on the *flavor* $s \in \{-1, 0, 1\}$ and on the *inverse temperature* $\beta > 0$ and is given by

$$\Pi_\alpha^{[s]} = \frac{e^{-\beta\varepsilon_\alpha}(1+sn_\alpha)}{\sum_{\alpha'} e^{-\beta\varepsilon_{\alpha'}}(1+sn_{\alpha'})}. \tag{5.16}$$

For $\beta = 0$ the NGF (with fitness of the m-faces) reduces to the neutral NGF model, i.e. $\Pi_\alpha^{[s]}$ reduces to Eq. (5.6).

Bianconi & Rahmede (2016)

The average of the generalized degree of the NGF over δ-faces of energy ε

$$\left\langle [k_{d,m}(\alpha) - 1] \,|\, \varepsilon_\alpha = \varepsilon \right\rangle$$

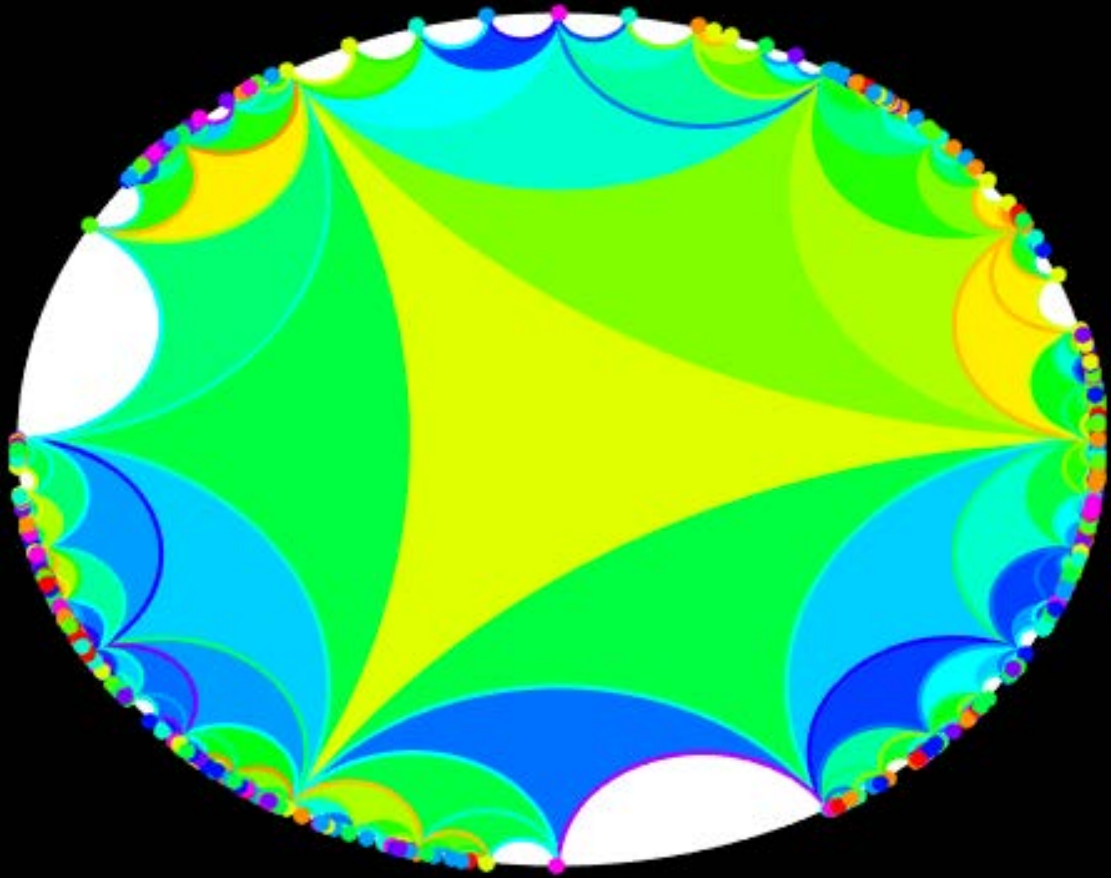
follows a regular pattern

Flavor	$s = -1$	$s = 0$	$s = 1$
$m = d - 1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$m = d - 2$	Boltzmann	Bose-Einstein	Bose-Einstein
$m \leq d - 3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

Manifolds in d=3

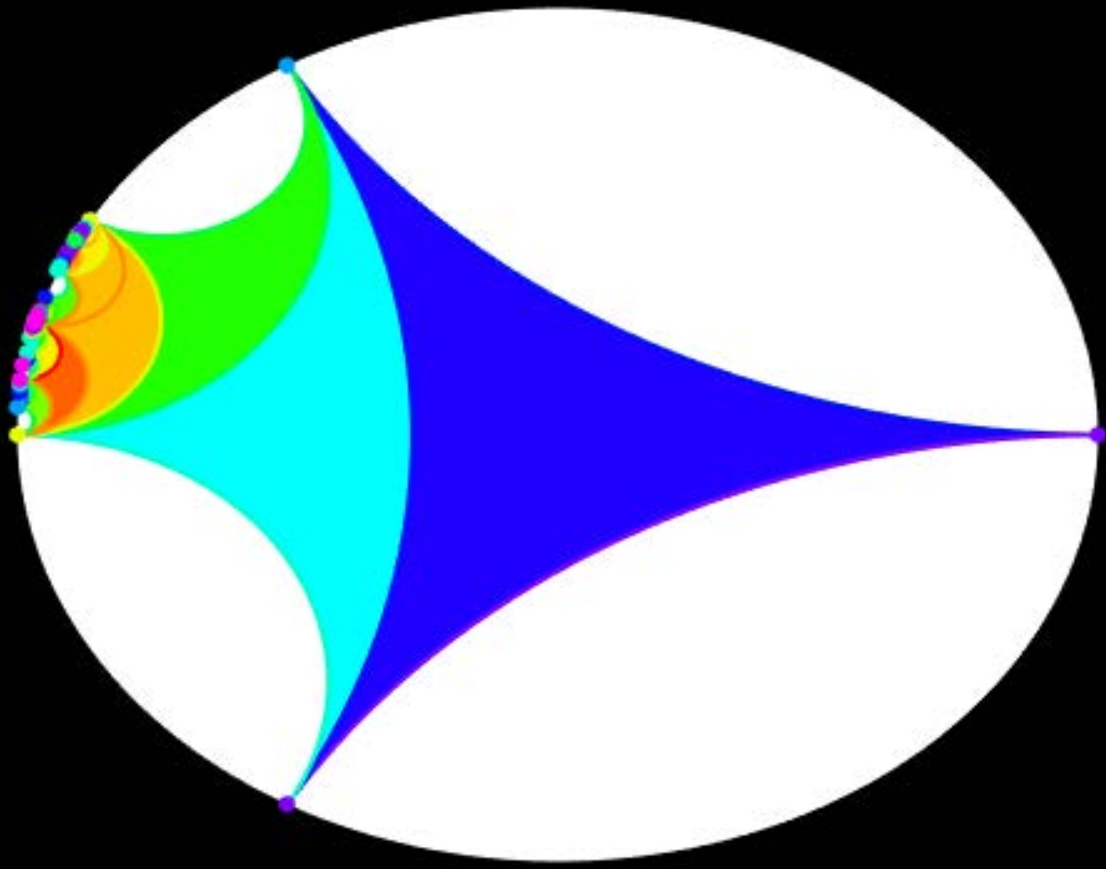
In NGF with s=-1 and d=3
also called
Complex Quantum Network Manifolds
the average of the generalized degree follow
the Fermi-Dirac, Boltzmann and Bose-Einstein distribution
respectively for
triangular faces, links and nodes

Emergent geometry at high temperature



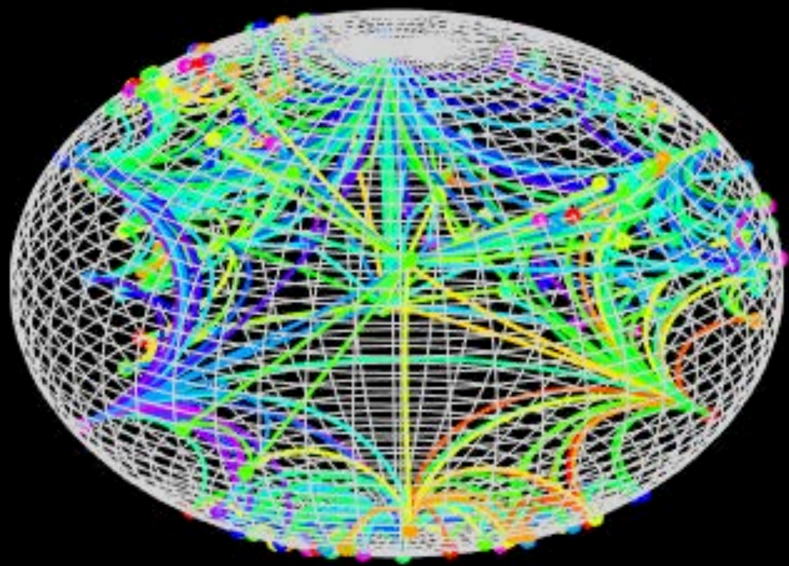
$s=-1$
$d=2$
$\beta=0.01$

Emergent geometry at low temperature



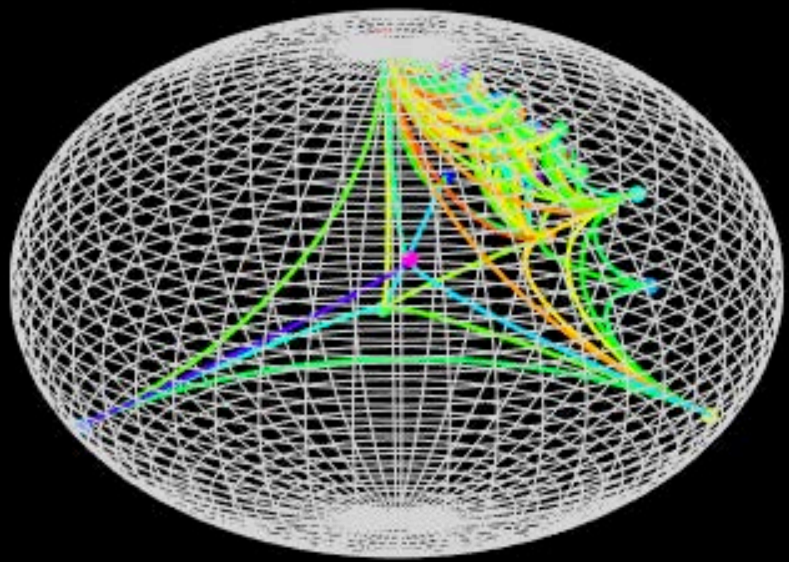
s=-1
d=2
β=5

Emergent geometry at high temperature

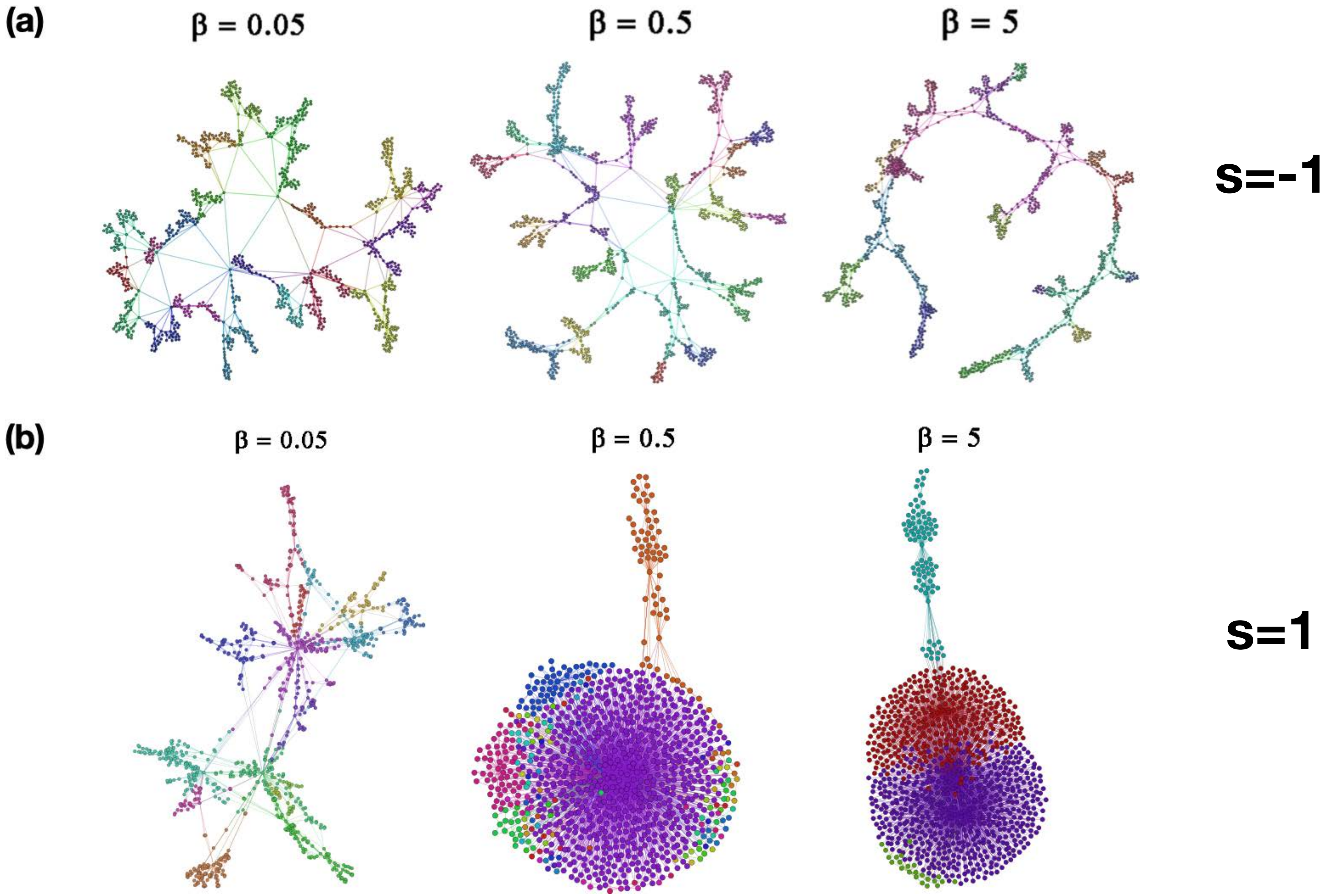


s=-1
d=3
β=0.01

Emergent geometry at low temperature

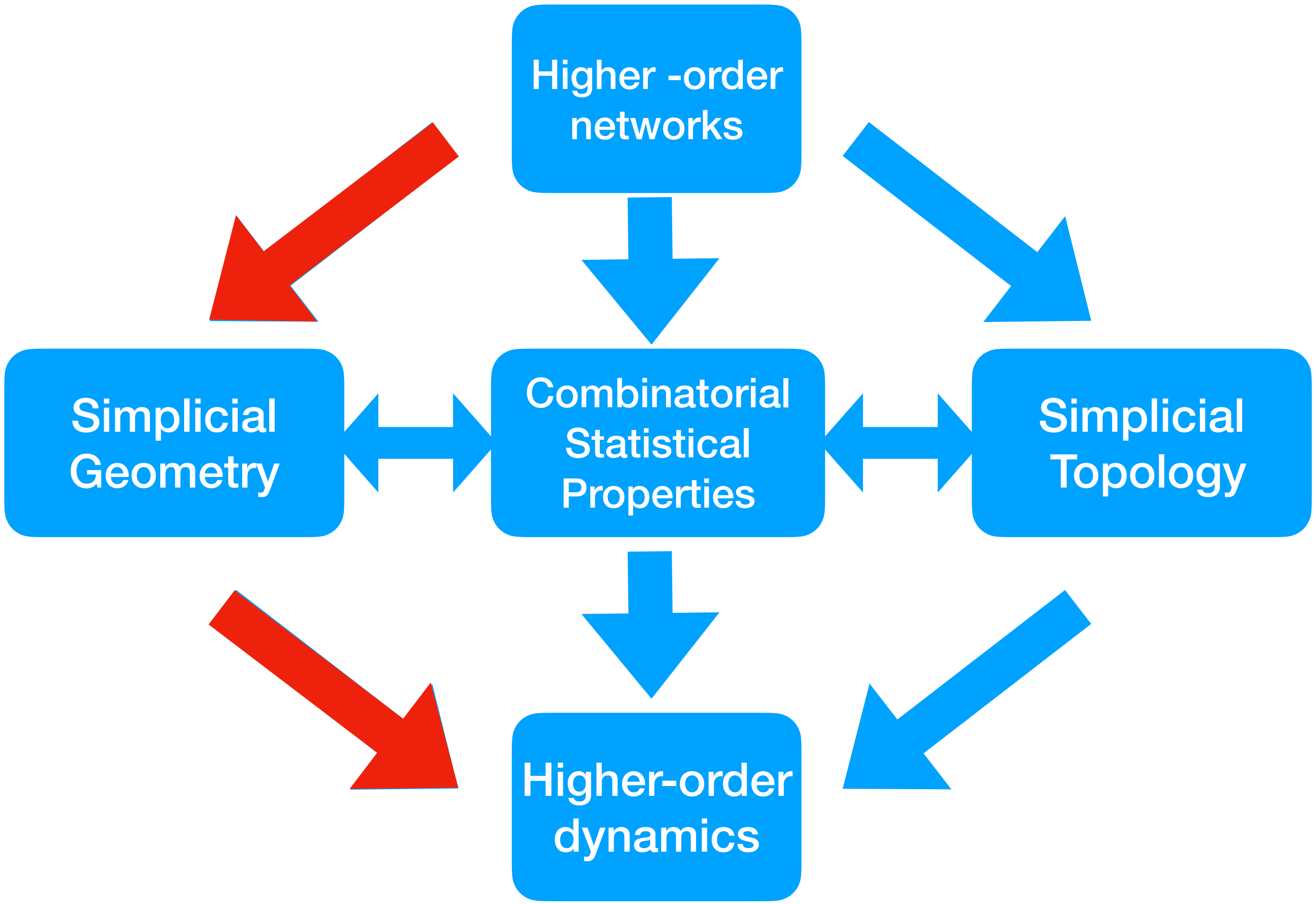


$s=-1$
$d=3$
$\beta=5$

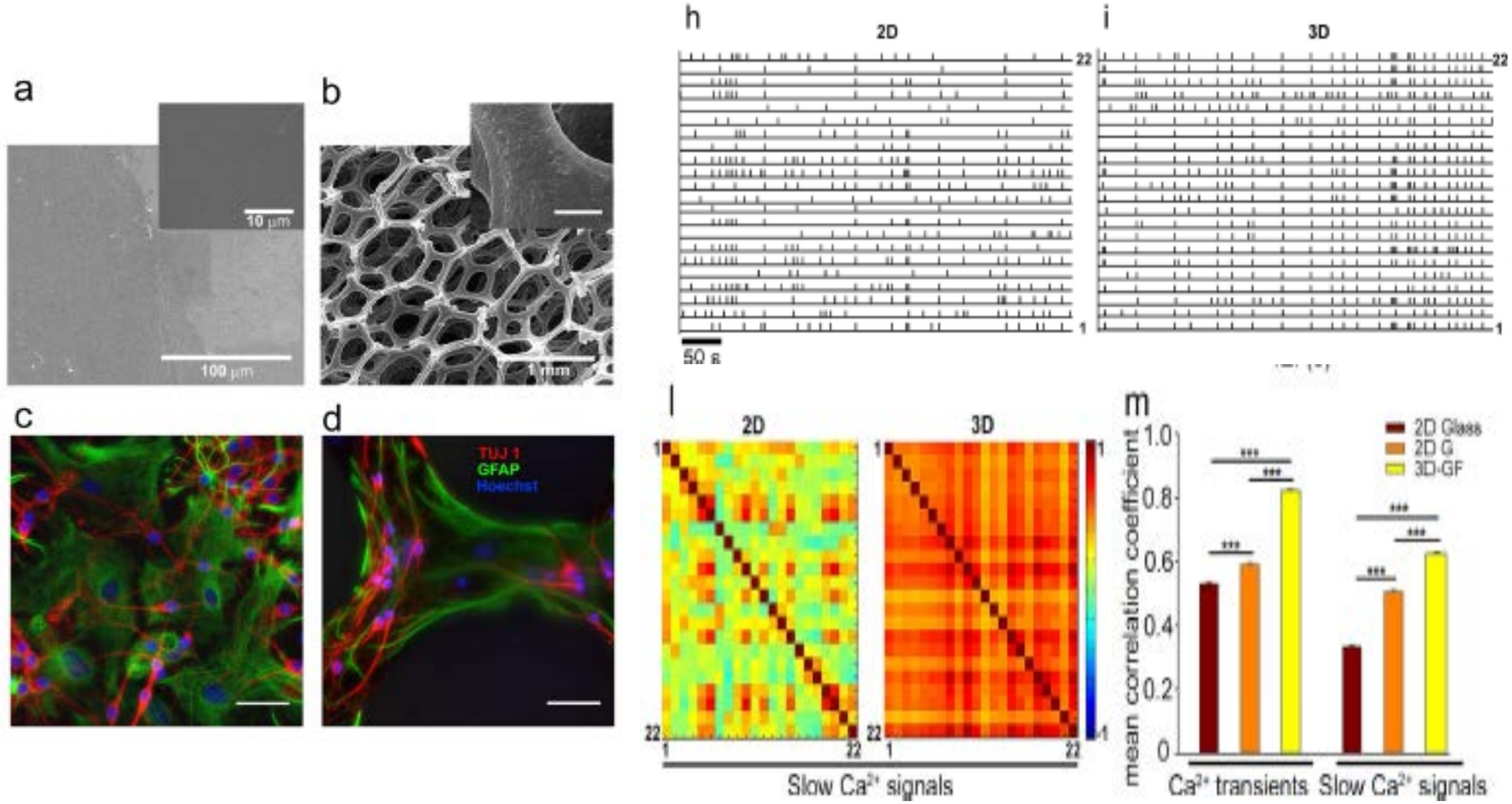


(a)
β = 0.05
β = 0.5
β = 5
s=-1
(b)
β = 0.05
β = 0.5
β = 5
s=1

Higher-order structure and dynamics

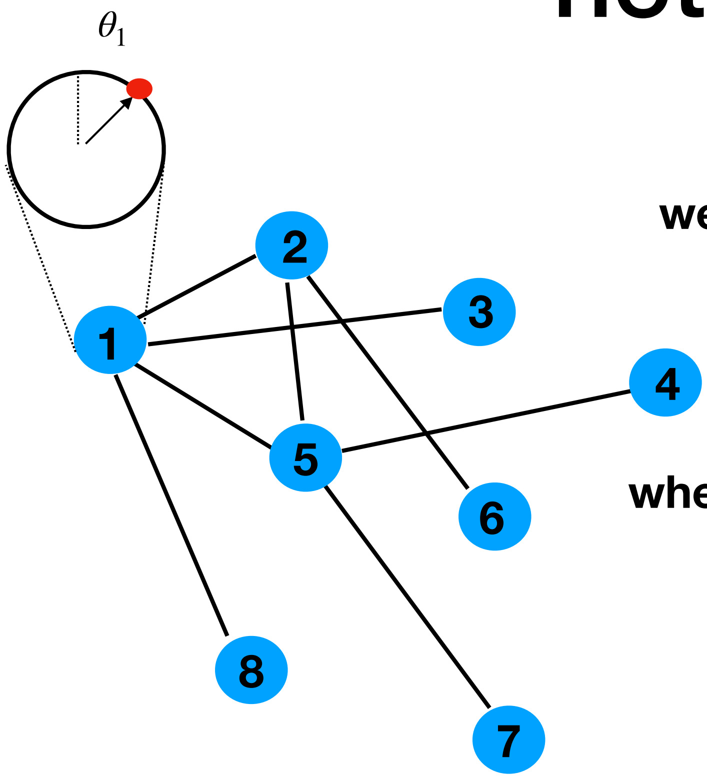


The role of dimensionality in neuronal dynamics



Uloa Severino et al. Scientific Reports (2016)

Kuramoto model on a network



Given a network of N nodes defined by an adjacency matrix a we assign to each node a phase obeying

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j - \theta_i\right)$$

where the internal frequencies of the nodes are drawn randomly from

$$\omega \sim \mathcal{N}(\Omega, 1)$$

and the coupling constant is σ

Order parameter for synchronization

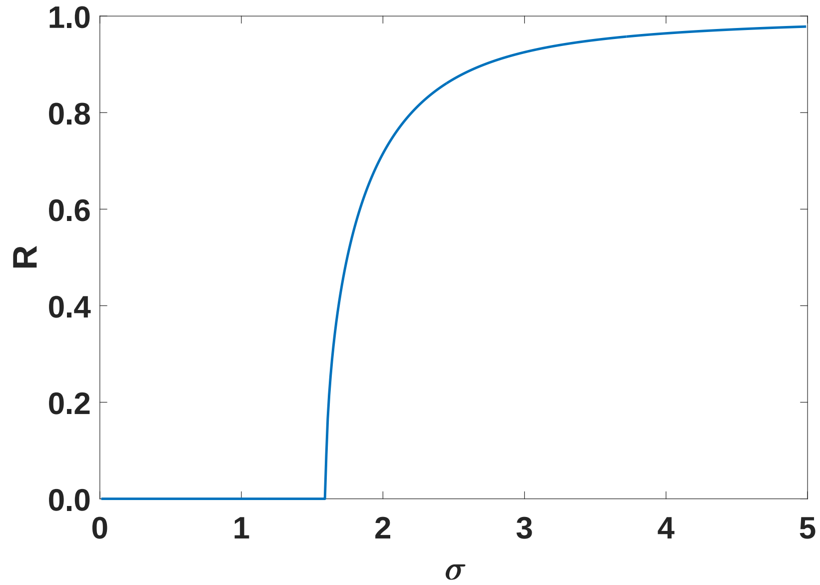
We consider the global order parameter R

$$R = \frac{1}{N}\left|\sum_{i=1}^{N} e^{\mathrm{i}\theta_i}\right|$$

which indicates the

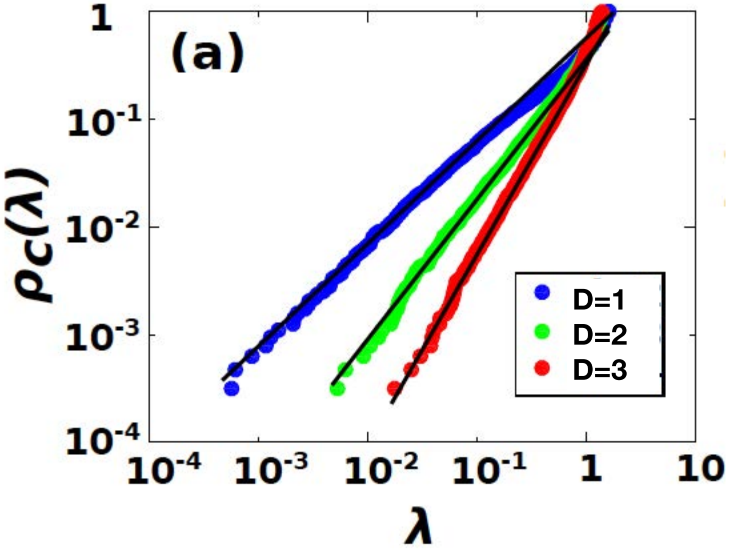
synchronisation transition

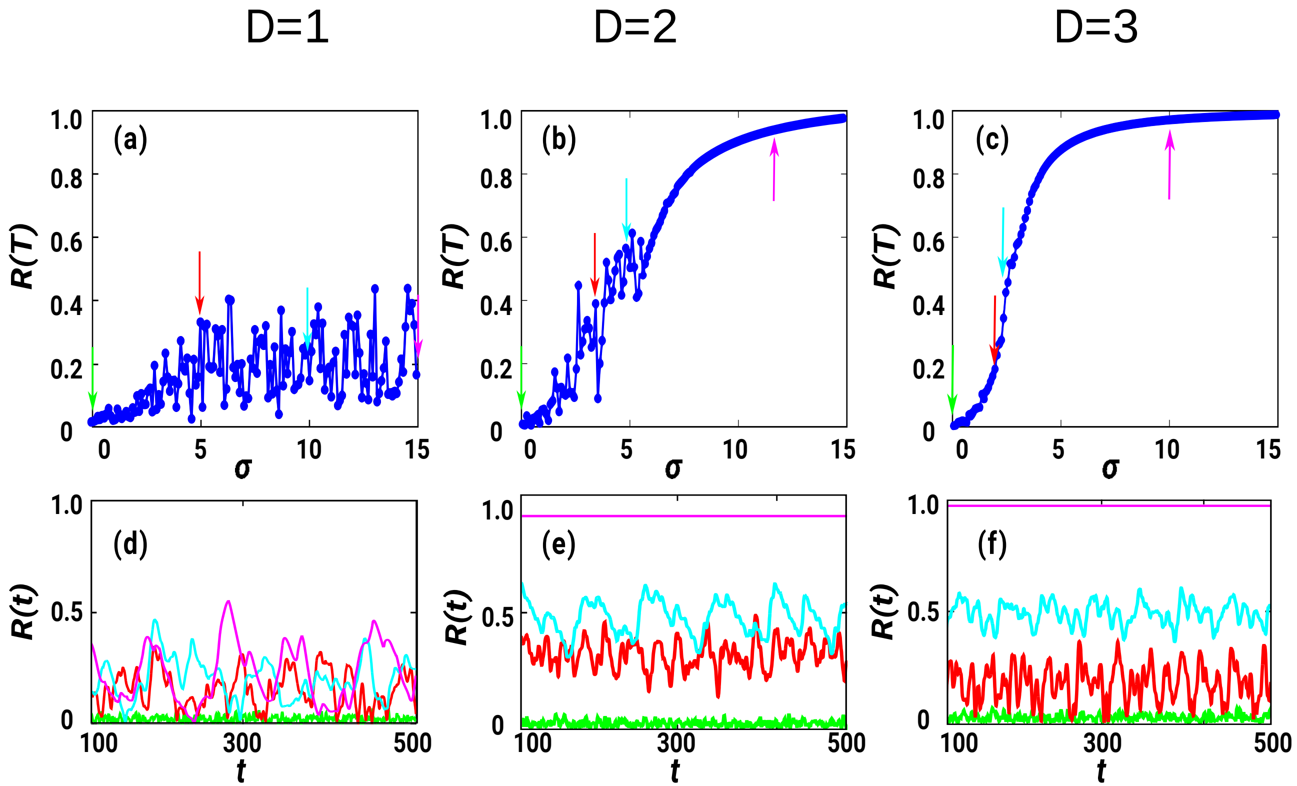
$$\begin{array}{ll} R \simeq 0 & \textbf{for } \sigma < \sigma_c \\ R \textbf{ finite} & \textbf{for } \sigma \geq \sigma_c \end{array}$$



Spectral dimension of geometric networks and synchronisation

NGFs spectral density





Millan et al. Sci. Rep. (2018); Millan et al. PRE (2019)

A geometric network displays a spectral dimension if the density of eigenvalues of the Graph Laplacian scales as

$$\rho(\lambda) \sim \lambda^{d_S/2-1} \text{ for } \lambda \ll 1$$

We consider the cumulative density of eigenvalues

$$\rho_c(\lambda) \sim \lambda^{d_S/2} \text{ for } \lambda \ll 1$$

Conclusions

- **Non-equilibrium models of simplicial complex is a fundamental approach to address the problem of emergent geometry and emergent community structure**

- **NGF display statistical properties depending on the dimension of the faces that are considered**

- **NGF display a dependence of their spectral dimension with the nature and dimension of the building block from which they are formed**

- **NGF provide an ideal tool to study the interplay between network geometry and dynamics**

References and collaborators

Topological characterisation of node neighbourhoods

Kartun-Giles, A.P. and Bianconi, G., 2019. Beyond the clustering coefficient: A topological analysis of node neighbourhoods in complex networks. *Chaos, Solitons & Fractals: X*, *1*, p.100004.

Simplicial complex models

1. **Courtney, O.T. and Bianconi, G., 2016. Generalized network structures: The configuration model and the canonical ensemble of simplicial complexes. *Physical Review E*, *93*(6), p.062311.**
2. **Z. Wu, G. Menichetti, C. Rahmede and G. Bianconi *Scientific Reports* 5, 10073 (2015).**
3. **Bianconi, G. and Rahmede, C., 2017. Emergent hyperbolic network geometry. *Scientific Reports*, *7*, p.41974.**
4. **Bianconi, G. and Rahmede, C., 2016. Network geometry with flavor: from complexity to quantum geometry. *Physical Review E*, *93*, p.032315.**
5. **Mulder, D. and Bianconi, G., 2018. Network geometry and complexity. *Jour. Stat. Phys.*, *173*, pp.783-805.**
6. **Courtney OT, Bianconi G. Weighted growing simplicial complexes. *Physical Review E* 2017, 95:062301.**

CODES repository :

•GitHub page : https://github.com/ginestrab (G. Bianconi)

Network Geometry with Flavor

Consider pure cell complexes formed by gluing identical regular polytopes along d-1 faces

- Starting from a single d-dimensional regular polytope

(1) GROWTH :

At every timestep we add a new d-dimension polytope glued to an existing (d-1)-face).

(2) ATTACHMENT:

The probability that the new polytope will be connected to a face α depends on the flavor s=-1,0,1 and is given by

$$\Pi_{\alpha}^{[s]} = \frac{(1 + sn_{\alpha})}{\sum_{\alpha'}(1 + sn_{\alpha'})}$$

Combinatorial properties of simplicial complexes

Configuration model of simplicial complexes

For background on Maximum Entropy Models of Networks see LTTC Course https://www.youtube.com/channel/UCsHAVdCU5XLaBYDXoINYZvg

Entropy of ensembles of simplicial complexes

To every simplicial complex we associate a probability

$$P(\mathscr{K})$$

The entropy of the ensemble of simplicial complexes is given by

$$S = -\sum_{\mathscr{K}} P(\mathscr{K}) \ln P(\mathscr{K})$$

Constraints

We might consider simplicial complex ensemble
with given
Expected generalized degrees of the nodes
or
Given generalized degrees of the nodes

Soft constraints

$$\sum_{\mathscr{K}} P(\mathscr{K}) \left[\sum_{\alpha \supset i} a_\alpha \right] = \bar{k}_{d,0}(i)$$

Hard constraints

$$\sum_{\alpha \supset i} a_\alpha = k_{d,0}(i)$$

[Courtney & Bianconi (2015)]

Maximum entropy ensembles

The maximum entropy ensembles
of simplicial complexes
are caracterized by a probability measure given by

Soft constraints

$$P(\mathscr{K}) = \frac{1}{Z} e^{-\sum_i \lambda_i \sum_{\alpha \supset i} a_\alpha}$$

Hard constraints

$$P(\mathscr{K}) = \frac{1}{\mathcal{N}} \delta\left(k_{d,0}(i), \sum_{\alpha \supset i} a_\alpha\right)$$

[Courtney & Bianconi (2015)]

Marginal

The probability of having a simplex μ is given by

$$\boxed{p_\mu = \frac{e^{-\sum_{r\subset\mu}\lambda_r}}{1+e^{-\sum_{r\subset\mu}\lambda_r}}}$$

Where the Lagrangian multipliers are fixed by the constraint

$$\bar{k}_{d,0}(i) = \sum_{\alpha\supset i} p_\alpha = \sum_{\alpha\supset i} \frac{e^{-\sum_{r\subset\alpha}\lambda_r}}{1+e^{-\sum_{r\subset\alpha}\lambda_r}}$$

[Courtney & Bianconi (2015)]

Structural cutoff

The simplified formula for p_μ

$$\boxed{p_\mu = d!\frac{\prod_{r\subset\mu} k_{d,0}(r)}{\left(\langle k_{d,0}(r)\rangle N\right)^d}}$$

is valid in presence of the structural cutoff

$$k_{d,0}(r) < K \textbf{ with } K = \left(\frac{\langle k_{d,0}(r)\rangle N}{d!}\right)^{1/(d+1)}$$

[Courtney & Bianconi (2015)]

Marginal probability

The marginal probability of a d-dimensional simplex μ is given by

$$p_\mu = \frac{e^{-\sum_{r\subset\mu}\lambda_r}}{1+e^{-\sum_{r\subset\mu}\lambda_r}}$$

In presence of a maximum degree K (the structural cutoff) the marginal can be written as

$$p_\mu = d!\frac{\prod_{r\subset\mu} k_{d,0}(r)}{\left(\langle k_{d,0}(r)\rangle N\right)^d}$$

where

$$K = \left[\frac{(\langle k_{d,0}(r)\rangle N)^d}{d!}\right]^{1/(d+1)}$$

[Courtney & Bianconi (2015)]

Case d=1

The marginal probability of a 1-dimensional simplex μ is given by

$$p_{ij} = \frac{e^{-\lambda_i - \lambda_j}}{1 + e^{-\lambda_i - \lambda_j}}$$

In presence of a maximum degree K (the structural cutoff) the marginal can be written as

$$p_{ij} = \frac{k_{d,0}(i) k_{d,0}(j)}{\left(\langle k_{d,0}(r) \rangle N\right)}$$ **where** $$K = \left[(\langle k_{d,0}(r) \rangle N) \right]^{1/2}$$

[Courtney & Bianconi (2015)]

Case d=2

The marginal probability of a 2-dimensional simplex μ is given by

$$p_{ijr} = \frac{e^{-\lambda_i - \lambda_j - \lambda_r}}{1 + e^{-\lambda_i - \lambda_j - \lambda_r}}$$

In presence of a maximum degree K (the structural cutoff) the marginal can be written as

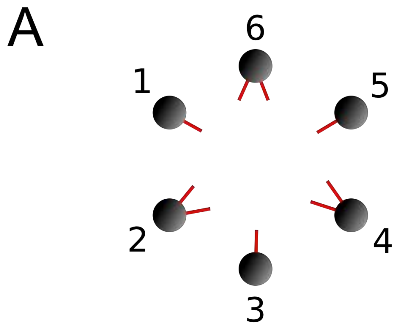
$$p_{ijr} = 2\frac{k_{d,0}(i)k_{d,0}(j)k_{d,0}(r)}{\left(\langle k_{d,0}(r)\rangle N\right)^2}$$

where

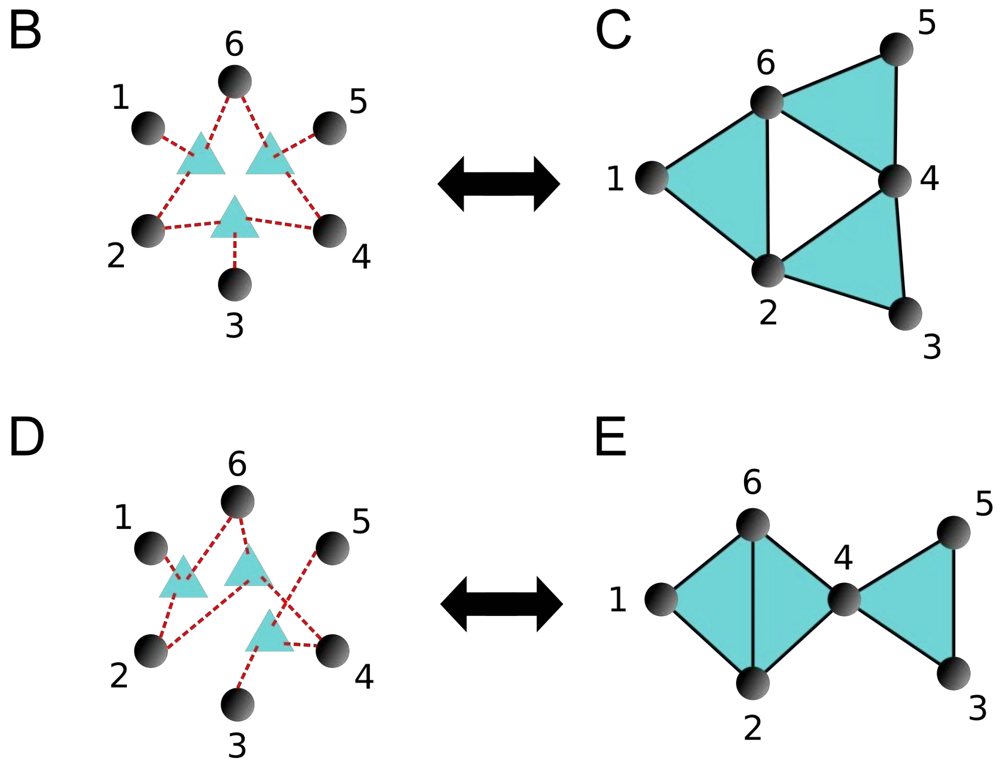
$$K = \frac{(\langle k_{d,0}(r)\rangle N)^{2/3}}{2^{1/3}}$$

[Courtney & Bianconi (2015)]

Ensemble of simplicial complexes



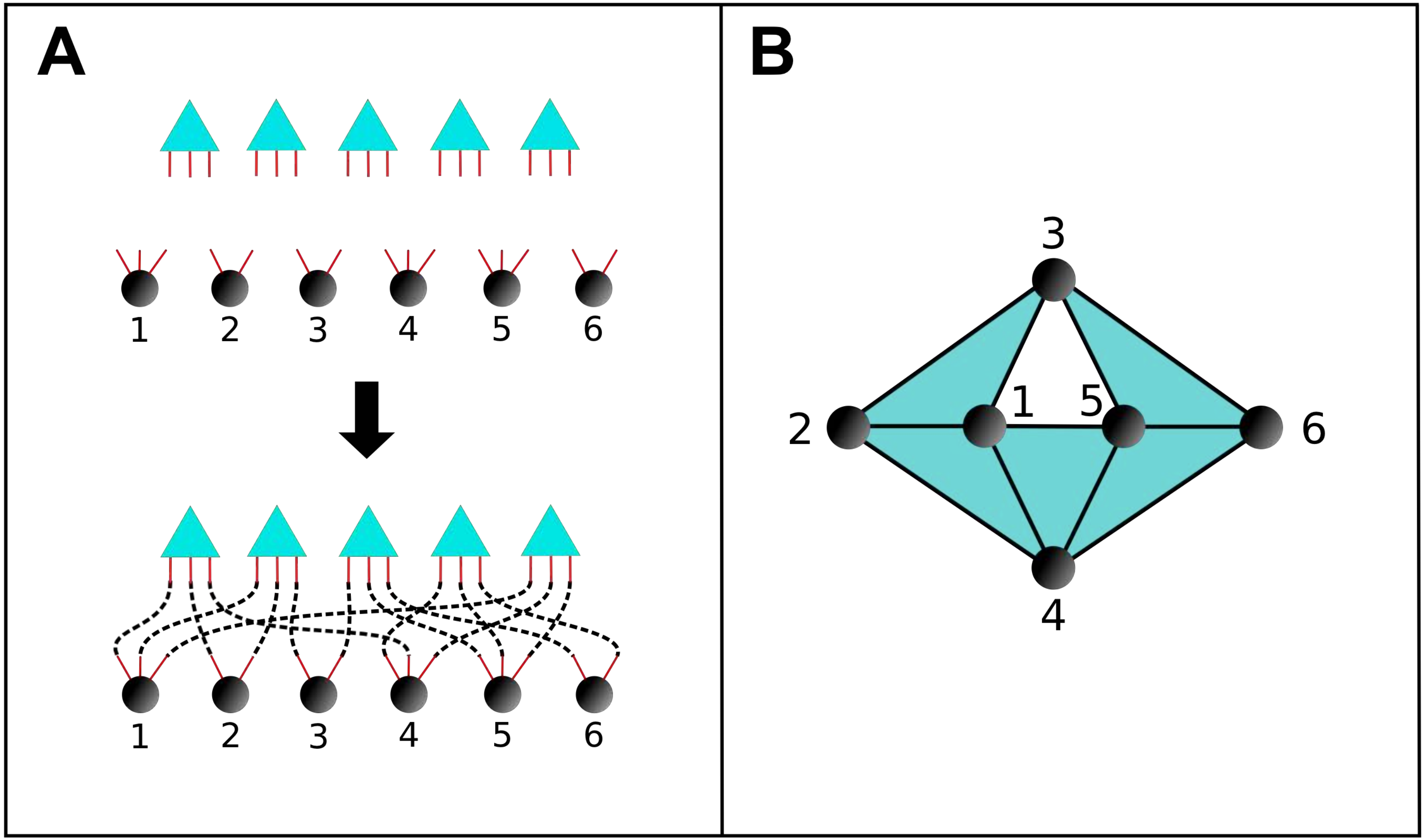
We consider an ensemble of
pure simplicial complexes
formed by d-dimensional simplicies and their faces
where each node has given generalized degree



- Given the generalized degree of the nodes there are in general multiple ways to realize the simplicial complex.
- The information encoded in the constraints is captured by the entropy of the ensemble

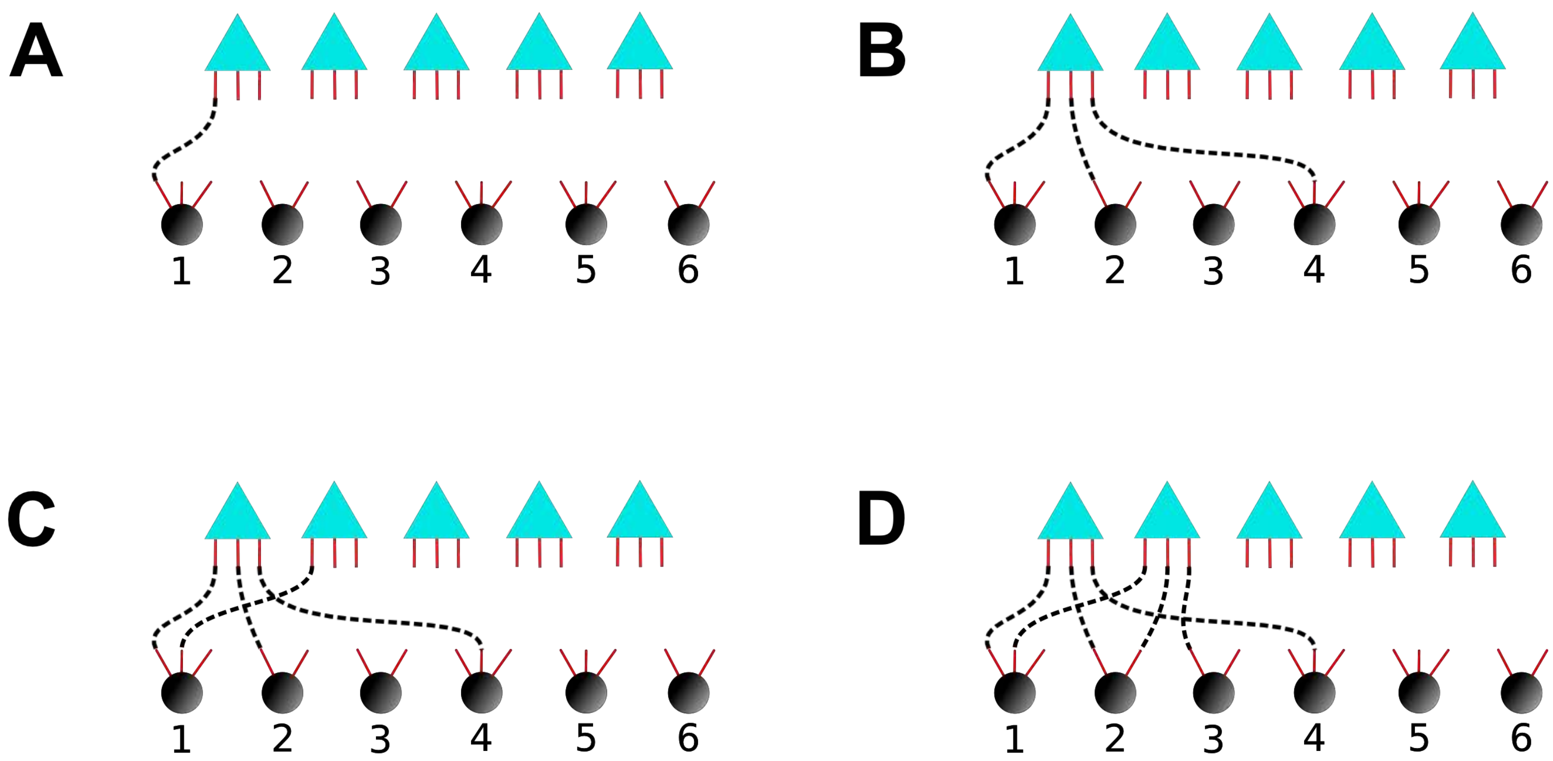
[Courtney & Bianconi (2015)]

Construction of a random simplicial complex

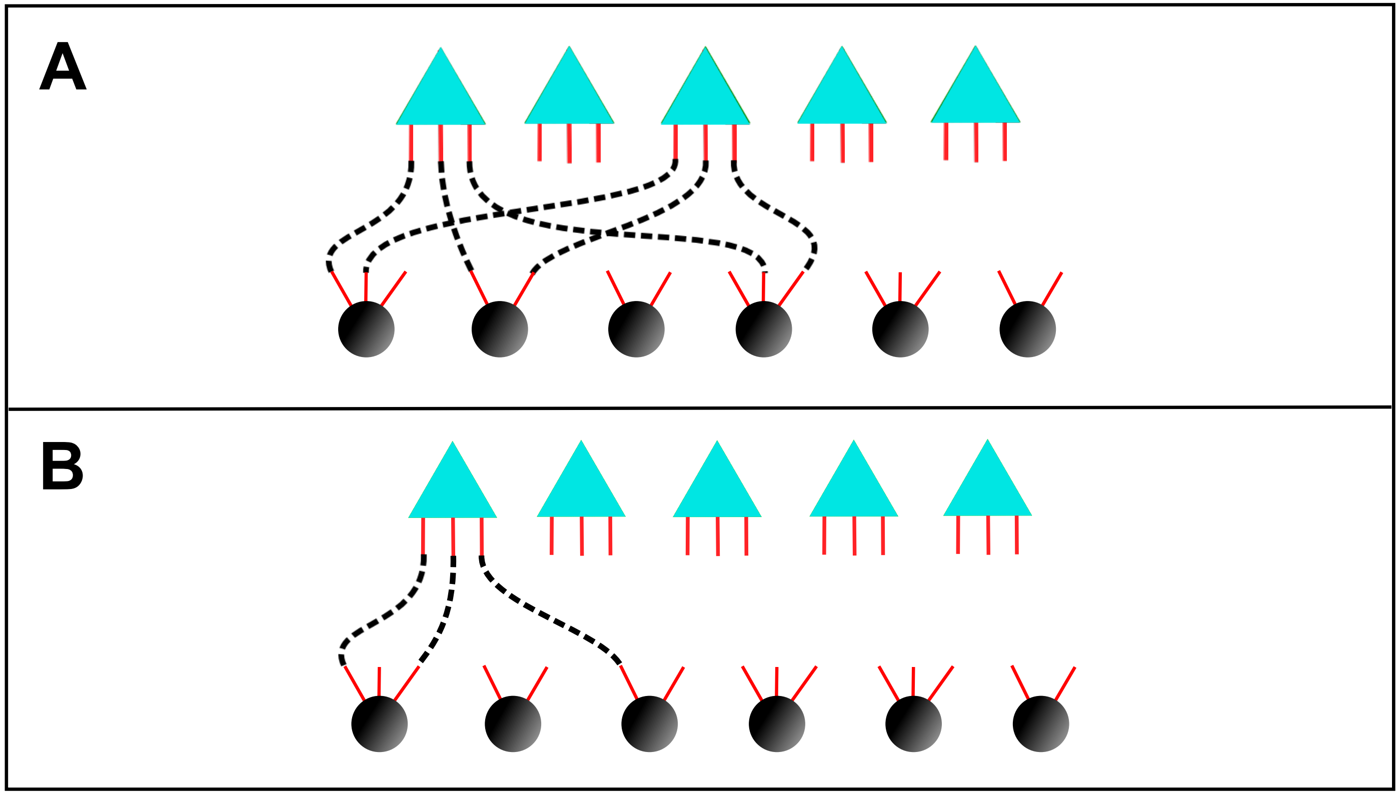


CODE AVAILABLE AT GITHUB **ginestrab**

Matching of the stubs



Illegal matchings



Entropy of network ensembles

Entropy of a canonical network ensemble with expected generalized degree sequence

$$S = -\sum_{\mu \in S_d(N)} \left[p_\mu \ln p_\mu + (1-p_\mu)\ln(1-p_\mu) \right]$$

Entropy of a microcanonical network ensemble with given generalized degree sequence of the nodes is given by

$$\Sigma = \ln \mathcal{N} = S - \Omega \qquad \Omega = -\sum_{r=1}^{N} \ln \frac{1}{k_{d,0}(r)!} (k_{d,0}(r))^{k_{d,0}(r)} e^{-k_{d,0}(r)}$$

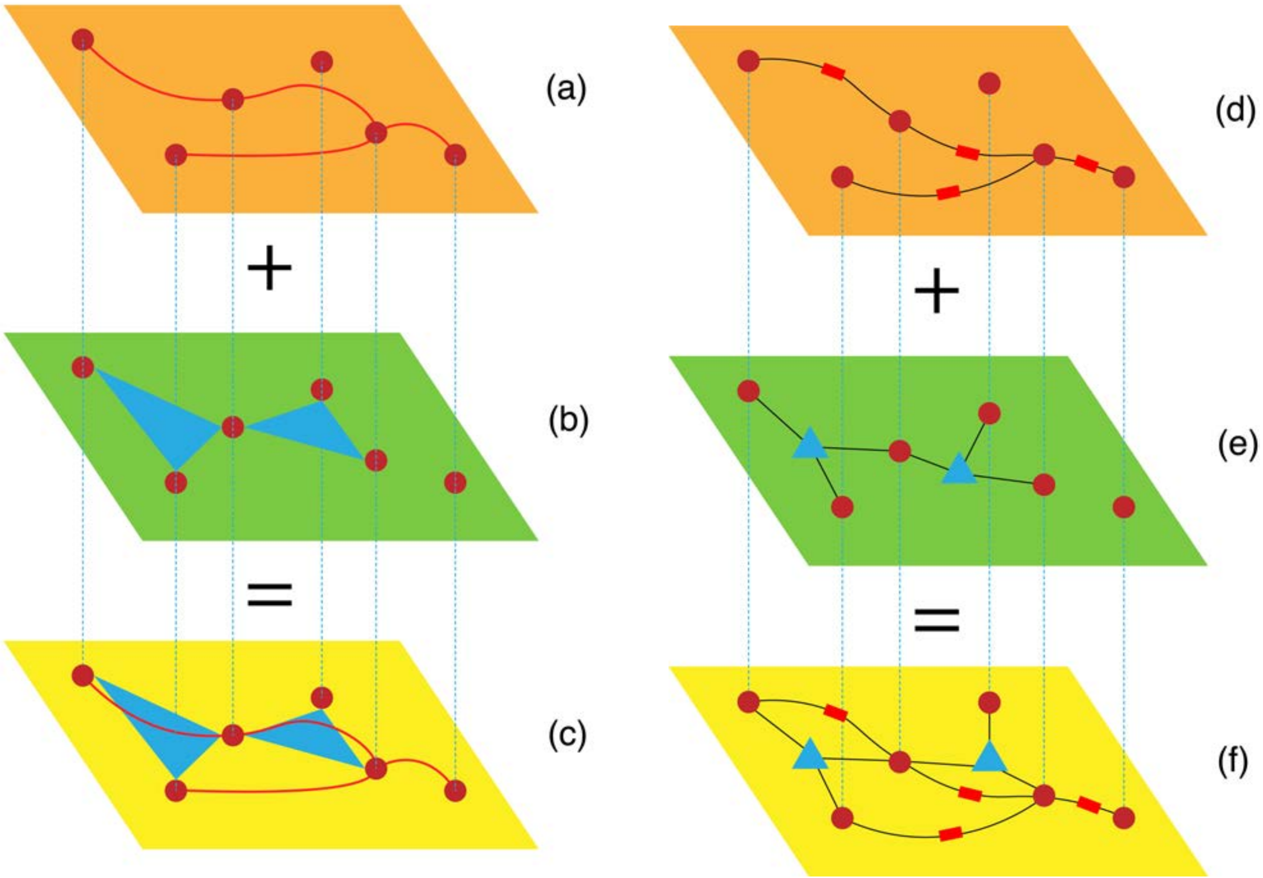
Where $\mathcal{N}$ is the total number of simplicial complexes in the ensemble

Asymptotic expression for the number of simplicial complexes with given generalized degrees of the nodes

$$\mathcal{N} \sim \frac{\left[(\langle k\rangle N)!\right]^{d(d+1)}}{\prod_{r=0}^{N} k_{d,0}(r)!} \frac{1}{(d!)^{\langle k\rangle N/(d+1)}} \exp\left(-\frac{d!}{2(d+1)(\langle k\rangle N)^{d-1}}\left(\frac{\langle k^2\rangle}{\langle k\rangle}\right)^{d+1}\right)$$

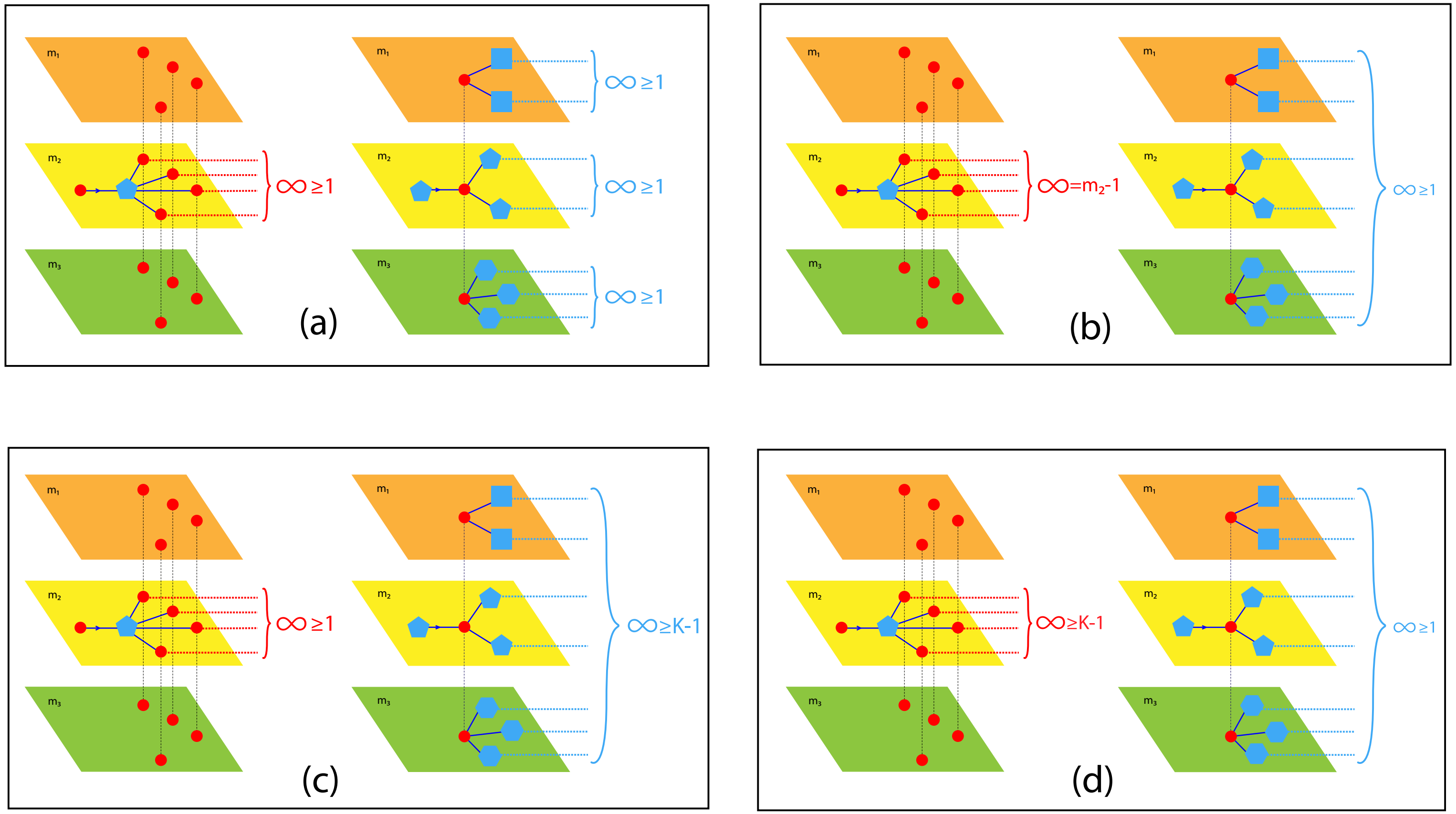
[Courtney & Bianconi (2015)]

From model of pure simplicial complexes to multiplex hypergraph



[Sun & Bianconi (2021)]

Multiplex hypergraphs can sustain non-trivial cooperative processes leading to discontinuous transitions



[Sun & Bianconi (2021)]