Higher-order networks

An introduction to simplicial complexes

Lesson III

Mathematics of Large Networks
Erdos Center, Budapest

29 May 2022

Ginestra Bianconi

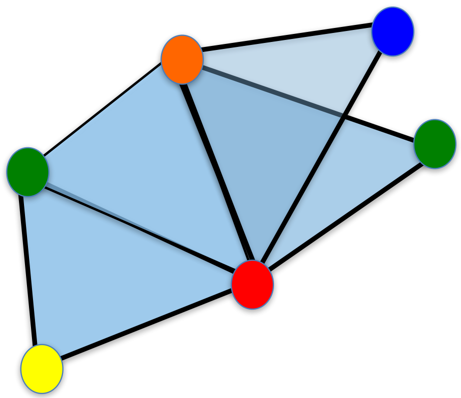
School of Mathematical Sciences, Queen Mary University of London
Alan Turing Institute

Queen Mary University of London

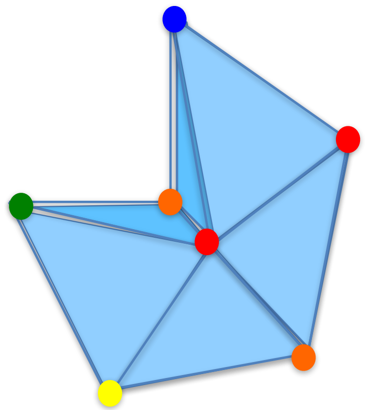
The Alan Turing Institute

Higher-order networks

Higher-order networks are characterising the interactions between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.



d=2 simplicial complex

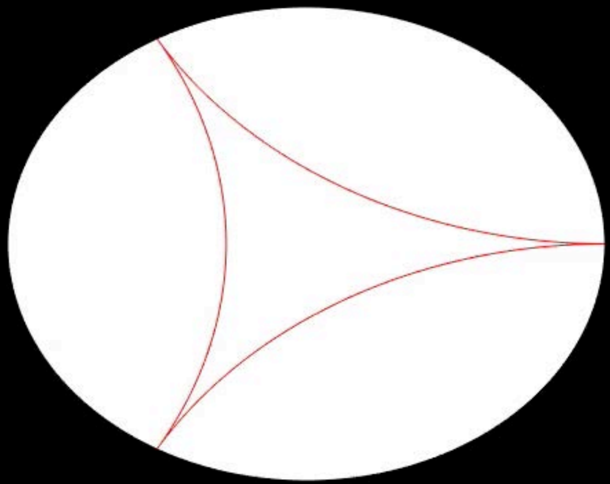


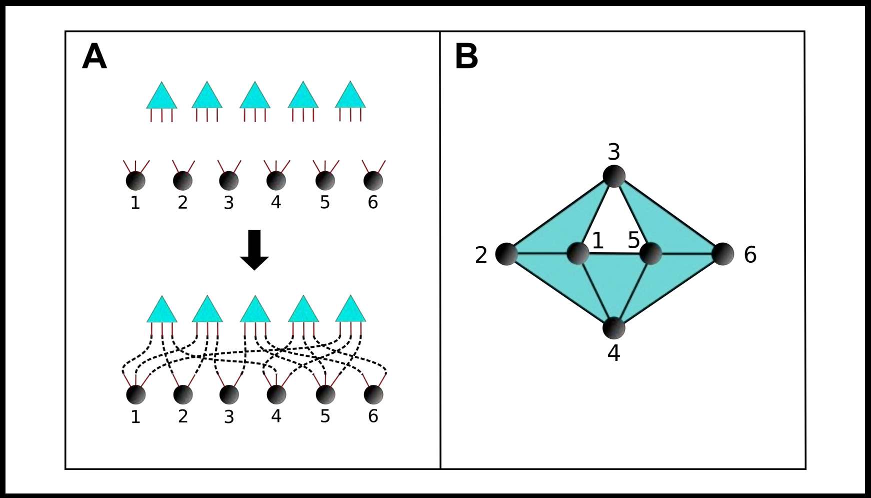
d=3 simplicial complex

Simplicial complex models

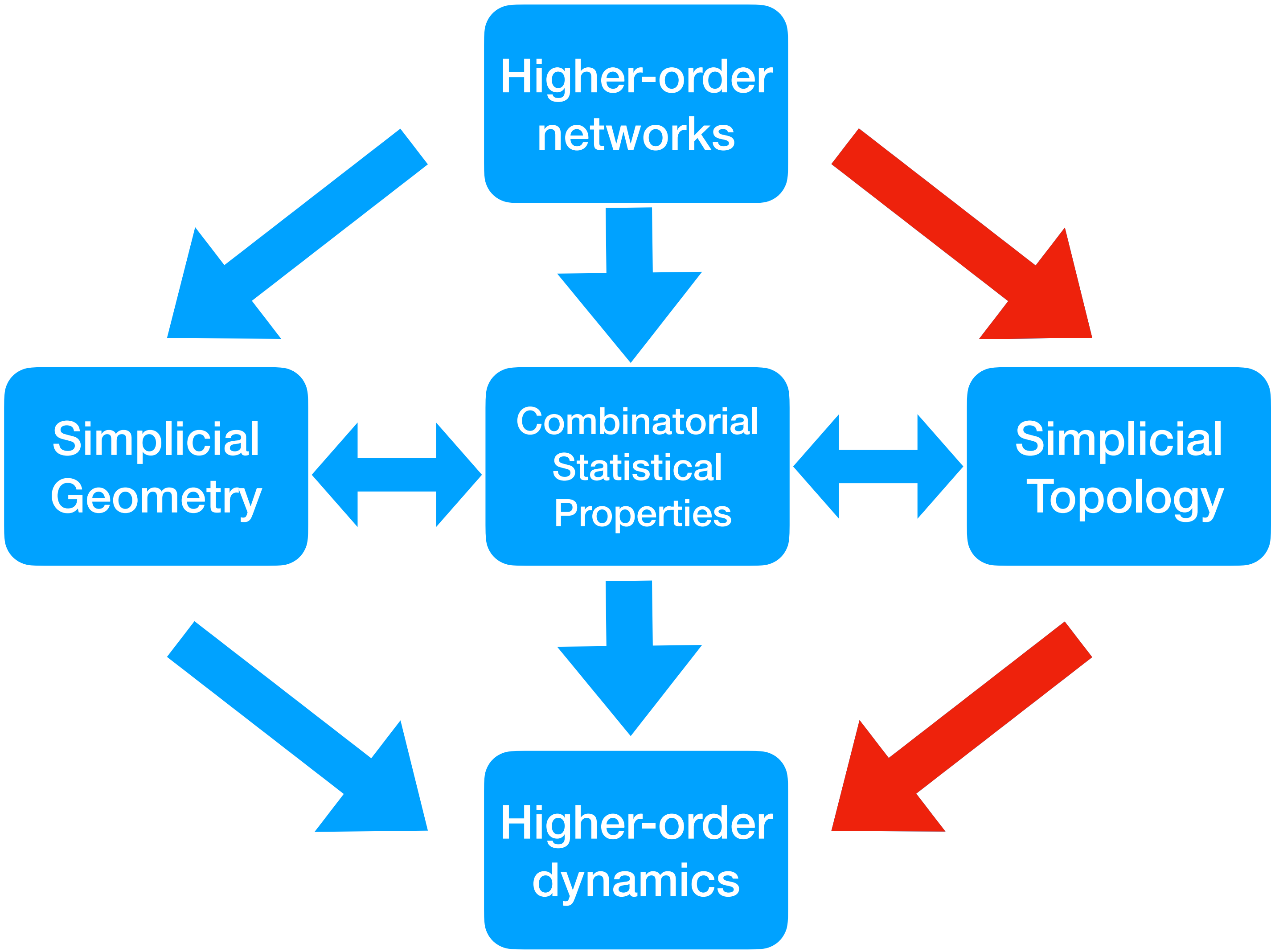
Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





Higher-order structure and dynamics



Lesson II:
Topology and higher-order dynamics

- Introduction to algebraic topology
- Higher-order operators and their properties
  1. Topological signals
  2. The Hodge Laplacian and Hodge decomposition
  3. The Dirac operator
- Simplicial synchronisation and higher-order Kumamoto model

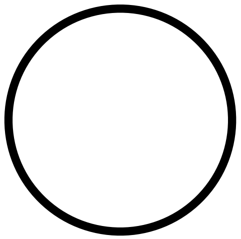
Introduction to Algebraic Topology

Betti numbers

Point

$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 0$

Circle

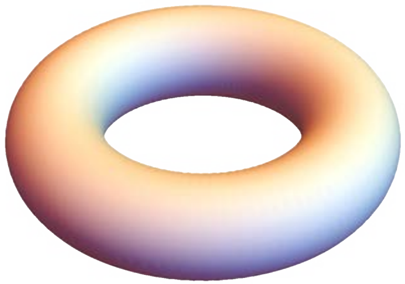


$\beta_0 = 1$
$\beta_1 = 1$
$\beta_2 = 0$

Sphere

$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 1$

Torus

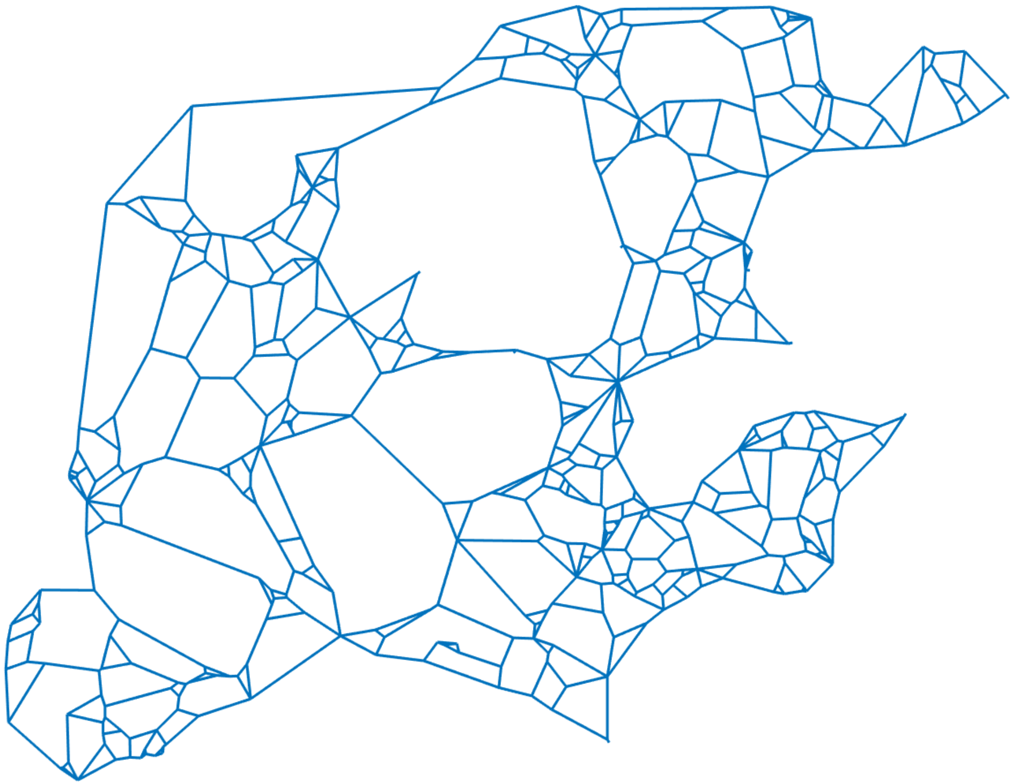


$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 1$

Euler characteristic

$$\chi = \sum_n (-1)^n \beta_n$$

Betti number 1



Fungi network from Sang Hoon Lee, et. al. Jour. Compl. Net. (2016)

Orientation of the simplex

A m-dimensional *oriented simplex* α is a set of $m+1$ nodes

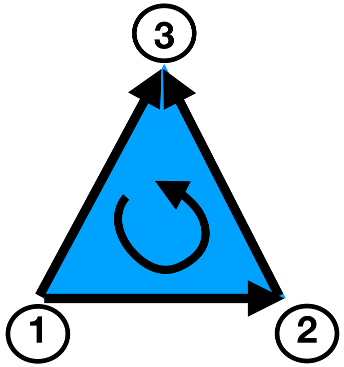
$$\alpha = [v_0, v_1, \dots, v_m], \tag{3.1}$$

associated to an orientation wuch that

$$[v_0, v_1, \dots, v_m] = (-1)^{\sigma(\pi)}[v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(m)}] \tag{3.2}$$

where $\sigma(\pi)$ indicates the parity of the permutation π.

Therefore we have



$$[i, j] = -[j, i]$$

$$[i, j, k] = [j, k, i] = [k, i, j] = -[j, i, k] = -[k, j, i] = -[i, k, j]$$

m-Chains

THE m-CHAINS

Given a simplicial complex, a m-chain C_m consists of the elements of a free abelian group with basis on the m-simplices of the simplicial complex.

Its elements can be represented as linear combinations of the of all oriented m-simplices

$$\alpha = [v_0, v_1, \ldots, v_m] \tag{3.6}$$

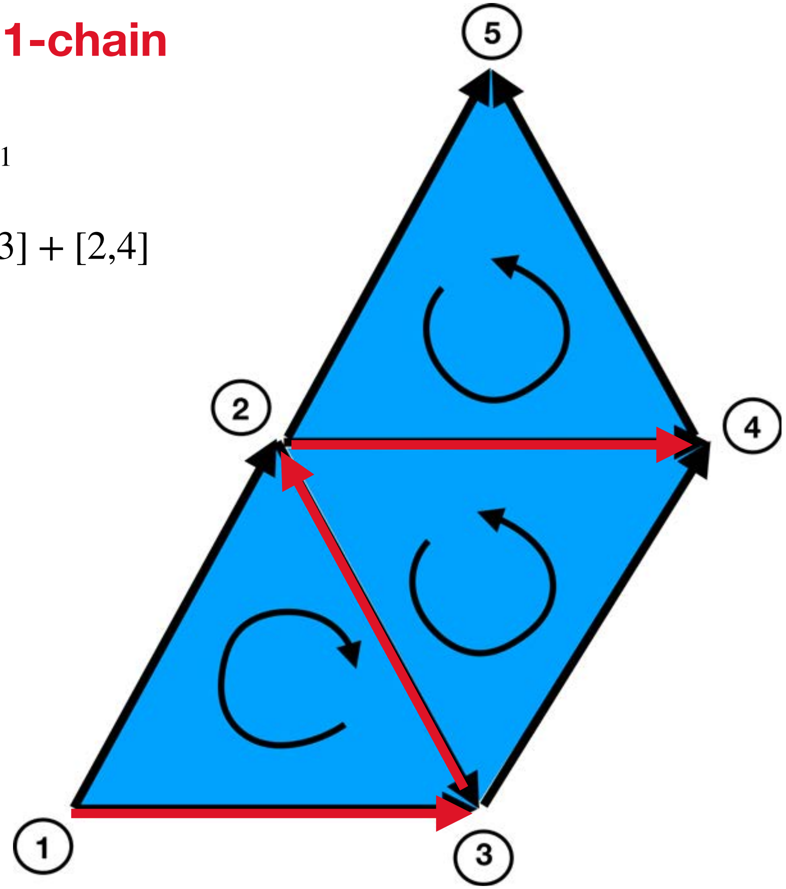
with coefficients in $\mathbb{Z}$.

Oriented simplicial complex and n-chains

Example of 1-chain

$$a \in \mathscr{C}_1$$

$$a = [1,3] - [2,3] + [2,4]$$



Boundary operator

THE BOUNDARY MAP

The boundary map ∂_m is a linear operator

$$\partial_m : C_m \to C_{m-1} \tag{3.8}$$

whose action is determined by the action on each m-simplex of the simplicial complex is given by

$$\partial_m[v_0, v_1 \dots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \dots, v_{p-1}, v_{p+1}, \dots, v_m]. \tag{3.9}$$

From this definition it follows that the $\mathrm{im}(\partial_m)$ corresponds to the space of $(m-1)$ boundaries and the $\ker(\partial_m)$ is formed by the cyclic m-chains.

Special groups

Boundary group $\hat{B}_m =$ **im**(∂_{m+1})
Cycle group $\hat{Z}_m =$ **ker**(∂_m)

Boundary operator

The boundary map ∂_n is a linear operator

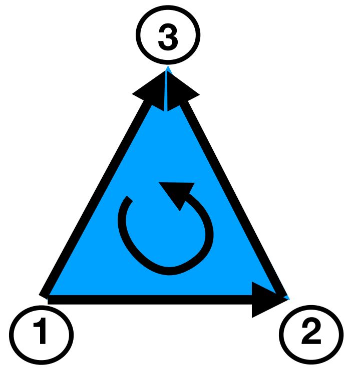
$$\partial_n : \mathscr{C}_n \to \mathscr{C}_{n-1}$$

whose action is determined by the action on each n-simplex of the simplicial complex

$$\boxed{\partial_n[i_0, i_1 \ldots, i_n] = \sum_{p=0}^{n} (-1)^p [i_0, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_n] .}$$

Therefore we have

$$\partial_1[1,2] = [2] - [1] .$$



$$\partial_2[1,2,3] = [2,3] - [1,3] + [1,2] .$$

The boundary of a boundary is null

The boundary operator has the property

$$\boxed{\partial_n \partial_{n+1} = 0 \quad \forall n \geq 1}$$

Which is usually indicated by saying that the boundary of the boundary is null.

This property follows directly from the definition of the boundary, as an example we have

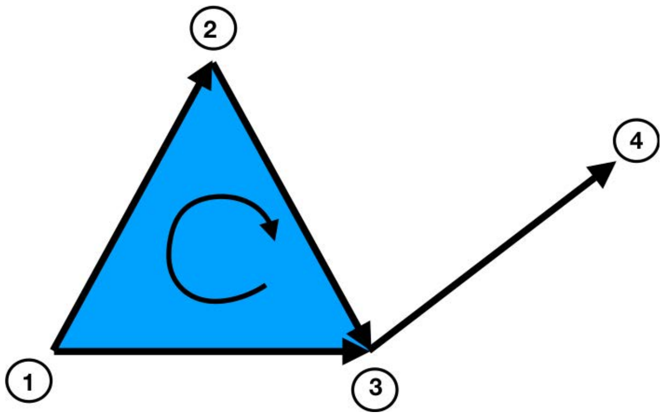
$$\partial_1 \partial_2 [i, j, k] = \partial_1([j, k] - [i, k] + [i, j]) = -[j] + [k] + [i] - [k] - [i] + [j] = 0.$$

Incidence matrices

Given a basis for the n simplices and n-1 simplices
the n-boundary operator

$$\partial_n[i_0, i_1 \ldots, i_n] = \sum_{p=0}^{n} (-1)^p [i_0, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_n] \,.$$

is captured by the incidence matrix $\mathbf{B}_{[n]}$



$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array} \,,$$

$$\mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array} \,.$$

Boundary of the boundary is null

In terms of the incidence matrices the relation

$$\partial_n \partial_{n+1} = 0 \quad \forall n \geq 1$$

Can be expressed as

$$\boxed{\mathbf{B}_{[n]}\mathbf{B}_{[n+1]} = \mathbf{0} \quad \forall n \geq 1 \qquad \mathbf{B}_{[n+1]}^{\top}\mathbf{B}_{[n]}^{\top} = \mathbf{0} \quad \forall n \geq 1}$$

Homology groups

THE HOMOLOGY GROUPS

The homology group $\mathcal{H}_m$ is the quotient space

$$\mathcal{H}_m = \frac{\ker(\partial_m)}{\mathrm{im}(\partial_{m+1})}, \tag{3.14}$$

denoting homology classes of m-cyclic chains that are in the $\ker(\partial_m)$ and they do differ by cyclic chains that are not boundaries of $(m+1)$-chains, i.e. they are in $\mathrm{im}(\partial_{m+1})$.

It follows that $a \in$ ker(∂_m) is in the same homology class than $a + b \in$ ker(∂_m) with $b \in$ im(∂_{m+1})

Betti numbers

BETTI NUMBERS

The Betti number β_m indicates the number of m-dimensional cavities of a simplicial complex and is given by the rank of the homology group $\mathcal{H}_m$, i.e.

$$\beta_m = \mathrm{rank}(\mathcal{H}_m) = \mathrm{rank}(\ker(\partial_m)) - \mathrm{rank}(\mathrm{im}(\partial_{m+1})). \tag{3.15}$$

Euler characteristic

THE EULER CHARACTERISTIC AND THE EULER-POINCARÉ FORMULA

The Euler characterisic χ is defined as the alternating sum of the number of m-dimensional simplices, i.e.

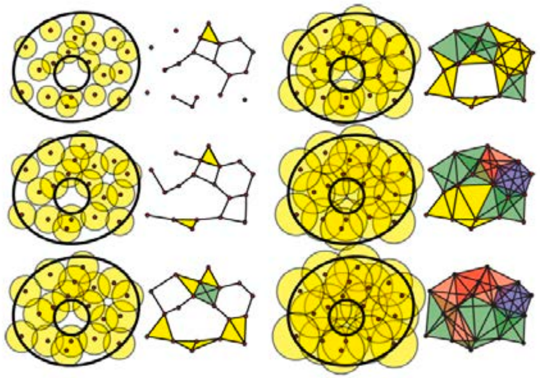
$$\chi = \sum_{m \geq 0} s_m, \tag{3.16}$$

where s_m is the number of m-dimensional simplices in the simplicial complex. According to the Euler-Poincaré formula, the Euler characteristic χ of a simplicial complex can be expressed in terms of the Betti numbers as

$$\chi = \sum_{m \geq 0} (-1)^m \beta_m. \tag{3.17}$$

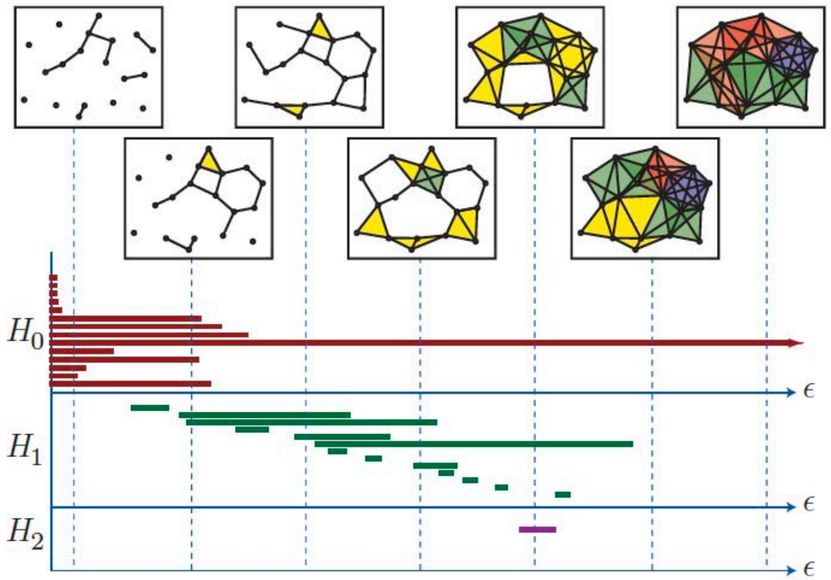
Persistent homology

Filtration: distance/weights



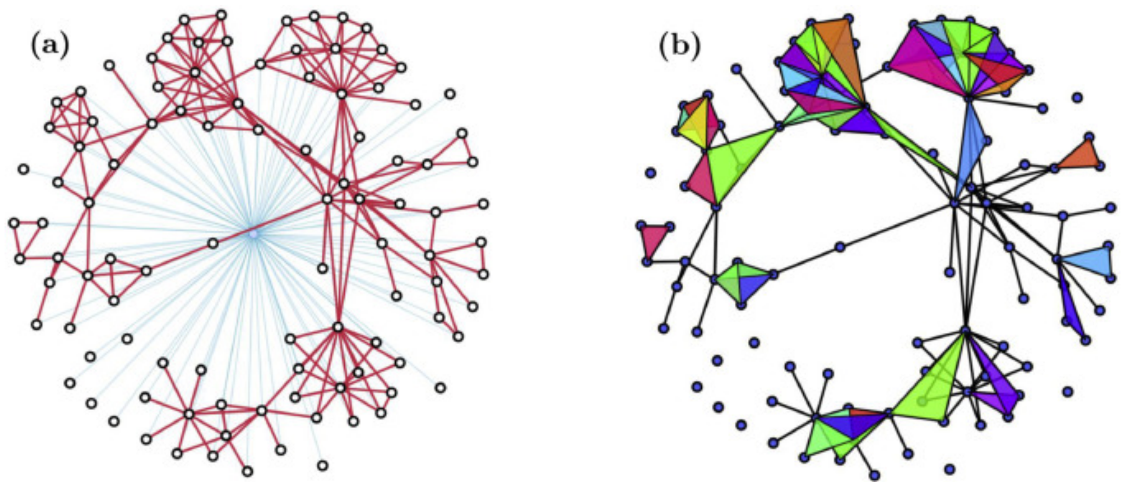
Ghrist 2008

Persistent homology Barcode



Topological clustering

The node neighbourhood is the clique simplicial complex formed by the set of all the neighbours of a node and their connections



Properties of the node

The degree k_i

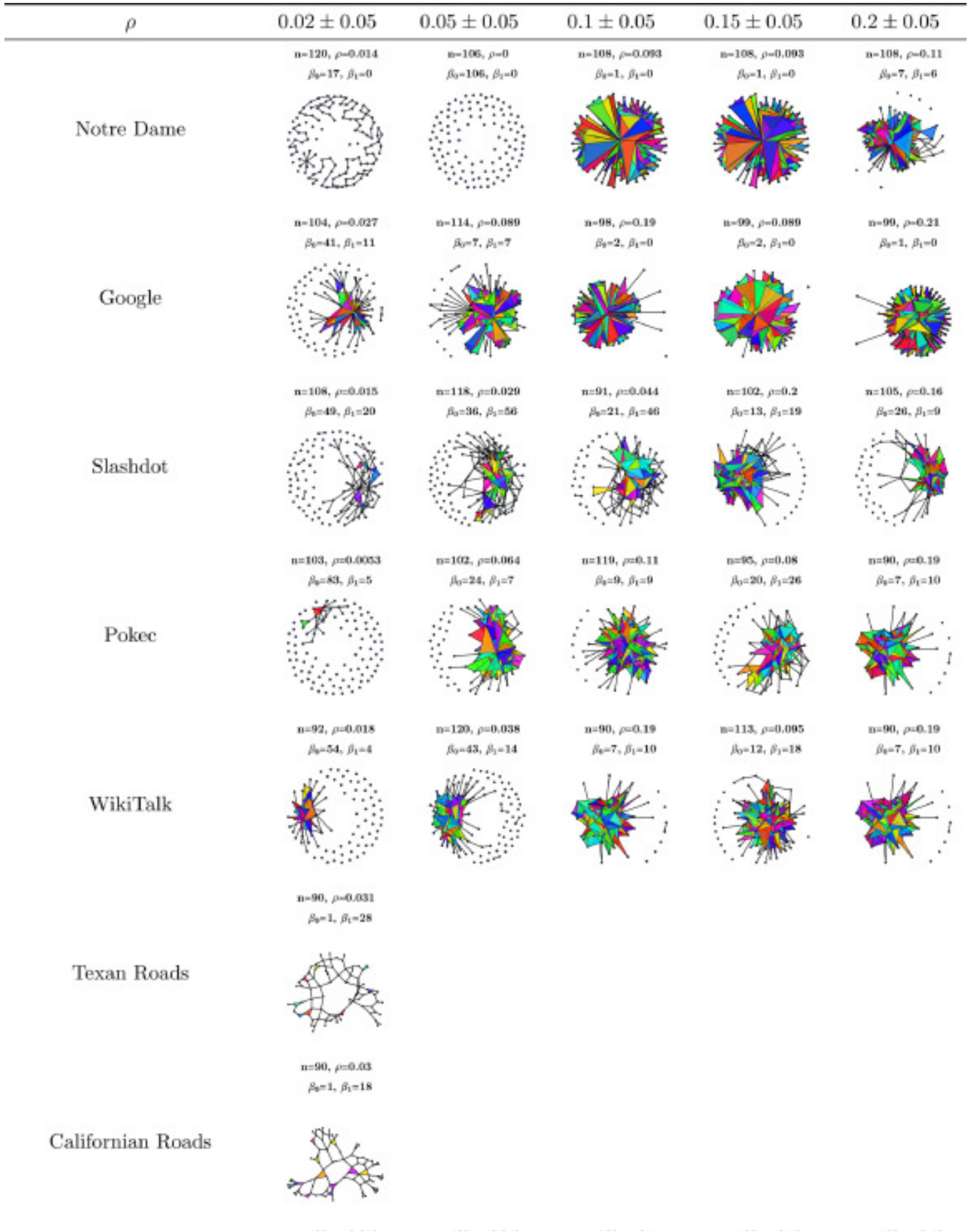
The local clustering coefficient C_i

AP Kartun-Giles et al. (2019)

Properties of the node neighbourhood

Number of nodes n

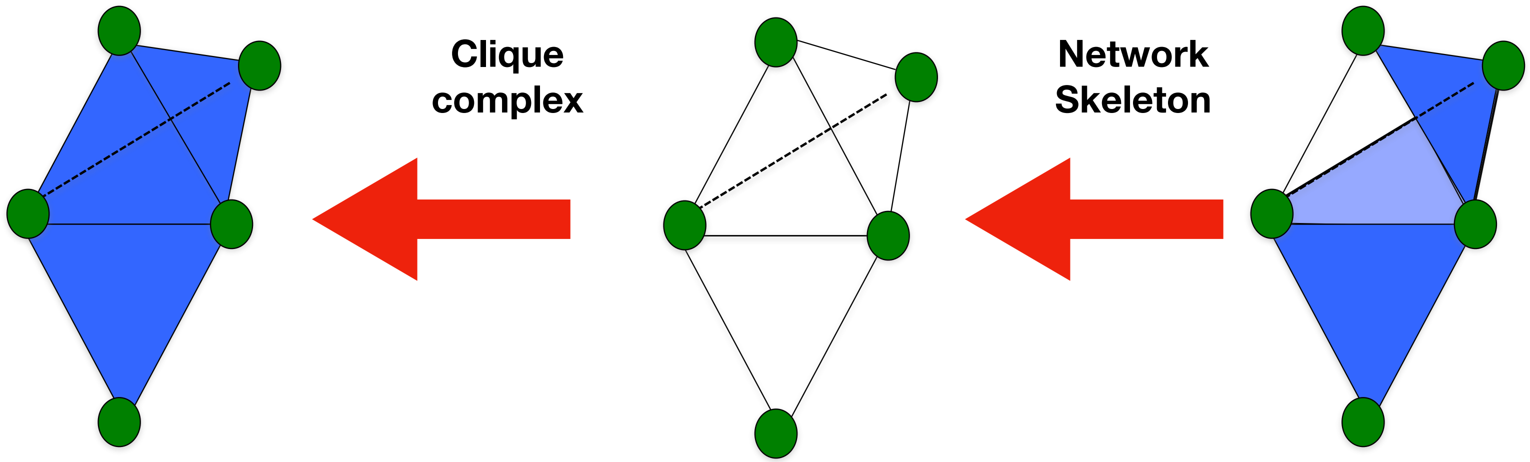
Density of the links ρ



Node neighbourhoods with the same number of nodes and the same density of links can have very different topology

AP Kartun-Giles et al. (2019)

The skeleton of a simplicial complex and its clique complex



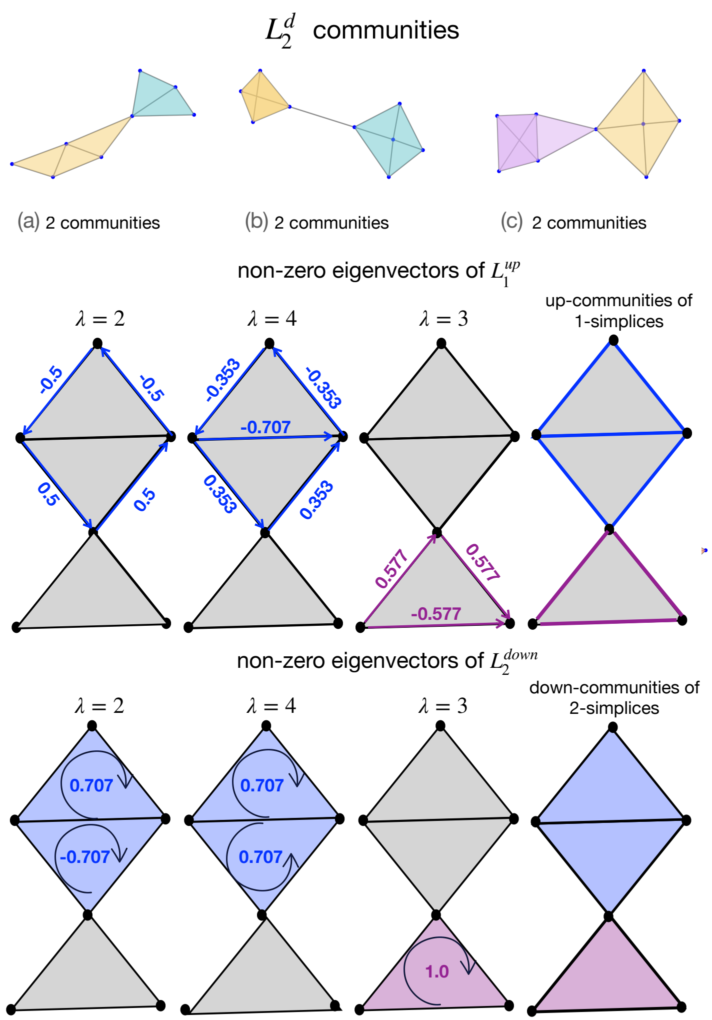
Attention!

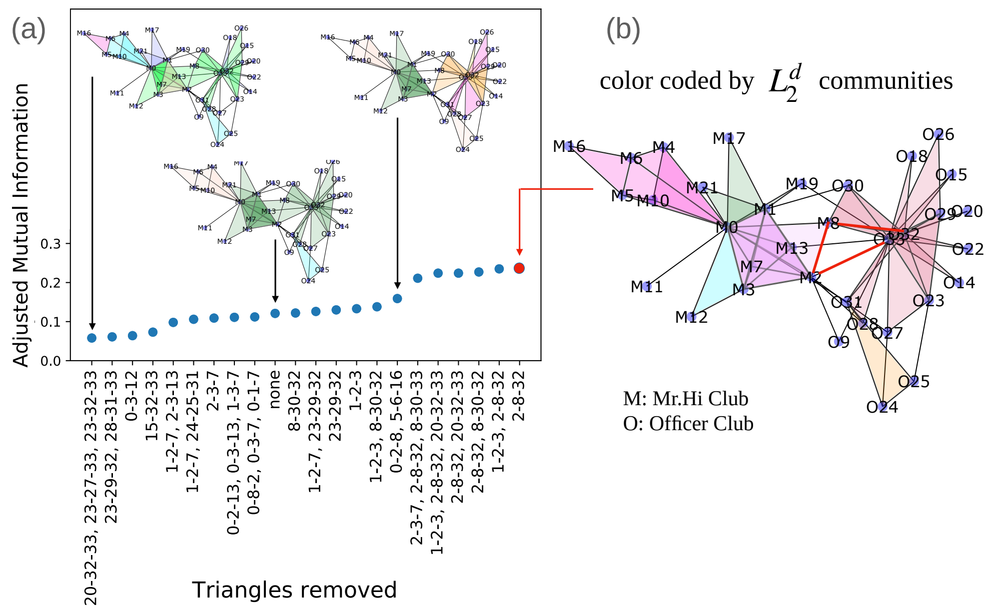
By concatenating the operations you are not guaranteed to return to the initial simplicial complex

Inference of higher-order interactions

Higher-order communities

L_2^d communities





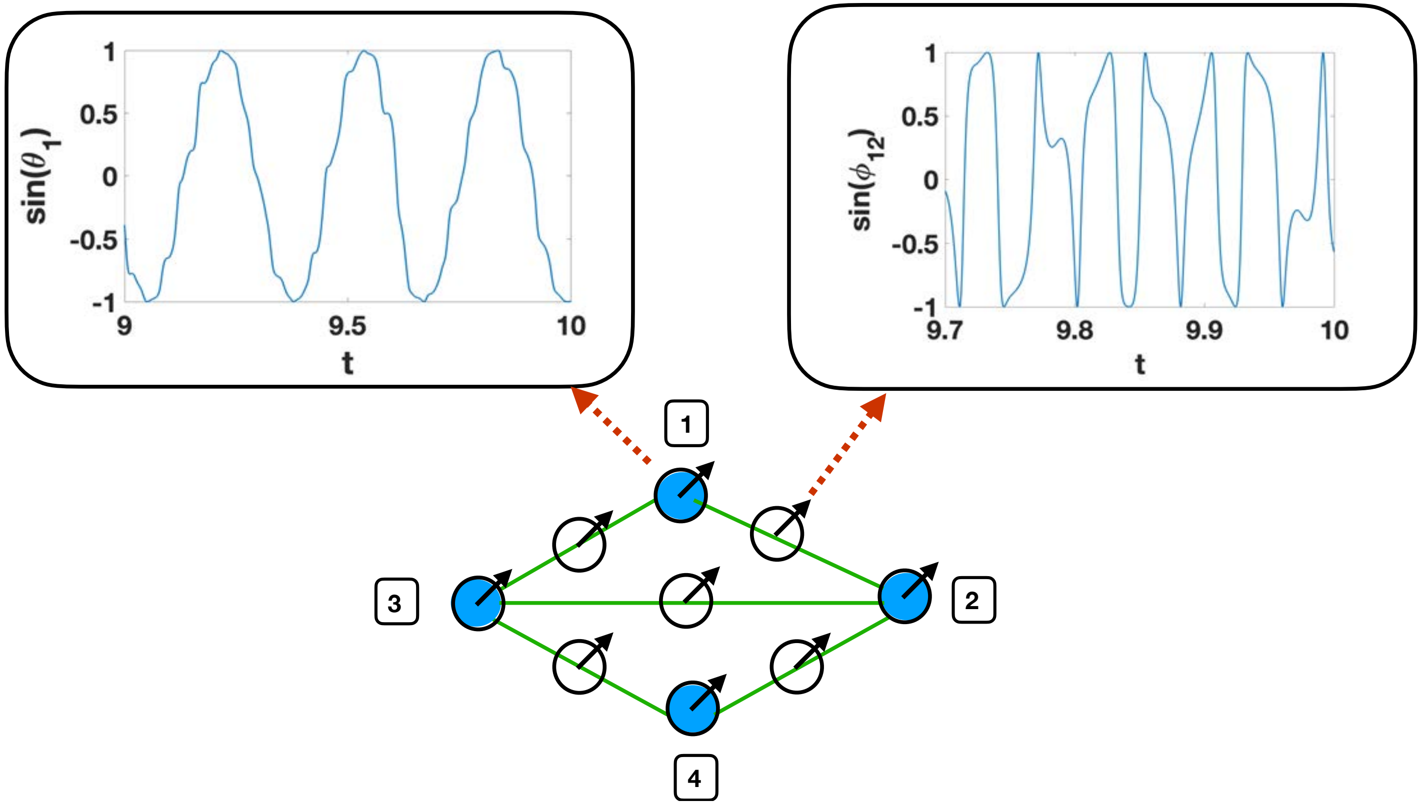
We can infer which higher-order interactions using higher-order communities and ground-truth community assignments

S. Khrisnagopal and GB (2021)

Topological signals,
Hodge Laplacian
And
Dirac operator

Topological signals

Simplicial complexes and networks can sustain dynamical variables (signals) not only defined on nodes but also defined on higher order simplices these signals are called ***topological signals***



Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are co-chains or vector fields

Graph Laplacian in terms of the incidence matrix

The graph Laplacian of elements

$$\left(L_{[0]}\right)_{ij} = \delta_{ij} k_i - a_{ij}$$

Can be expressed in terms of the 1-incidence matrix

as

$$\boxed{\mathbf{L}_{[0]} = \mathbf{B}_{[1]} \mathbf{B}_{[1]}^{\top} .}$$

Higher-order Laplacian

The higher order Laplacians can be defined in terms of the incidence matrices as

$$\boxed{\mathbf{L}_{[n]} = \mathbf{B}_{[n]}^\top \mathbf{B}_{[n]} + \mathbf{B}_{[n+1]} \mathbf{B}_{[n+1]}^\top .}$$

The dimension of the ker$(\mathbf{L}_{[n]})$ is the n-Betti number β_n

The higher order Laplacian can be decomposed as

$$\mathbf{L}_{[n]} = \mathbf{L}_{[n]}^{down} + \mathbf{L}_{[n]}^{up},$$

with

$$\mathbf{L}_{[n]}^{down} = \mathbf{B}_{[n]}^\top \mathbf{B}_{[n]},$$
$$\mathbf{L}_{[n]}^{up} = \mathbf{B}_{[n+1]} \mathbf{B}_{[n+1]}^\top .$$

Higher-order Laplacian

The higher order Laplacians can be defined in terms of the incidence matrices as

$$\boxed{\mathbf{L}_{[n]} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]} + \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}.}$$

The dimension of the $\ker(\mathbf{L}_{[n]})$ is the n-Betti number β_n

The higher order Laplacian can be decomposed as

$$\mathbf{L}_{[n]} = \mathbf{L}_{[n]}^{down} + \mathbf{L}_{[n]}^{up},$$

with

$$\mathbf{L}_{[n]}^{down} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]},$$
$$\mathbf{L}_{[n]}^{up} = \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}.$$

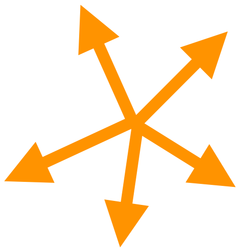
Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

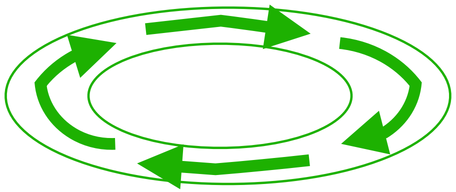
in a irrotational, harmonic and solenoidal components

$$\boxed{\mathbb{R}^{D_n} = \textbf{im}(\mathbf{B}_{[n]}^\top) \oplus \textbf{ker}(\mathbf{L}_{[n]}) \oplus \textbf{im}(\mathbf{B}_{[n+1]})}$$

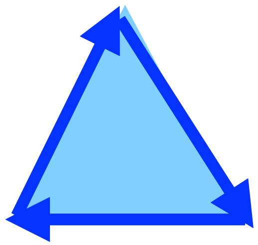
which in the case of topological signals of the links can be sketched as



Irrotational component
Gradient Flow

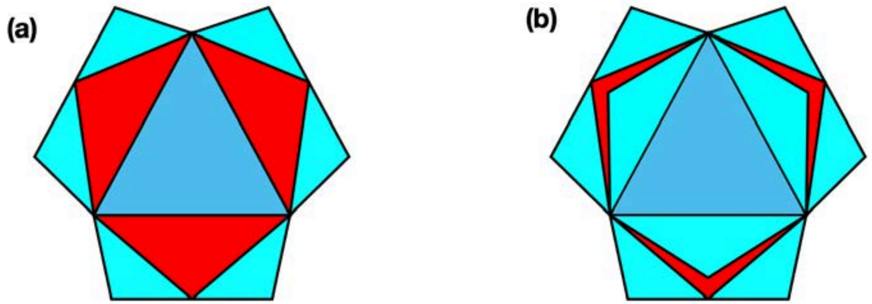


Harmonic component



Solenoidal component
Curl Flow

Apollonian and pseudo-fractal simplicial complexes



- We start at time t=1 with a single d-simplex
- At each time t>1, we glue a d-simplex
    A. to every (d-1)-face added at the previous time (Apollonian simplicial complexes)
    B. to every (d-1)-face of the simplicial complex (pseudo-fractal simplicial complexes)

Higher-order spectral dimension

NGFs, Apollonian and pseudo-fractal network

do not have just a single spectral dimension

but they display a vector of spectral dimensions

$$\mathbf{d_S} = (d_S^{[0]}, d_S^{[1]}, \ldots, d_S^{[d-2]})$$

with one spectral dimension for each m-order up-Laplacian

Higher-order spectral dimension of Apollonian and Pseudo-fractal networks

Apollonian simplicial complexes

d/m	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$m=d-3$	—	3.738 13	4.5742	5.199 79	5.700 72	6.119 32	6.479 49	6.795 96
$m=d-4$	—	—	7.399 62	8.482 12	9.356 64	10.0913	10.7253	11.2833
$m=d-5$	—	—	—	11.729	12.9719	14.0179	14.9217	15.7178
$m=d-6$	—	—	—	—	16.5732	17.9293	19.1017	20.1346
$m=d-7$	—	—	—	—	—	21.8337	23.2741	24.5434
$m=d-8$	—	—	—	—	—	—	27.4423	28.9478
$m=d-9$	—	—	—	—	—	—	—	33.3496

Pseudo-fractal simplicial complexes

d/m	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$m=d-2$	3.169 93	4.0	4.643 86	5.169 93	5.614 71	6.0	6.339 85	6.643 86
$m=d-3$	—	5.315 62	5.869 24	6.280 83	6.605 35	6.871 91	7.0975	7.292 81
$m=d-4$	—	—	8.376 10	8.997 32	9.497 05	9.915 47	10.276	10.5934
$m=d-5$	—	—	—	12.7140	13.7232	14.4689	15.057	15.5463
$m=d-6$	—	—	—	—	17.3048	18.5860	19.5562	20.3283
$m=d-7$	—	—	—	—	—	22.2618	23.7403	24.897
$m=d-8$	—	—	—	—	—	—	27.5667	29.1935
$m=d-9$	—	—	—	—	—	—	—	33.1841

[M. Reitz, G. Bianconi (2020)]

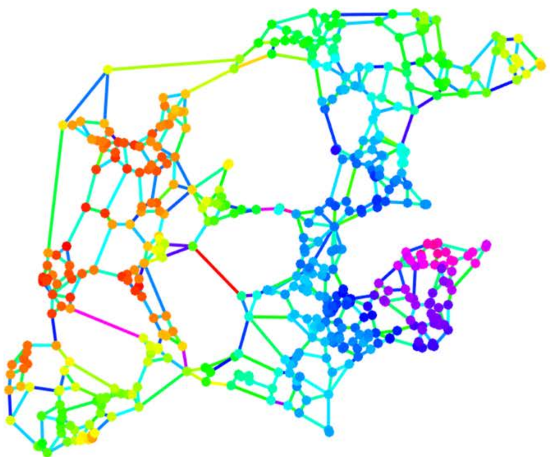
Numerical evidence shows that also NGF have different spectral dimension of higher-order Laplacians

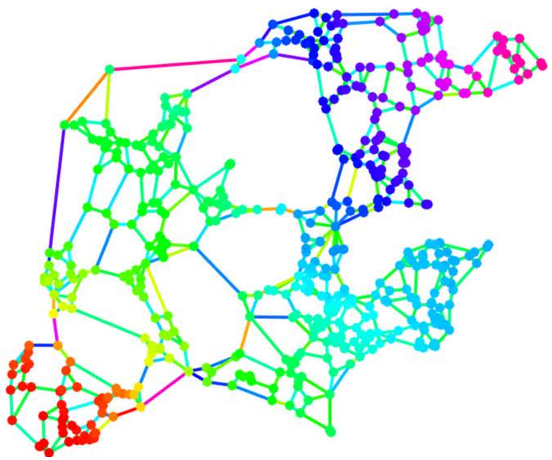
[J.J. Torres, G. Bianconi (2020)]

Topological Dirac operator

How to treat the interaction between topological signals of different dimensions coexisting in the same network topology?

G. Bianconi,
Topological Dirac equation on networks and simplicial complexes (2021)





Topological spinor

On a network we consider the topological spinor

$$\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\chi} \end{pmatrix}$$

Characterising the dynamical state of the topological signals of the network, being a vector with a block structure formed by a

0-cochain and a 1-cochain

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}, \quad \boldsymbol{\chi} = \begin{pmatrix} \chi_{\ell_1} \\ \chi_{\ell_2} \\ \vdots \\ \chi_{\ell_L} \end{pmatrix}.$$

Topological Dirac operator on a network

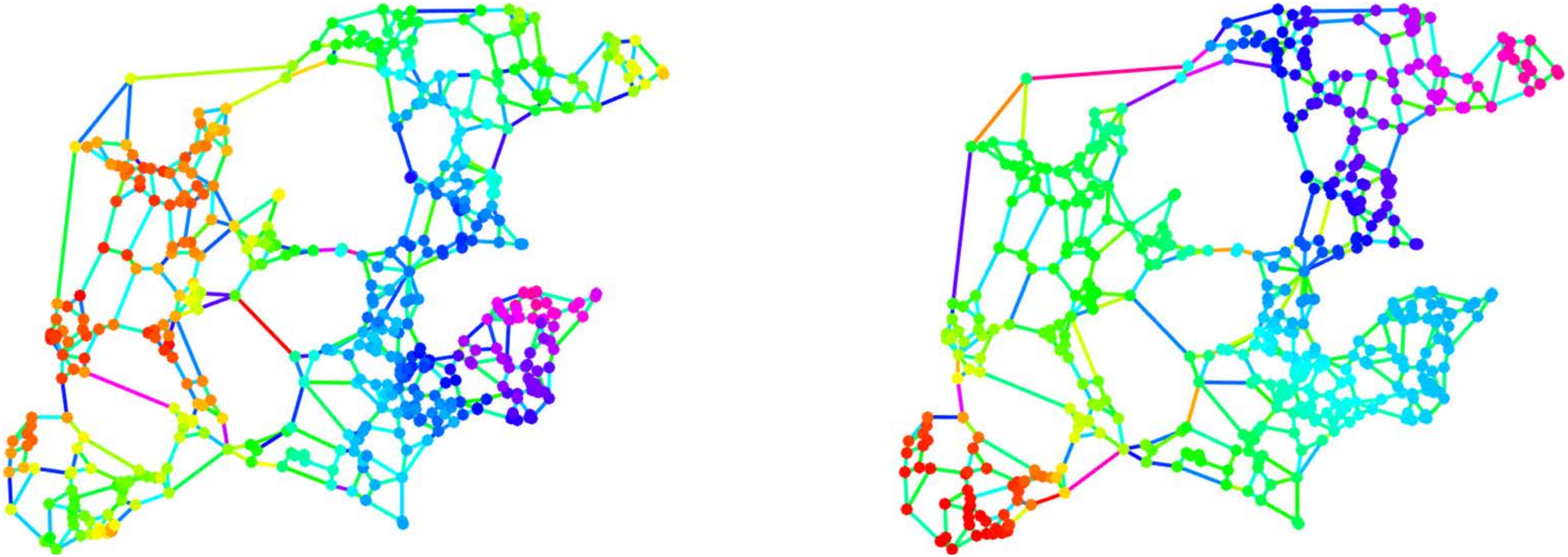
We define the Dirac operator of a network is defined as

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b\mathbf{B}_{[1]} \\ b^{\star}\mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

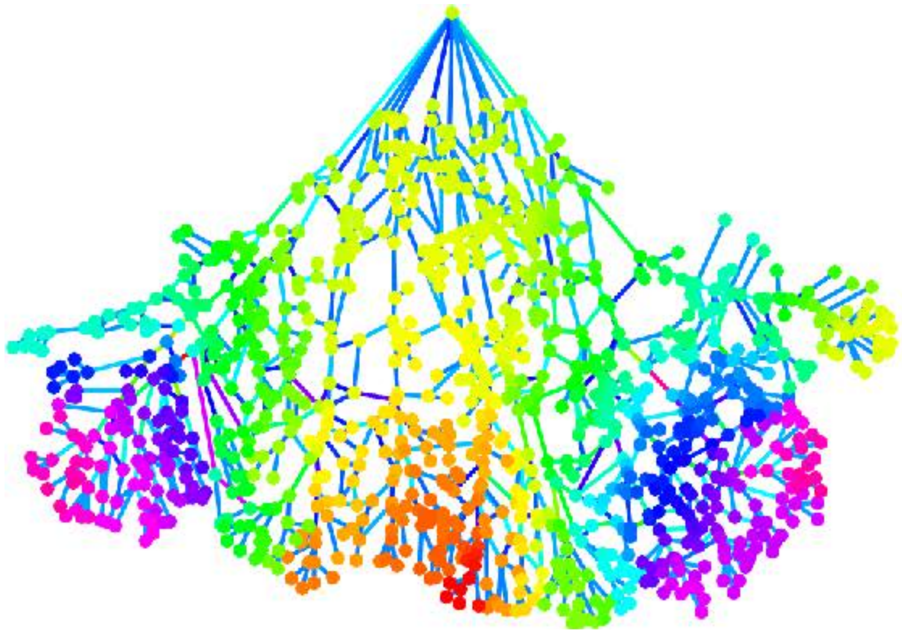
with $b \in \mathbb{C}, \ \ |b| = 1.$

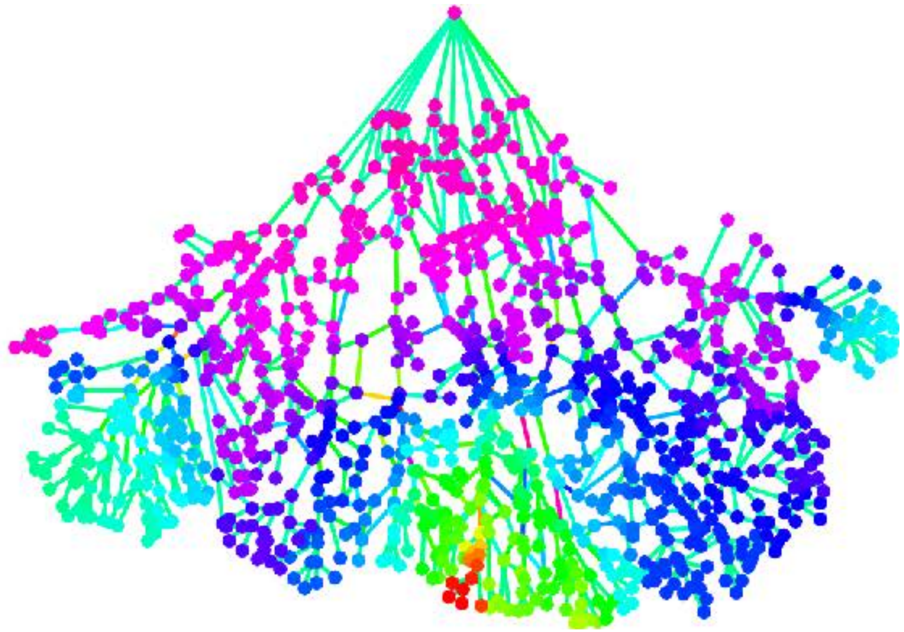
We have the notable property that $\mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]}^{down} \end{pmatrix}$

Energy eigenstates of the Topological Dirac Operator on real networks



Energy eigenstates of the Topological Dirac Operator on real networks





Topological Dirac operator on a simplicial complex

The Topological Dirac operator can be extended to higher-dimensional simplices. For instance on a 3-dimensional simplex it is given by

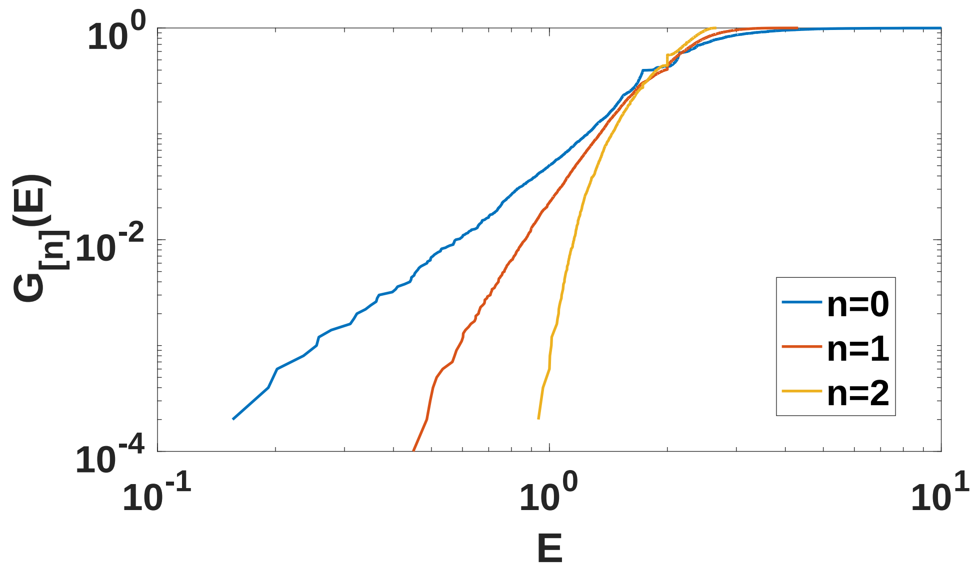
$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b_{[1]}\mathbf{B}_{[1]} & \mathbf{0} & \mathbf{0} \\ b^{\star}_{[1]}\mathbf{B}^{\top}_{[1]} & \mathbf{0} & b_{[2]}\mathbf{B}_{[2]} & \mathbf{0} \\ \mathbf{0} & b^{\star}_{[2]}\mathbf{B}^{\top}_{[2]} & \mathbf{0} & b_{[3]}\mathbf{B}_{[3]} \\ \mathbf{0} & \mathbf{0} & b^{\star}_{[3]}\mathbf{B}^{\top}_{[3]} & \mathbf{0} \end{pmatrix}$$

Topological Dirac equation on simplicial complexes

- The topological Dirac equation can be extended to simplicial complexes, in the case of zero mass the eigenstates are given by

$$E\boldsymbol{\psi} = \mathbf{D}\boldsymbol{\psi}$$

- It can be shown that thanks to the Hodge decomposition this equation leads to a multi-band spectrum of the energy states.

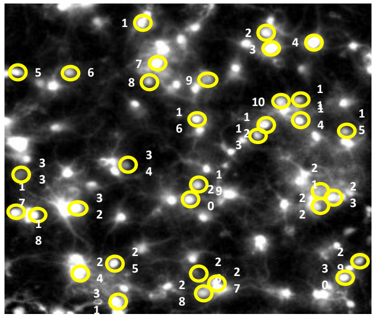


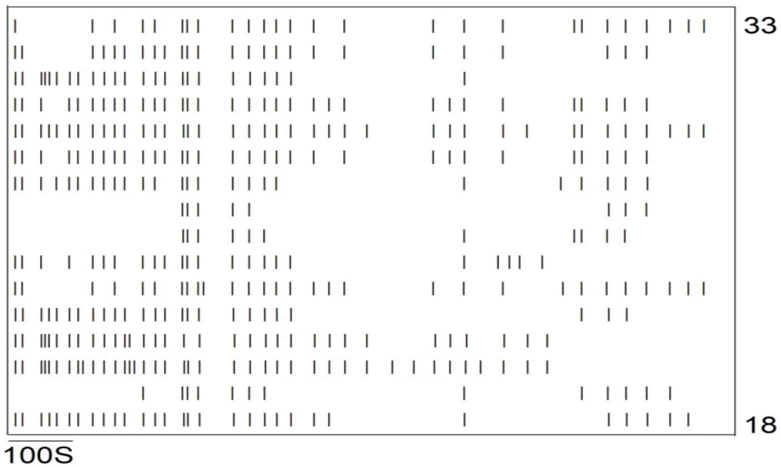
Multi-band eigenspectrum of the Topological Dirac equation on a 3-dimensional NGF

Kumamoto Model on a network

Synchronization is a fundamental dynamical process

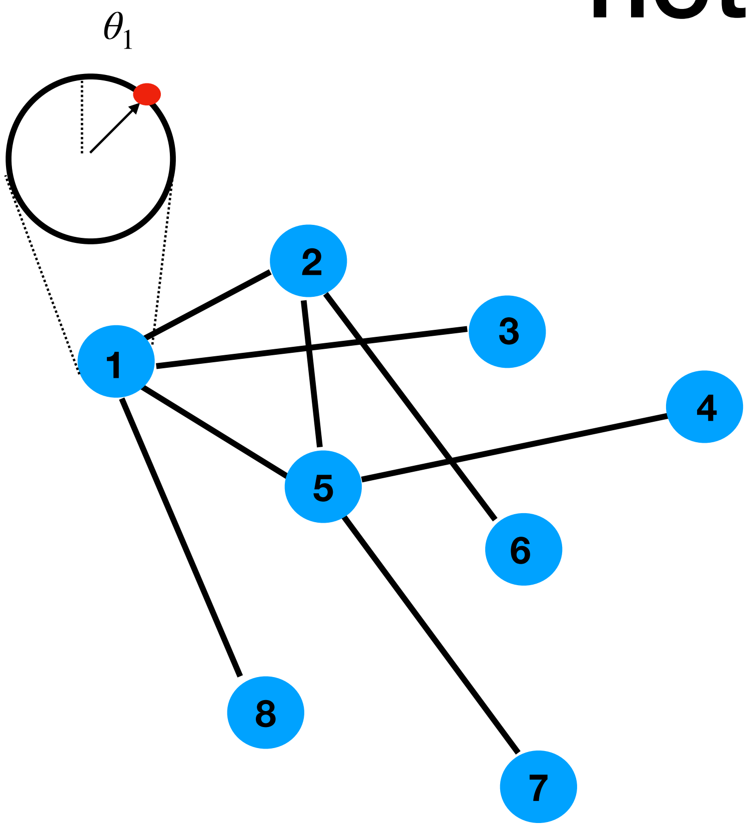
NEURONS





FIREFLIES

Kuramoto model on a network



Given a network of N nodes
defined by an adjacency matrix a
we assign to each node a phase obeying

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j - \theta_i\right)$$

where the internal frequencies of the nodes
are drawn randomly from

$$\omega \sim \mathcal{N}(\Omega, 1)$$

and the coupling constant is σ

Order parameter for synchronization

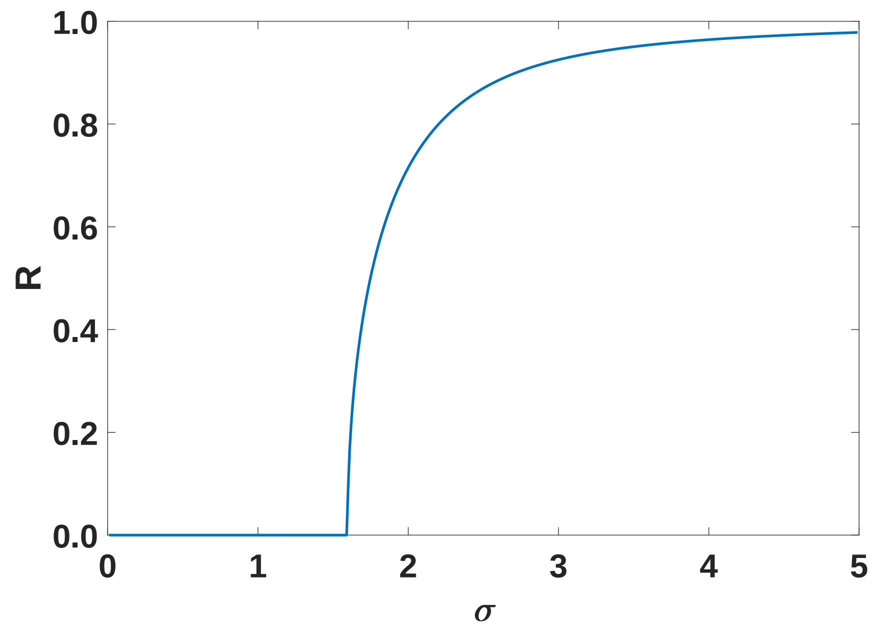
We consider the global order parameter R

$$\boxed{R = \frac{1}{N}\left|\sum_{i=1}^{N} e^{\mathrm{i}\,\theta_i}\right|}$$

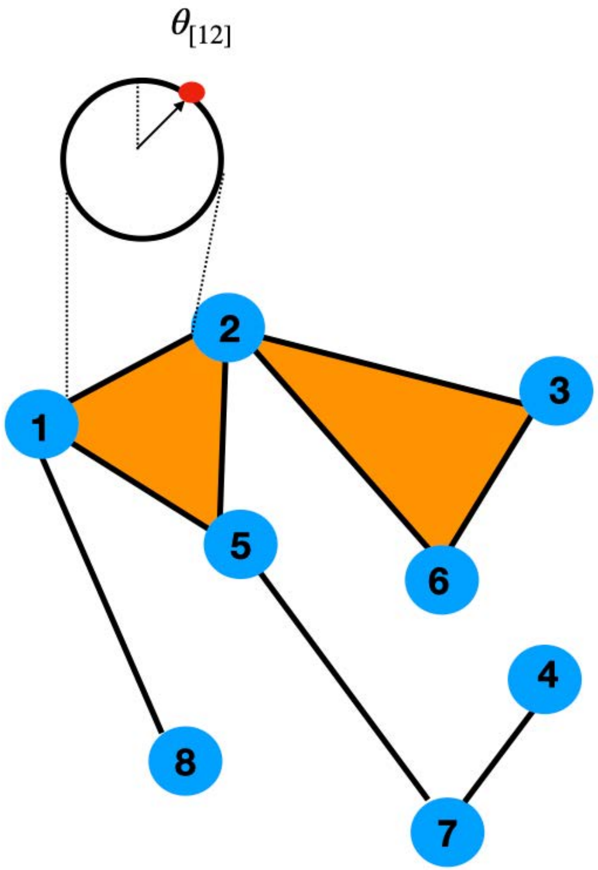
which indicates the

synchronisation transition

$$\boxed{\begin{array}{ll} R \simeq 0 & \textbf{for } \sigma < \sigma_c \\ R \textbf{ finite} & \textbf{for } \sigma \geq \sigma_c \end{array}}$$



The higher-order simplicial Kuramoto model



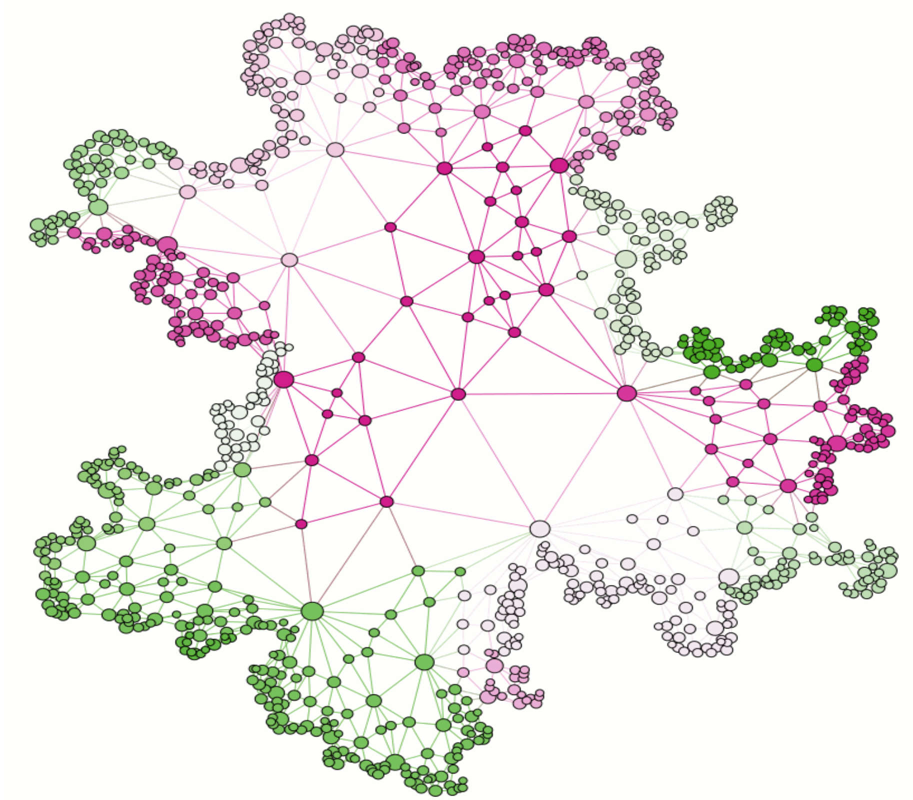
How to define the higher-order Kuramoto model coupling higher dimensional topological signals?

Explosive higher-order Kuramoto model on simplicial complexes

A. P. Millán, J. J. Torres, and G.Bianconi,
***Physical Review Letters*, *124*, 218301 (2020)**

Topological signals

Simplicial complexes can sustain dynamical variables (signals)
not only defined on nodes but also defined on higher order simplices
these signals are called *topological signals*



Standard Kuramoto model in terms of incidence matrices

The standard Kuramoto model, can be expressed in terms

of the incidence matrix $\mathbf{B}_{[1]}$ as

$$\boxed{\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}}$$

where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

Topological signals

We associate to each

n-dimensional simplex α a phase ϕ_α

For instance for n=1 we might associate to each link a oscillating flux

The vector of phases is indicated by

$$\boxed{\boldsymbol{\phi} = (\ldots, \phi_\alpha \ldots)^\top}$$

Simplicial synchronisation

We propose to study the higher-order Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi},}$$

where is the vector of phases associated to n-simplices

and the topological signals ad their internal frequencies are indicated by

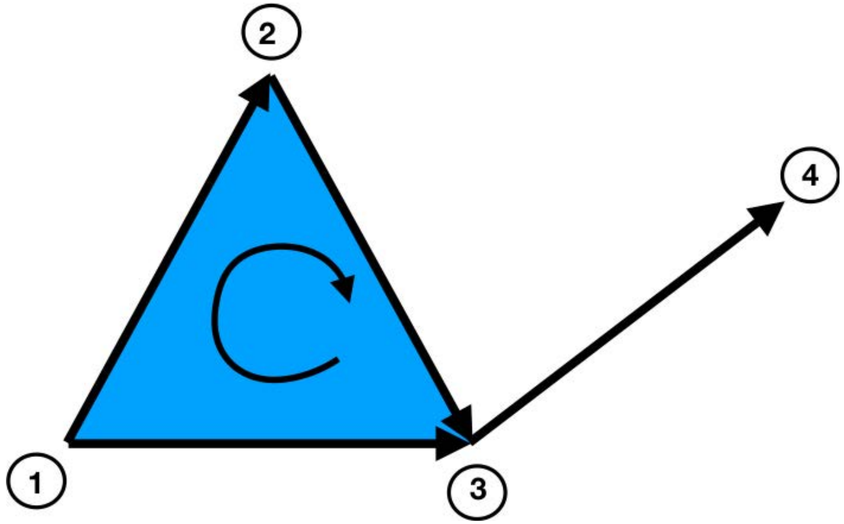
$$\boldsymbol{\phi} = (\ldots, \theta_{\alpha} \ldots)^{\top}$$

$$\hat{\boldsymbol{\omega}} = (\ldots, \hat{\omega}_{\alpha} \ldots)^{\top}$$

with the internal frequencies

$$\hat{\omega}_{\alpha} \sim \mathcal{N}(\Omega, 1)$$

Topologically induced many-body interactions



$$\dot{\phi}_{[12]} = \hat{\omega}_{[12]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[12]} - \phi_{[23]}) + \sin(\phi_{[13]} + \phi_{[12]})\right],$$
$$\dot{\phi}_{[13]} = \hat{\omega}_{[13]} + \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[13]} + \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$
$$\dot{\phi}_{[23]} = \hat{\omega}_{[23]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[23]} - \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$
$$\dot{\phi}_{[34]} = \hat{\omega}_{[34]} - \sigma \left[\sin(\phi_{[34]}) - \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

If we define a higher-order Kuramoto model on

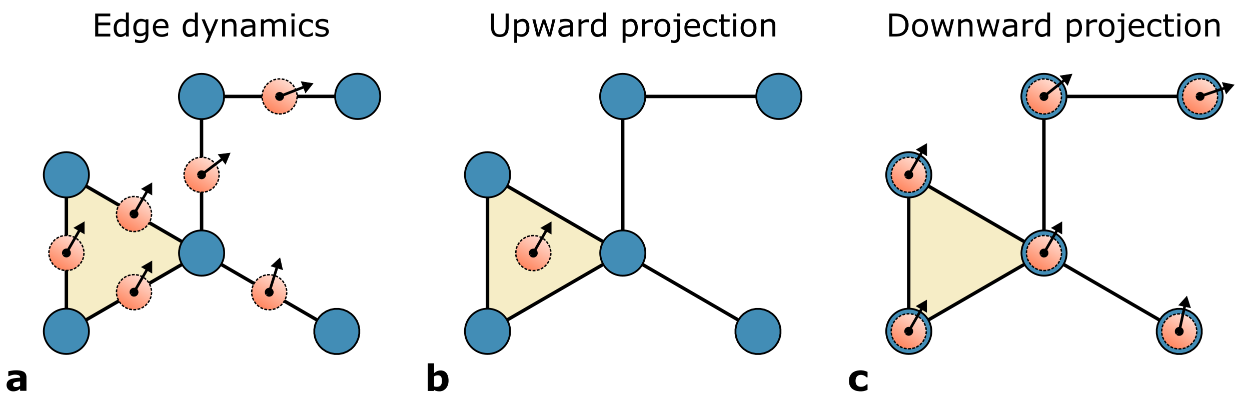
n-simplices,

(let us say links, n=1) a key question is:

What is the dynamics induced

on (n-1) faces and (n+1) faces?

i.e. what is the dynamics induced on nodes and triangles?



Projected dynamics on n-1 and n+1 faces

A natural way to project the dynamics is to use the incidence matrices obtaining

$$\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\phi}$$ **Discrete curl**

$$\boldsymbol{\phi}^{[-]} = \mathbf{B}_{[n]}\boldsymbol{\phi}$$ **Discrete divergence**

Projected dynamics on n-1 and n+1 faces

Thanks to Hodge decomposition,

the projected dynamics

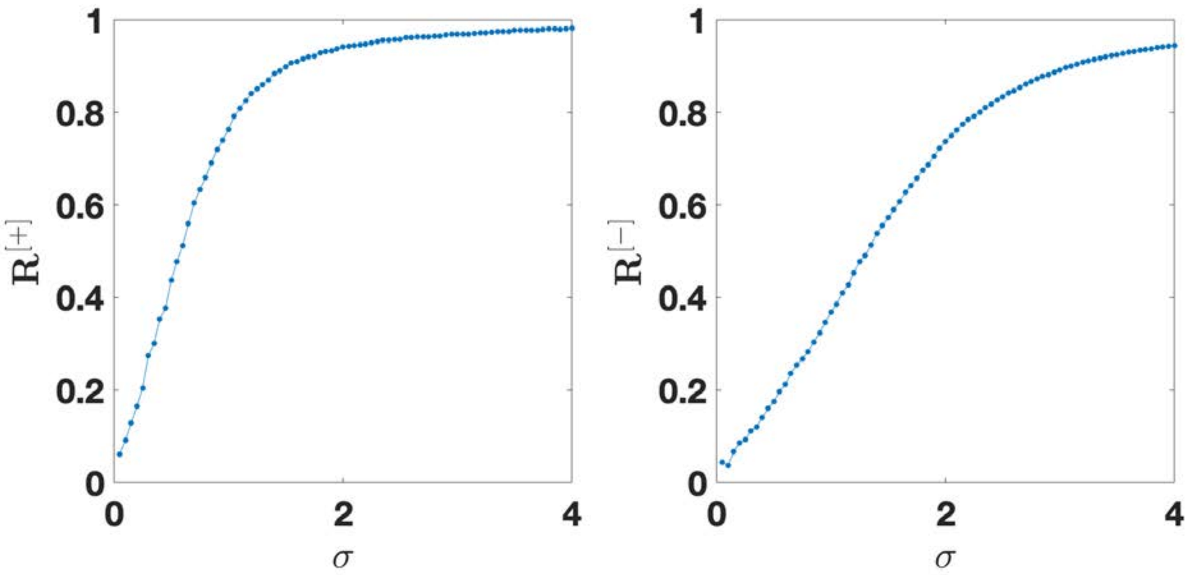
on the (n-1) and (n+1) faces

decouple

$$\boxed{\begin{aligned}\dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[n+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[n+1]}^{[down]} \sin(\boldsymbol{\phi}^{[+]}) \\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[n]}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[n-1]}^{[up]} \sin(\boldsymbol{\phi}^{[-]})\end{aligned}}$$

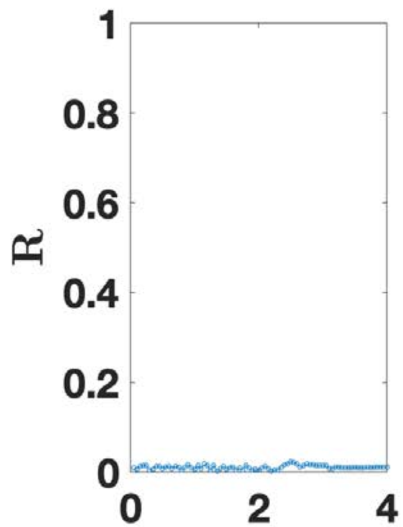
Simplicial Synchronization transition

$$R^{[+]} = \frac{1}{N_{n+1}} \left| \sum_{\alpha=1}^{N_{n+1}} e^{\,i\phi_\alpha^{[+]}} \right| \qquad R^{[-]} = \frac{1}{N_{n-1}} \left| \sum_{\alpha=1}^{N_{n-1}} e^{\,i\phi_\alpha^{[-]}} \right|$$



Order parameters using the n-dimensional phases

$$\boxed{R = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{i\phi_\alpha}\right|}$$



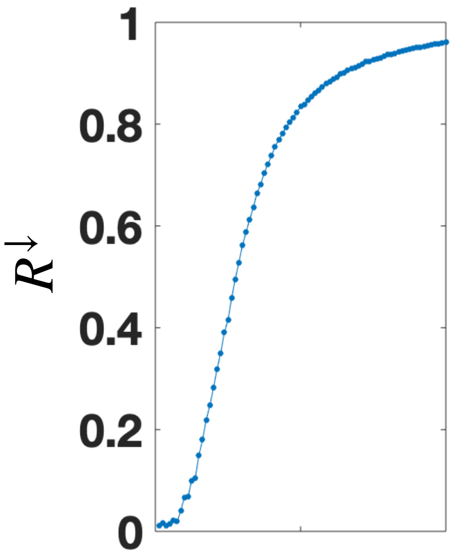
Order parameters using the n-dimensional phases

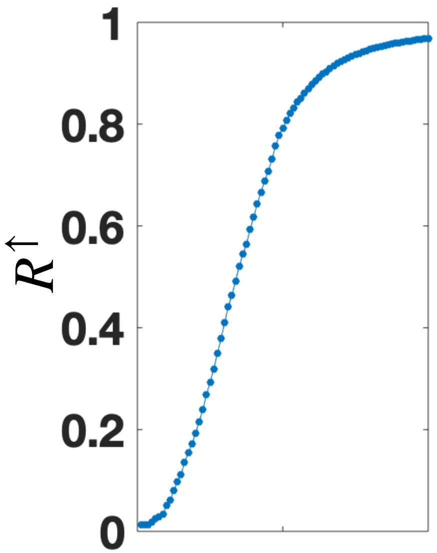
$$\boldsymbol{\phi}^{\downarrow} = \mathbf{L}_{[n]}^{down}\boldsymbol{\phi}$$

$$\boxed{R^{\downarrow} = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{i\,\phi_\alpha^{\downarrow}}\right|}$$

$$\boldsymbol{\phi}^{\uparrow} = \mathbf{L}_{[n]}^{up}\boldsymbol{\phi}$$

$$\boxed{R^{\uparrow} = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{i\,\phi_\alpha^{\uparrow}}\right|}$$





Only if we perform

the correct topological filtering

of the topological signal

we can reveal higher-order synchronisation

Explosive simplicial synchronisation

We propose the Explosive Higher-order Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\phi} - \sigma R^{[+]}\mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]}\boldsymbol{\phi}}$$

Projected dynamics

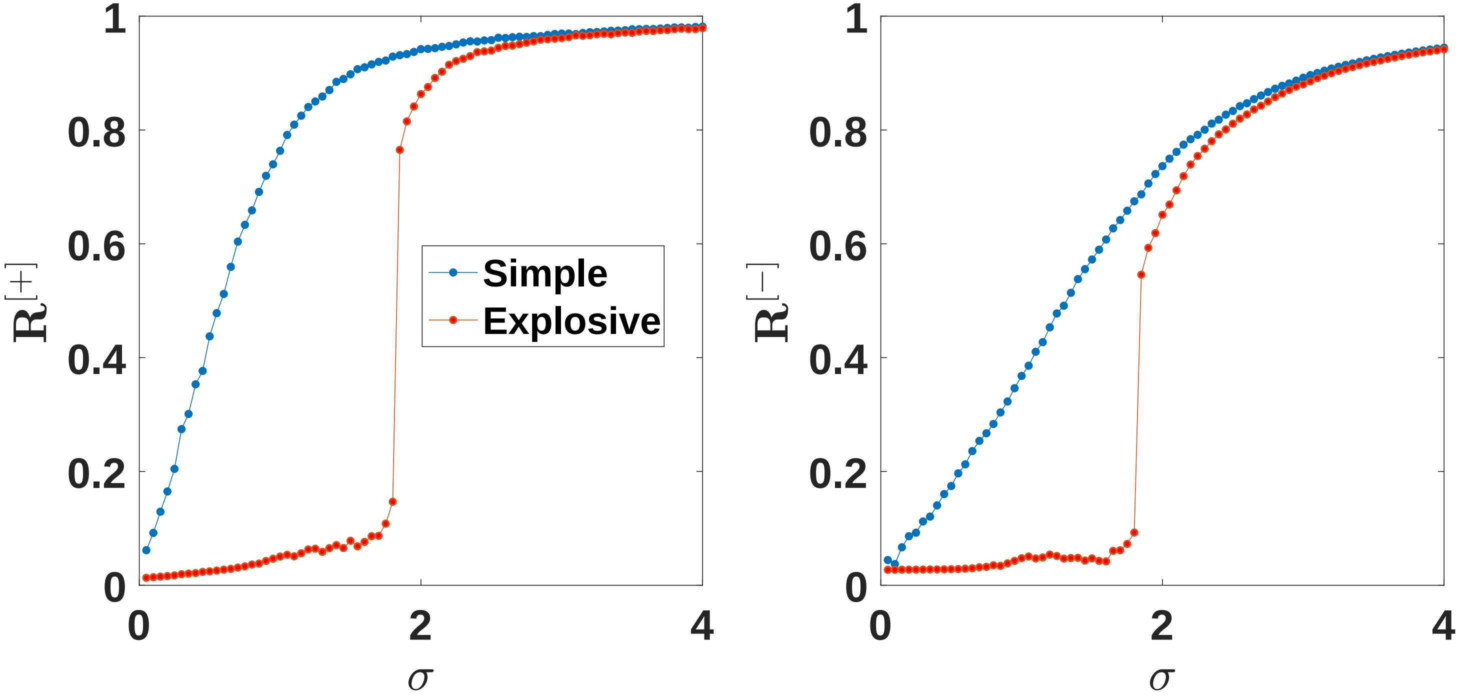
The projected dynamics on

(n+1) and (n-1) are now coupled

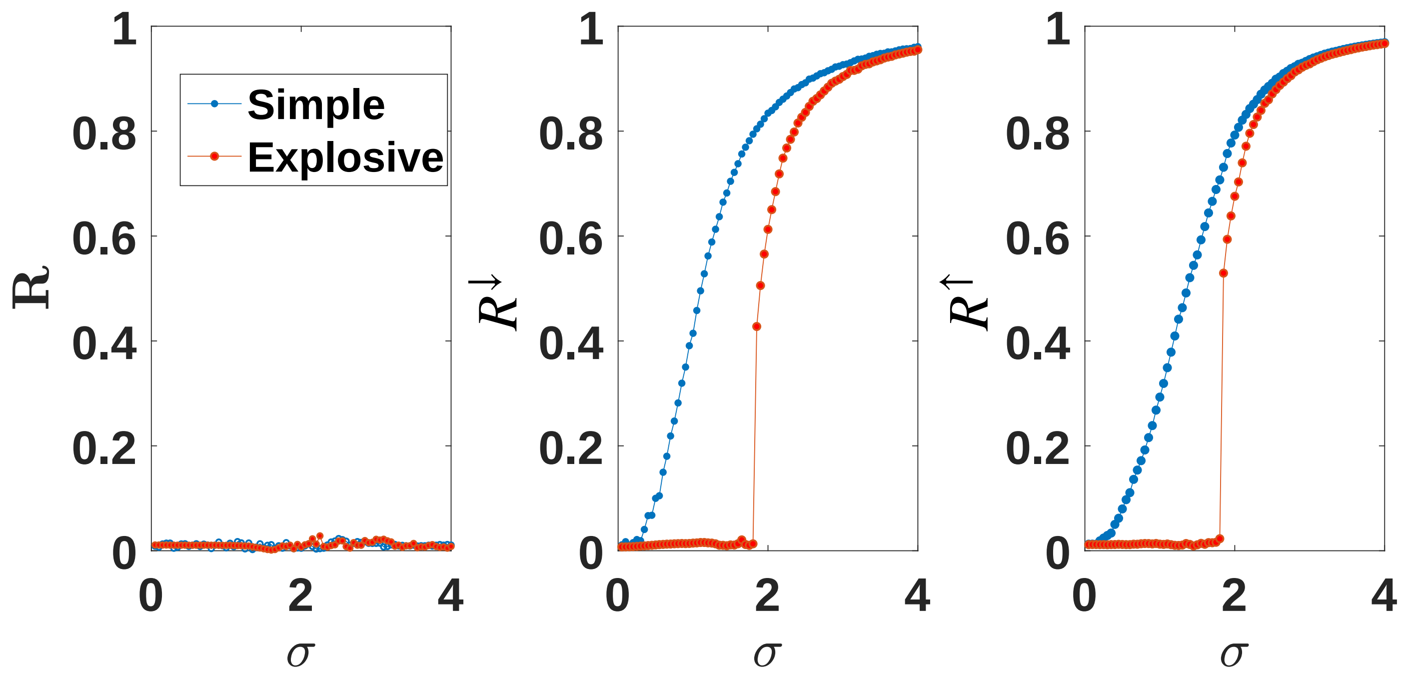
by their order parameters

$$\boxed{\begin{aligned}\dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[n+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{L}_{[n+1]}^{[down]}\sin(\boldsymbol{\phi}^{[+]})\\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[n]}\hat{\boldsymbol{\omega}} - \sigma R^{[+]}\mathbf{L}_{[n-1]}^{[up]}\sin(\boldsymbol{\phi}^{[-]})\end{aligned}}$$

The explosive simplicial synchronisation transition



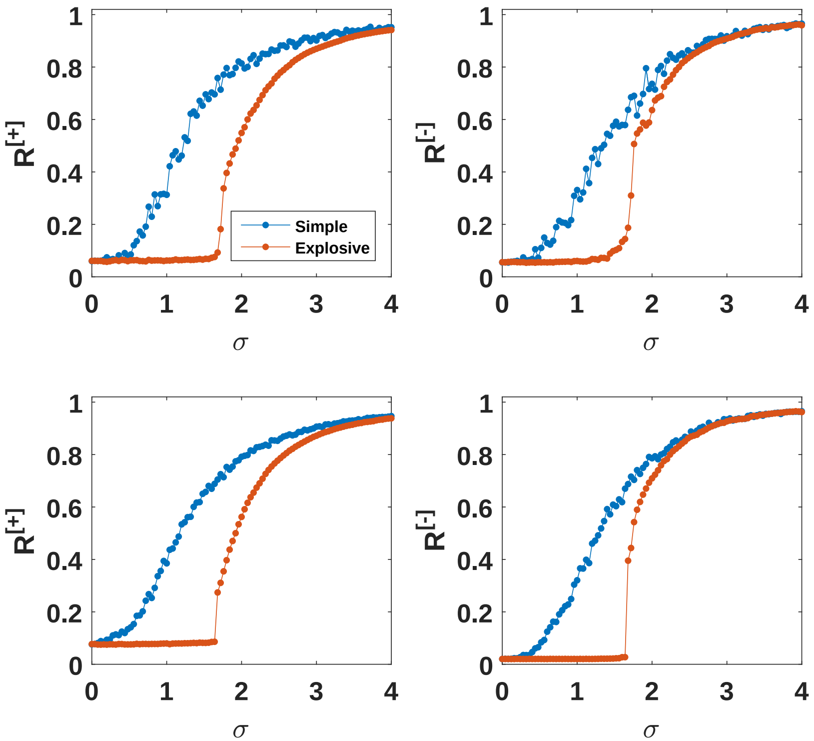
Order parameters associated to n-faces



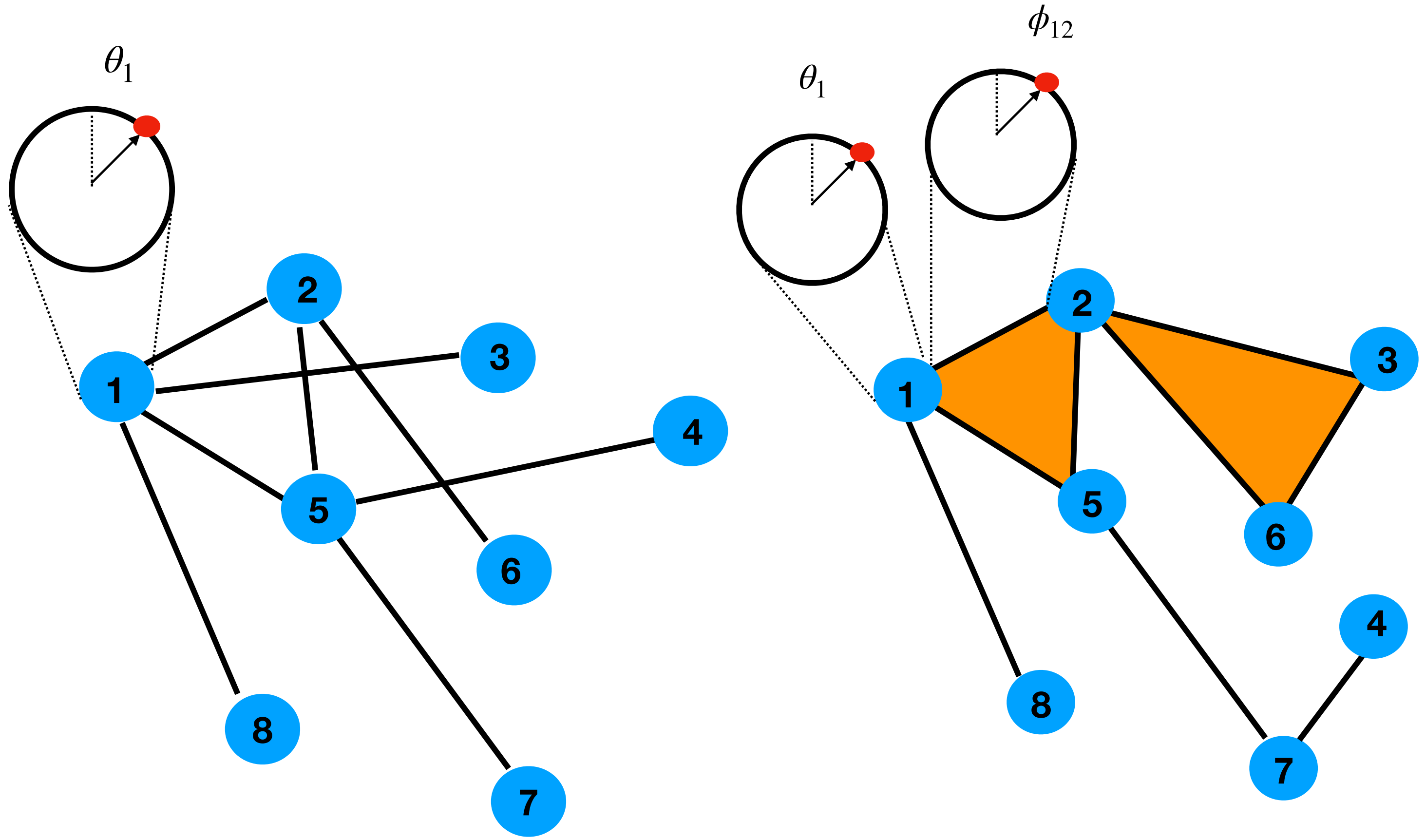
Higher-order synchronisation on real Connectomes

Homo sapiens Connectome

C.elegans Connectome



Coupling topological signals of different dimension

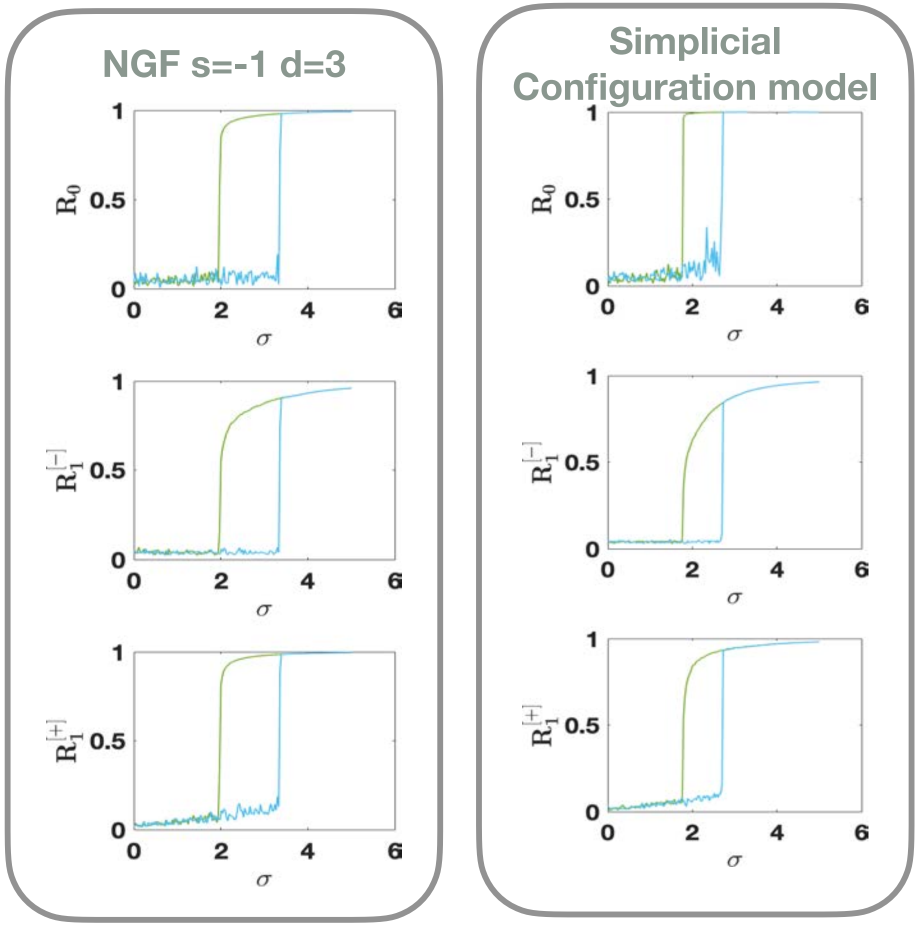


R. Ghorbanchian, J. Restrepo, J.J. Torres and G. Bianconi (2020)

Explosive synchronisation of globally coupled topological signals

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R_1^{[-]} \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^\top \boldsymbol{\theta}$$

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R_1^{[+]} R_0 \mathbf{B}_{[n]}^\top \sin \mathbf{B}_{[n]} \boldsymbol{\phi} - \sigma R_1^{[-]} \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^\top \boldsymbol{\phi}$$



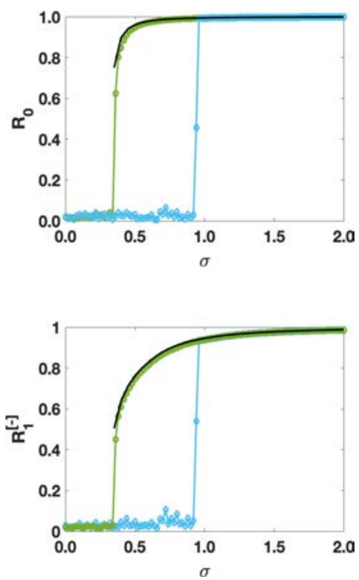
Annealed solution on random networks

The annealed solution captures the backward transition

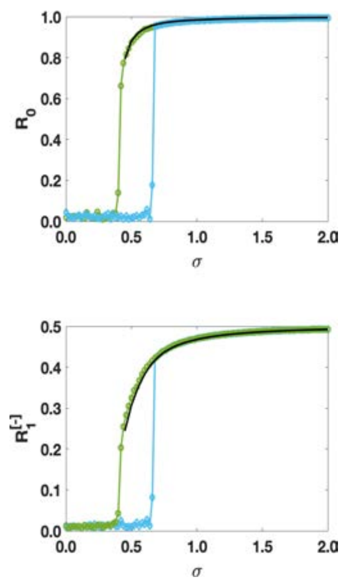
Reveals that the transition is discontinuous

Gives very reliable results for connected networks that are not too sparse

Poisson network



Power-law network



Dirac synchronisation

Dirac synchronization

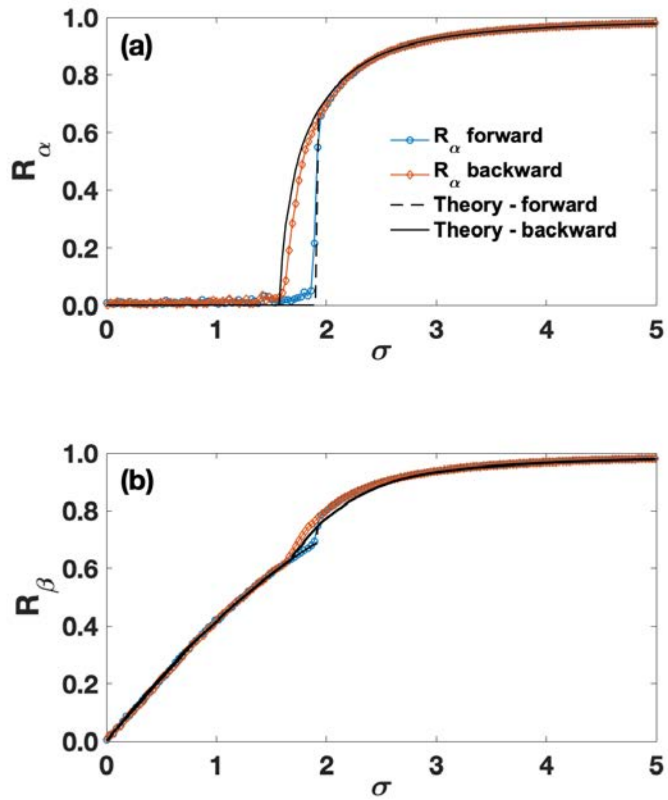
couples topological signals

of different dimensions locally and topologically

using the Dirac operator

Dirac synchronisation is explosive
with a thermodynamically histeresis loop

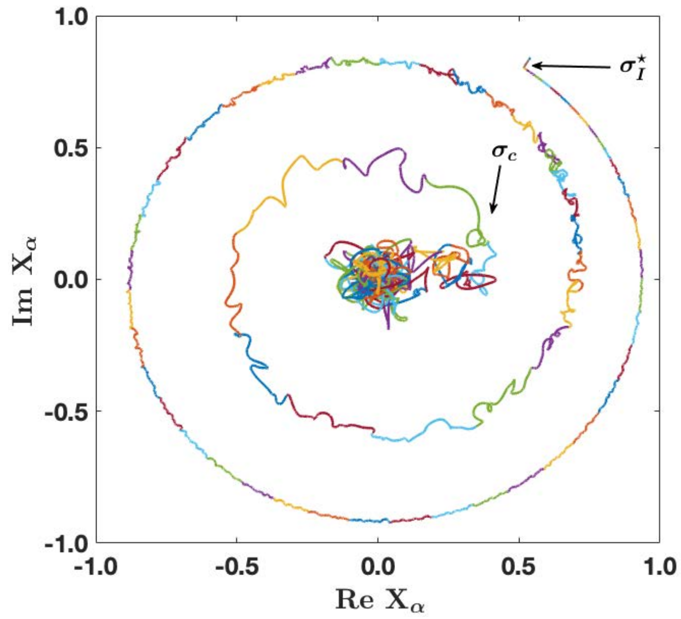
The order parameter involves
a linear combination of
signals of the nodes and signals of the links
(projected on the nodes)



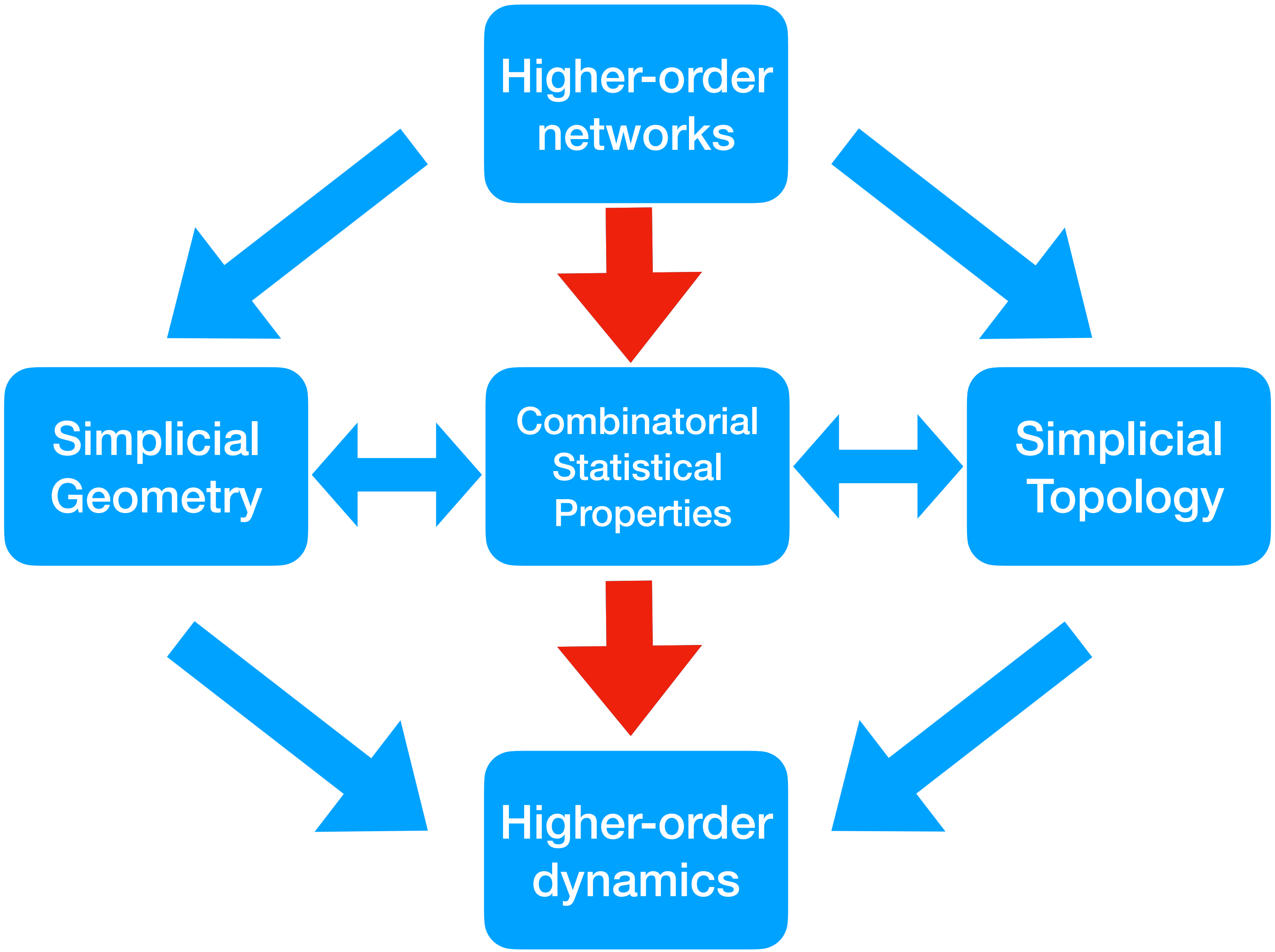
Dirac synchronisation

Dirac synchronisation leads
to the emergence of rhythmic phase in which
the order parameter acquires spontaneously
a dynamical phase in the rotating frame,
i.e. in the frame
in which in average the intrinsic phases
have zero average.

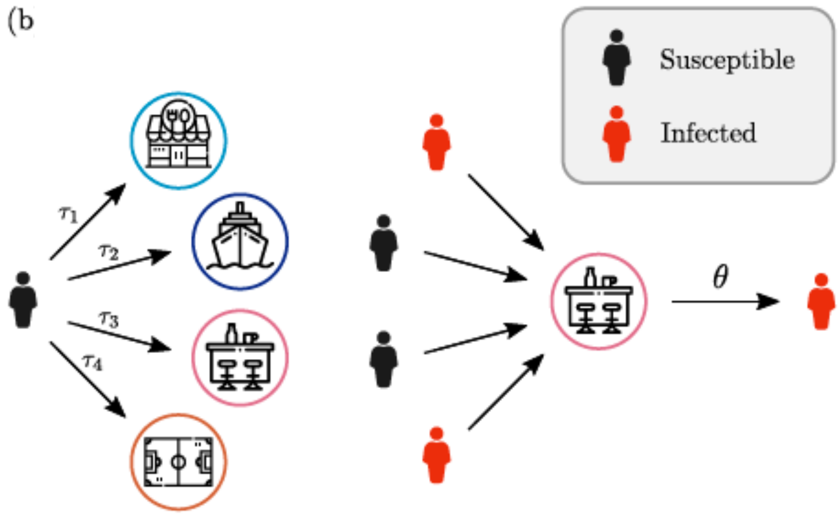
The rhythmic phase in the Dirac synchronisation
sheds light
on topological mechanisms
for the emergence of brain rhythms



Higher-order structure and dynamics



Co-location and non-linear infection kernels in epidemic spreading processes



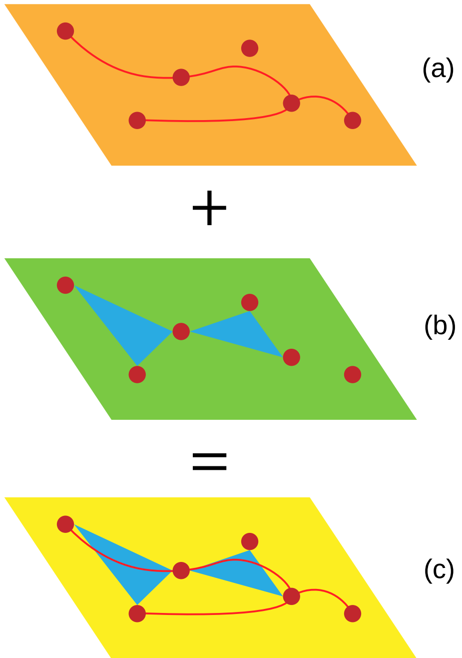
Co-location affects epidemic spreading

It can be modelled by a temporal hypergraph

Threshold effects are important factors that can lead to non-linear infection kernels

G. St-Onge et al. Phys. Rev. Lett. (2021)

Multiplex Hypergraphs

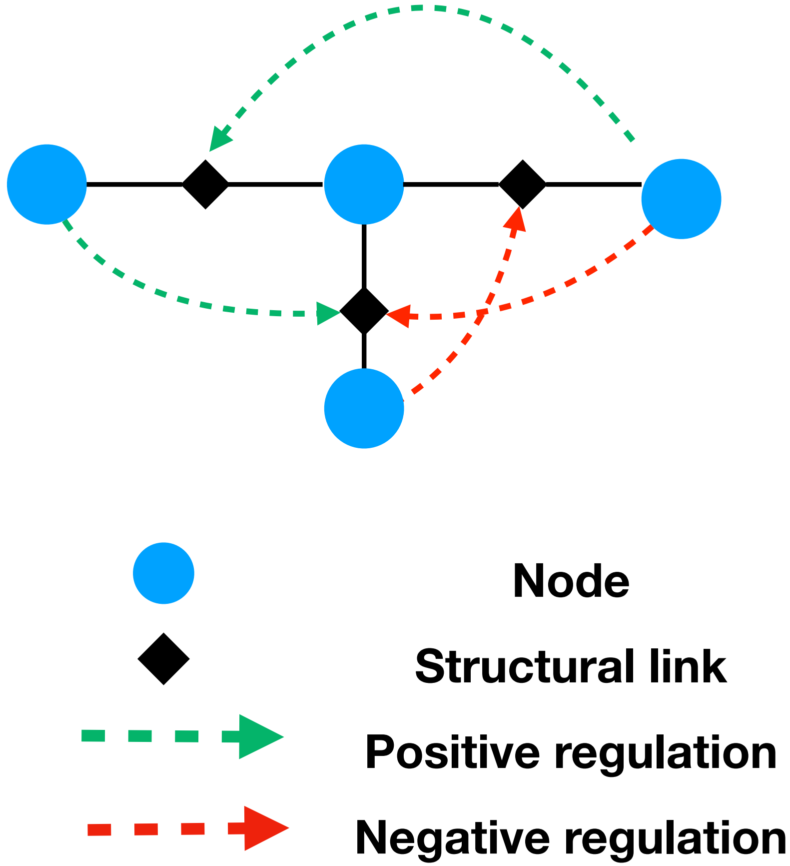


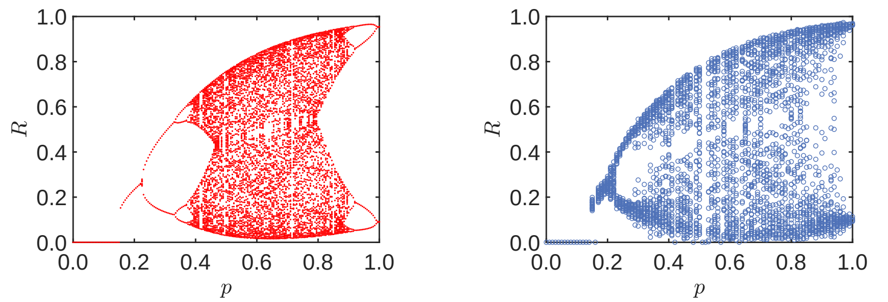
**Multiplex Hypergraphs
are formed by layers each capturing
interaction of a given order**

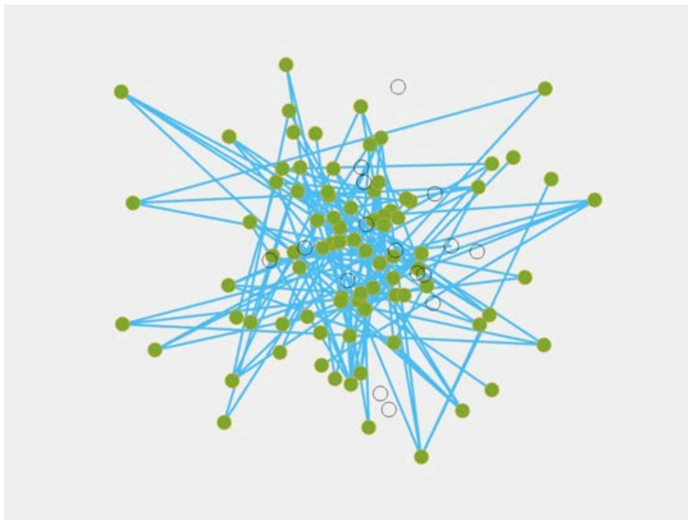
**Higher-order percolation problems
including cooperative effects
are discontinuous**

H. Sun and GB PRE (2021)

Triadic interactions induce blinking and chaos in connectivity of higher-order networks







H. Sun, F. Radicchi, J. Kurths and G. Bianconi (2022)

Conclusions

Simplicial synchronisation
is able to capture the synchronisation of
topological signals of higher dimension.

It can be detected by monitoring
the irrotational and the solenoidal components of
the topological signal.

Dirac synchronisation coupling locally
topologically signals of different dimensions
is explosive
and gives rise of rhythmic phase

References and collaborators

The Dirac operator
Bianconi, Ginestra. "The topological Dirac equation of networks and simplicial complexes." *JPhys Complexity 2*, 035022 (2021).

Higher-order simplicial Kumamoto model
Millán, A.P., Torres, J.J. and Bianconi, G., 2020. Explosive higher-order Kuramoto dynamics on simplicial complexes. *Physical Review Letters*, *124*(21), p.218301.

Globally Coupled dynamics of nodes and links
Ghorbanchian, Reza, Juan G. Restrepo, Joaquín J. Torres, and Ginestra Bianconi. "Higher-order simplicial synchronization of coupled topological signals." *Communications Physics* 4, no. 1 (2021): 1-13.

Topological synchronization is explosive
Calmon, Lucille, Juan G. Restrepo, Joaquín J. Torres, and Ginestra Bianconi. "Topological synchronization: explosive transition and rhythmic phase." *arXiv preprint arXiv:2107.05107* (2021).